ELEMENTS OF ARITHMETIC.

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ARITHMETIC.

BY

AUGUSTUS DE MORGAN,
OF TRINITY COLLEGE CAMBRIDGE;
FELLOW OF THE ROYAL ASTRONOMICAL SOCIETY, AND OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY,
PROFESSOR OF MATHEMATICS IN UNIVERSITY COLLEGE, LONDON.

"Hominis studiosi est intelligere, quas utilitates proprie afferat arithmetica his, qui solidam et perfectam doctrinam in cæteris philosophiæ partibus explicant. Quod enim vulgo dicunt, principium esse dimidium totius, id vel maxime in philosophiæ partibus conspicitur."—MELANCTHON.

"Ce n'est point par la routine qu on s'instruit, c'est par sa propre réflexion; et il est essentiel de contracter l'habitude de se rendre raison de ce qu'on fait: cette habitude s'acquiert plus facilement qu'on ne pense; et une fois acquise, elle ne se perd plus.'—CONDILLAC.

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PREFACE.

The preceding editions of this work were published in 1830, 1832, 1835, and 1840. This fifth edition differs from the three preceding, as to the body of the work, in nothing which need prevent the four, or any two of them, from being used together in a class. But it is considerably augmented by the addition of eleven new Appendixes,* relating to matters on which it is most desirable that the advanced student should possess information. The first Appendix, on *Computation*, and the sixth, on *Decimal Money*, should be read and practised by every student with as much attention as any part of the work. The mastery of the rules for instantaneous conversion of the usual fractions of a pound sterling into decimal fractions, gives the possessor the greater part of the advantage which he would derive from the introduction of a decimal coinage.

At the time when this work was first published, the importance of establishing arithmetic in the young mind upon reason and demonstration, was not admitted by many. The case is now altered: schools exist in which rational

* Some separate copies of these Appendixes are printed, for those who may desire to add them to the former editions.

arithmetic is taught, and mere rules are made to do no more than their proper duty. There is no necessity to advocate a change which is actually in progress, as the works which are published every day sufficiently shew. And my principal reason for alluding to the subject here, is merely to warn those who want nothing but routine, that this is not the book for their purpose.

A. DE MORGAN.

London, May 1, 1846.

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ELEMENTS OF ARITHMETIC.

BOOK I.

PRINCIPLES OF ARITHMETIC.

SECTION I.

NUMERATION.

1. IMAGINE a multitude of objects of the same kind assembled together; for example, a company of horsemen. One of the first things that must strike a spectator, although unused to counting, is, that to each man there is a horse. Now, though men and horses are things perfectly unlike, yet, because there is one of the first kind to every one of the second, one man to every horse, a new notion will be formed in the mind of the observer, which we express in words by saying that there is the same *number* of men as of horses. A savage, who had no other way of counting, might remember this number by taking a pebble for each man. Out of a method as rude as this has sprung our system of calculation, by the steps which are pointed out in the following articles. Suppose that there are two companies of horsemen, and a person wishes to know in which of them is the greater number, and also to be able to recollect how many there are in each.

2. Suppose that while the first company passes by, he drops a pebble into a basket for each man whom he sees. There is no connexion between the pebbles and the horsemen but this, that for every horseman

there is a pebble; that is, in common language, the *number* of pebbles and of horsemen is the same. Suppose that while the second company passes, he drops a pebble for each man into a second basket: he will then have two baskets of pebbles, by which he will be able to convey to any other person a notion of how many horsemen there were in each company. When he wishes to know which company was the larger, or contained most horsemen, he will take a pebble out of each basket, and put them aside. He will go on doing this as often as he can, that is, until one of the baskets is emptied. Then, if he also find the other basket empty, he says that both companies contained the same number of horsemen; if the second basket still contain some pebbles, he can tell by them how many more were in the second than in the first.

3. In this way a savage could keep an account of any numbers in which he was interested. He could thus register his children, his cattle, or the number of summers and winters which he had seen, by means of pebbles, or any other small objects which could be got in large numbers. Something of this sort is the practice of savage nations at this day, and it has in some places lasted even after the invention of better methods of reckoning. At Rome, in the time of the republic, the prætor, one of the magistrates, used to go every year in great pomp, and drive a nail into the door of the temple of Jupiter; a way of remembering the number of years which the city had been built, which probably took its rise before the introduction of writing.

4. In process of time, names would be given to those collections of pebbles which are met with most frequently. But as long as small numbers only were required, the most convenient way of reckoning them would be by means of the fingers. Any person could make with his two hands the little calculations which would be necessary for his purposes, and would name all the different collections of the fingers. He would thus get words in his own language answering to one, two, three, four, five, six, seven, eight, nine, and ten. As his wants increased, he would find it necessary to give names to larger numbers; but here he would be stopped by the immense quantity of words which

he must have, in order to express all the numbers which he would be obliged to make use of. He must, then, after giving a separate name to a few of the first numbers, manage to express all other numbers by means of those names.

5. I now shew how this has been done in our own language. The English names of numbers have been formed from the Saxon: and in the following table each number after ten is written down in one column, while another shews its connexion with those which have preceded it.

One		eleven	ten and one*
two		twelve	ten and two
three		thirteen	ten and three
four		fourteen	ten and four
five		fifteen	ten and five
six		sixteen	ten and six
seven		seventeen	ten and seven
eight		eighteen	ten and eight
nine		nineteen	ten and nine
ten		twenty	two tens
twenty-one	two tens and one	fifty	five tens
twenty-two	two tens and two	sixty	six tens
&c. &c.	&c. &c.	seventy	seven tens
thirty	three tens	eighty	eight tens
&c.	&c.	ninety	nine tens
forty	four tens	a hundred	ten tens
&c.	&c.		
a hundred and one	ten tens and one		
&c.	&c.		
a thousand	ten hundreds		
ten thousand			

* It has been supposed that *eleven* and *twelve* are derived from the Saxon for *one left* and *two left* (meaning, after ten is removed); but there seems better reason to think that *leven* is a word meaning ten, and connected with *decem*.

a hundred thousand	
a million	ten hundred thousand or one thousand thousand
ten millions	
a hundred millions	
&c.	

6. Words, written down in ordinary language, would very soon be too long for such continual repetition as takes place in calculation. Short signs would then be substituted for words; but it would be impossible to have a distinct sign for every number: so that when some few signs had been chosen, it would be convenient to invent others for the rest out of those already made. The signs which we use are as follow:

0	1	2	3	4	5	6	7	8	9
nought	one	two	three	four	five	six	seven	eight	nine

I now proceed to explain the way in which these signs are made to represent other numbers.

7. Suppose a man first to hold up one finger, then two, and so on, until he has held up every finger, and suppose a number of men to do the same thing. It is plain that we may thus distinguish one number from another, by causing two different sets of persons to hold up each a certain number of fingers, and that we may do this in many different ways. For example, the number fifteen might be indicated either by fifteen men each holding up one finger, or by four men each holding up two fingers and a fifth holding up seven, and so on. The question is, of all these contrivances for expressing the number, which is the most convenient? In the choice which is made for this purpose consists what is called the method of *numeration.*

8. I have used the foregoing explanation because it is very probable that our system of numeration, and almost every other which is used in the world, sprung from the practice of reckoning on the fingers, which children usually follow when first they begin to count. The method which I have described is the rudest possible; but, by a little

alteration, a system may be formed which will enable us to express enormous numbers with great ease.

9. Suppose that you are going to count some large number, for example, to measure a number of yards of cloth. Opposite to yourself suppose a man to be placed, who keeps his eye upon you, and holds up a finger for every yard which he sees you measure. When ten yards have been measured he will have held up ten fingers, and will not be able to count any further unless he begin again, holding up one finger at the eleventh yard, two at the twelfth, and so on. But to know how many have been counted, you must know, not only how many fingers he holds up, but also how many times he has begun again. You may keep this in view by placing another man on the right of the former, who directs his eye towards his companion, and holds up one finger the moment he perceives him ready to begin again, that is, as soon as ten yards have been measured. Each finger of the first man stands only for one yard, but each finger of the second stands for as many as all the fingers of the first together, that is, for ten. In this way a hundred may be counted, because the first may now reckon his ten fingers once for each finger of the second man, that is, ten times in all, and ten tens is one hundred (5).* Now place a third man at the right of the second, who shall hold up a finger whenever he perceives the second ready to begin again. One finger of the third man counts as many as all the ten fingers of the second, that is, counts one hundred. In this way we may proceed until the third has all his fingers extended, which will signify that ten hundred or one thousand have been counted (5). A fourth man would enable us to count as far as ten thousand, a fifth as far as one hundred thousand, a sixth as far as a million, and so on.

10. Each new person placed himself towards your left in the rank opposite to you. Now rule columns as in the next page, and to the right of them all place in words the number which you wish to represent; in the first column on the right, place the number of fingers

* The references are to the preceding articles.

which the first man will be holding up when that number of yards has been measured. In the next column, place the fingers which the second man will then be holding up; and so on.

	7th.	6th.	5th.	4th.	3d.	2d.	1st.	
I.						5	7	fifty-seven.
II.					1	0	4	one hundred and four.
III.					1	1	0	one hundred and ten.
IV.				2	3	4	8	two thousand three hundred and forty-eight.
V.			1	5	9	0	6	fifteen thousand nine hundred and six.
VI.		1	8	7	0	0	4	one hundred and eighty-seven thousand and four.
VII.	3	6	9	7	2	8	5	three million, six hundred and ninety-seven thousand, two hundred and eighty-five.

11. In I. the number fifty-seven is expressed. This means (5) five tens and seven. The first has therefore counted all his fingers five times, and has counted seven fingers more. This is shewn by five fingers of the second man being held up, and seven of the first. In II. the number one hundred and four is represented. This number is (5) ten tens and four. The second person has therefore just reckoned all his fingers once, which is denoted by the third person holding up one finger; but he has not yet begun again, because he does not hold up a finger until the first has counted ten, of which ten only four are completed. When all the last-mentioned ten have been counted, he then holds up one finger, and the first being ready to begin again, has no fingers extended, and the number obtained is eleven tens, or ten tens and one ten, or one hundred and ten. This is the case in III. You will now find no difficulty with the other numbers in the table.

12. In all these numbers a figure in the first column stands for only as many yards as are written under that figure in (6). A figure in the second column stands, not for as many yards, but for as many tens of yards; a figure in the third column stands for as many hundreds of yards; in the fourth column for as many thousands of yards; and so on: that is, if we suppose a figure to move from any column to the one

on its left, it stands for ten times as many yards as before. Recollect this, and you may cease to draw the lines between the columns, because each figure will be sufficiently well known by the *place* in which it is; that is, by the number of figures which come upon the right hand of it.

13. It is important to recollect that this way of writing numbers, which has become so familiar as to seem the *natural* method, is not more natural than any other. For example, we might agree to signify one ten by the figure of one with an accent, thus, $1'$; twenty or two tens by $2'$; and so on: one hundred or ten tens by $1''$; two hundred by $2''$; one thousand by $1'''$; and so on: putting Roman figures for accents when they become too many to write with convenience. The fourth number in the table would then be written $2'''\ 3''\ 4'\ 8$, which might also be expressed by $8\ 4'\ 3''\ 2'''$, $4'\ 8\ 3''\ 2'''$; or the order of the figures might be changed in any way, because their meaning depends upon the accents which are attached to them, and not upon the place in which they stand. Hence, a cipher would never be necessary; for 104 would be distinguished from 14 by writing for the first $1''4$, and for the second $1'4$. The common method is preferred, not because it is more exact than this, but because it is more simple.

14. The distinction between our method of numeration and that of the ancients, is in the meaning of each figure depending partly upon the place in which it stands. Thus, in 44444 each four stands for four of *something;* but in the first column on the right it signifies only four of the pebbles which are counted; in the second, it means four collections of ten pebbles each; in the third, four of one hundred each; and so on.

15. The things measured in (11) were yards of cloth. In this case one yard of cloth is called the *unit.* The first figure on the right is said to be in the *units' place*, because it only stands for so many units as are in the number that is written under it in (6). The second figure is said to be in the *tens'* place, because it stands for a number of tens of units. The third, fourth, and fifth figures are in the places of the *hundreds*, *thousands*, and *tens of thousands*, for a similar reason.

16. If the quantity measured had been acres of land, an acre of land would have been called the *unit*, for the unit is *one* of the things which are measured. Quantities are of two sorts; those which contain an exact number of units, as 47 yards, and those which do not, as 47 yards and a half. Of these, for the present, we only consider the first.

17. In most parts of arithmetic, all quantities must have the same unit. You cannot say that 2 yards and 3 feet make 5 *yards* or 5 *feet*, because 2 and 3 make 5; yet you may say that 2 *yards* and 3 *yards* make 5 *yards*, and that 2 *feet* and 3 *feet* make 5 *feet*. It would be absurd to try to measure a quantity of one kind with a unit which is a quantity of another kind; for example, to attempt to tell how many yards there are in a gallon, or how many bushels of corn there are in a barrel of wine.

18. All things which are true of some numbers of one unit are true of the same numbers of any other unit. Thus, 15 pebbles and 7 pebbles together make 22 pebbles; 15 acres and 7 acres together make 22 acres, and so on. From this we come to say that 15 and 7 make 22, meaning that 15 things of the same kind, and 7 more of the same kind as the first, together make 22 of that kind, whether the kind mentioned be pebbles, horsemen, acres of land, or any other. For these it is but necessary to say, once for all, that 15 and 7 make 22. Therefore, in future, on this part of the subject I shall cease to talk of any particular units, such as pebbles or acres, and speak of numbers only. A number, considered without intending to allude to any particular things, is called an *abstract* number: and it then merely signifies repetitions of a unit, or the *number of times* a unit is repeated.

19. I will now repeat the principal things which have been mentioned in this chapter.

I. Ten signs are used, one to stand for nothing, the rest for the first nine numbers. They are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The first of these is called a *cipher*.

II. Higher numbers have not signs for themselves, but are signified by placing the signs already mentioned by the side of each other, and agreeing that the first figure on the right hand shall keep the value which it has when it stands alone; that the second on the right hand shall mean ten times as many as it does when it stands alone; that the third figure shall mean one hundred times as many as it does when it stands alone; the fourth, one thousand times as many; and so on.

III. The right-hand figure is said to be in the *units' place*, the next to that in the *tens' place*, the third in the *hundreds' place*, and so on.

IV. When a number is itself an exact number of tens, hundreds, or thousands, &c., as many ciphers must be placed on the right of it as will bring the number into the place which is intended for it. The following are examples:

Fifty, or five tens, 50: seven hundred, 700.

Five hundred and twenty-eight thousand, 528000.

If it were not for the ciphers, these numbers would be mistaken for 5, 7, and 528.

V. A cipher in the middle of a number becomes necessary when any one of the denominations, units, tens, &c. is wanting. Thus, twenty thousand and six is 20006, two hundred and six is 206. Ciphers might be placed at the beginning of a number, but they would have no meaning. Thus 026 is the same as 26, since the cipher merely shews that there are no hundreds, which is evident from the number itself.

20. If we take out of a number, as 16785, any of those figures which come together, as 67, and ask, what does this sixty-seven mean? of what is it sixty-seven? the answer is, sixty-seven of the same collections as the 7, when it was in the number; that is, 67 hundreds. For the 6 is 6 thousands, or 6 ten hundreds, or sixty hundreds; which, with the 7, or 7 hundreds, is 67 hundreds: similarly, the 678 is 678 tens. This number may then be expressed either as

1 ten-thousand 6 thousands 7 hundreds 8 tens and 5;

or 16 thousands 78 tens and 5; or 1 ten thousand 678 tens and 5;

or 167 hundreds 8 tens and 5; or 1678 tens and 5, and so on.

21. EXERCISES.

I. Write down the signs for;—four hundred and seventy-six; two thousand and ninety-seven; sixty-four thousand three hundred and fifty; two millions seven hundred and four; five hundred and seventy-eight millions of millions.

II. Write at full length 53, 1805, 1830, 66707, 180917324, 66713721, 90976390, 25000000.

III. What alteration takes place in a number made up entirely of nines, such as 99999, by adding one to it?

IV. Shew that a number which has five figures in it must be greater than one which has four, though the first have none but small figures in it, and the second none but large ones. For example, that 10111 is greater than 9879.

22. You now see that the convenience of our method of numeration arises from a few simple signs being made to change their value as they change the column in which they are placed. The same advantage arises from counting in a similar way all the articles which are used in every-day life. For example, we count money by dividing it into pounds, shillings, and pence, of which a shilling is 12 pence, and a pound 20 shillings, or 240 pence. We write a number of pounds, shillings, and pence in three columns, generally placing points between the columns. Thus, 263 pence would not be written as 263, but as £1 . 1 . 11, where £ shews that the 1 in the first column is a pound. Here is a *system of numeration* in which a number in the second column on the right means 12 times as much as the same number in the first; and one in the third column is twenty times as great as the same in the second, or 240 times as great as the same in the first. In each of the tables of measures which you will hereafter meet with, you will see a separate system of numeration, but the methods of calculation for all will be the same.

23. In order to make the language of arithmetic shorter, some other signs are used. They are as follow:

I. 15+38 means that 38 is to be added to 15, and is the same thing as 53. This is the *sum* of 15 and 38, and is read fifteen *plus* thirty-eight (*plus* is the Latin for *more*).

II. 64—12 means that 12 is to be taken away from 64, and is the same thing as 52. This is the *difference* of 64 and 12, and is read sixty-four *minus* twelve (*minus* is the Latin for *less*).

III. 9×8 means that 8 is to be taken 9 times, and is the same thing as 72. This is the *product* of 9 and 8, and is read nine *into* eight.

IV. $\frac{108}{6}$ means that 108 is to be divided by 6, or that you must find out how many sixes there are in 108; and is the same thing as 18. This is the *quotient* of 108 and 6; and is read a hundred and eight *by* six.

V. When two numbers, or collections of numbers, with the foregoing signs, are the same, the sign = is put between them. Thus, that 7 and 5 make 12, is written in this way, 7+5=12. This is called an *equation*, and is read, seven *plus* five *equals* twelve. It is plain that we may construct as many equations as we please. Thus:

$$7+9-3=12+1\ ;\ \frac{12}{2}-1+3\times2=11, \text{ and so on.}$$

24. It often becomes necessary to speak of something which is true not of any one number only, but of all numbers. For example, take 10 and 7; their sum* is 17, their difference is 3. If this sum and difference be added together, we get 20, which is twice the greater of the two numbers first chosen. If from 17 we take 3, we get 14, which is twice the less of the two numbers. The same thing will be found to hold good of any two numbers, which gives this general proposition,—If the sum and difference of two numbers be added together, the result is twice the greater of the two; if the difference be taken from the sum, the result is twice the lesser of the two. If, then, we take *any* numbers, and call them the first number and the second number, and let the first number be the greater; we have

(1st No.+2d No.)+(1st No.−2d No.)=twice 1st No.

(1st No.+2d No.)−(1st No.−2d No.)=twice 2d No.

The brackets here enclose the things which must be first done, before the signs which join the brackets are made use of. Thus,

* Any little computations which occur in the rest of this section may be made on the fingers, or with counters.

$8-(2+1)\times(1+1)$ signifies that $2+1$ must be taken $1+1$ times, and the product must be subtracted from 8. In the same manner, any result made from two or more numbers, which is true whatever numbers are taken, may be represented by using first No., second No., &c., to stand for them, and by the signs in (23). But this may be much shortened; for as first No., second No., &c., may mean any numbers, the letters *a* and *b* may be used instead of these words; and it must now be recollected that *a* and *b* stand for two numbers, provided only that *a* is greater than *b*. Let twice *a* be represented by $2a$, and twice *b* by $2b$. The equations then become

$$(a+b)+(a-b)=2a, \text{ and } (a+b)-(a-b)=2b.$$

This may be explained still further, as follows:

25. Suppose a number of sealed packets, marked *a*, *b*, *c*, *d*, &c., on the outside, each of which contains a distinct but unknown number of counters. As long as we do not know how many counters each contains, we can make the letter which belongs to each stand for its number, so as to talk of *the number a*, instead of the number in the packet marked *a*. And because we do not know the numbers, it does not therefore follow that we know nothing whatever about them; for there are some connexions which exist between all numbers, which we call *general properties* of numbers. For example, take any number, multiply it by itself, and subtract one from the result; and then subtract one from the number itself. The first of these will always contain the second exactly as many times as the original number increased by one. Take the number 6; this multiplied by itself is 36, which diminished by one is 35: again, 6 diminished by 1 is 5; and 35 contains 5, 7 times, that is, 6+1 times. This will be found to be true of any number, and, when proved, may be said to be true of the number contained in the packet marked *a*, or of the number *a*. If we represent *a* multiplied by itself by *aa*,* we have, by (23)

$$\frac{aa-1}{a-1}=a+1.$$

* This should be (23) $a\times a$, but the sign $\times$ is unnecessary here. It is used with numbers, as in 2×7, to prevent confounding this, which is 14, with 27.

26. When, therefore, we wish to talk of a number without specifying any one in particular, we use a letter to represent it. Thus: Suppose we wish to reason upon what will follow from dividing a number into three parts, without considering what the number is, or what are the parts into which it is divided. Let a stand for the number, and b, c, and d, for the parts into which it is divided. Then, by our supposition,

$$a = b+c+d.$$

On this we can reason, and produce results which do not belong to any particular number, but are true of all. Thus, if one part be taken away from the number, the other two will remain, or

$$a-b = c+d.$$

If each part be doubled, the whole number will be doubled, or

$$2a = 2b+2c+2d.$$

If we diminish one of the parts, as d, by a number x, we diminish the whole number just as much, or

$$a-x = b+c+(d-x).$$

27. EXERCISES.

What is $a+2b-c$, where $a=12$, $b=18$, $c=7$?—*Answer*, 41.

What is $\dfrac{aa-bb}{a-b}$, where $a=6$ and $b=2$?—*Ans.* 8.

What is the difference between $(a+b)(c+d)$ and $a+bc+d$, for the following values of a, b, c, and d?

a	b	c	d	*Ans.*
1	2	3	4	10
2	12	7	1	25
1	1	1	1	1

SECTION II.

ADDITION AND SUBTRACTION.

28. There is no process in arithmetic which does not consist entirely in the increase or diminution of numbers. There is then nothing which might not be done with collections of pebbles. Probably, at first, either these or the fingers were used. Our word *calculation* is derived from the Latin word *calculus*, which means a pebble. Shorter ways of counting have been invented, by which many calculations, which would require long and tedious reckoning if pebbles were used, are made at once with very little trouble. The four great methods are, Addition, Subtraction, Multiplication, and Division; of which, the last two are only ways of doing several of the first and second at once.

29. When one number is increased by others, the number which is as large as all the numbers together is called their *sum*. The process of finding the sum of two or more numbers is called ADDITION, and, as was said before, is denoted by placing a cross (+) between the numbers which are to be added together.

Suppose it required to find the sum of 1834 and 2799. In order to add these numbers, take them to pieces, dividing each into its units, tens, hundreds, and thousands:

1834 is 1 thous. 8 hund. 3 tens and 4;
2799 is 2 thous. 7 hund. 9 tens and 9.

Each number is thus broken up into four parts. If to each part of the first you add the part of the second which is under it, and then put together what you get from these additions, you will have added 1834 and 2799. In the first number are 4 units, and in the second 9: these will, when the numbers are added together, contribute 13 units to the sum. Again, the 3 tens in the first and the 9 tens in the second will contribute 12 tens to the sum. The 8 hundreds in the first and the 7 hundreds in the second will add 15 hundreds to the sum; and the thousand in the first with the 2 thousands in the second will contribute 3 thousands to the sum; therefore the sum required is

3 thousands, 15 hundreds, 12 tens, and 13 units.

To simplify this result, you must recollect that—

13 units are . . . . 1 ten and 3 units.
12 tens are . . . . . 1 hund. and 2 tens.
15 hund. are 1 thous. and 5 hund.
3 thous. are 3 thous.

Now collect the numbers on the right-hand side together, as was done before, and this will give, as the sum of 1834 and 2799,

4 thousands, 6 hundreds, 3 tens, and 3 units,

which (19) is written 4633.

30. The former process, written with the signs of (23) is as follows:

$$1834 = 1\times1000 + 8\times100 + 3\times10 + 4$$
$$2799 = 2\times1000 + 7\times100 + 9\times10 + 9$$

Therefore,

$$1834 + 2799 = 3\times1000 + 15\times100 + 12\times10 + 13$$

But

$$13 = 1\times10 + 3$$
$$12\times10 = 1\times100 + 2\times10$$
$$15\times100 = 1\times1000 + 5\times100$$
$$3\times1000 = 3\times1000$$

Therefore,

$$1834 + 2799 = 4\times1000 + 6\times100 + 3\times10 + 3$$
$$= 4633.$$

31. The same process is to be followed in all cases, but not at the same length. In order to be able to go through it, you must know how to add together the simple numbers. This can only be done by memory; and to help the memory you should make the following table three or four times for yourself:

	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14	15	16
8	9	10	11	12	13	14	15	16	17
9	10	11	12	13	14	15	16	17	18

The use of this table is as follows: Suppose you want to find the sum of 8 and 7. Look in the left-hand column for either of them, 8, for example; and look in the top column for 7. On the same line as 8, and underneath 7, you find 15, their sum.

32. When this table has been thoroughly committed to memory, so that you can tell at once the sum of any two numbers, neither of which exceeds 9, you should exercise yourself in adding and subtracting two numbers, one of which is greater than 9 and the other less. You should write down a great number of such sentences as the following, which will exercise you at the same time in addition, and in the use of the signs mentioned in (23).

$12+6=18 \qquad 22+6=28 \qquad 19+8=27$

$54+9=63 \qquad 56+7=63 \qquad 22+8=30$

$100-9=91 \qquad 27-8=19 \qquad 44-6=38$, &c.

33. When the last two articles have been thoroughly studied, you will be able to find the sum of any numbers by the following process,* which is the same as that in (29).

* In this and all other processes, the student is strongly recommended to look at and follow the first Appendix.

RULE I. Place the numbers under one another, units under units, tens under tens, and so on.

II. Add together the units of all, and part the *whole* number thus obtained into units and tens. Thus, if 85 be the number, part it into 8 tens and 5 units; if 136 be the number, part it into 13 tens and 6 units (20).

III. Write down the units of this number under the units of the rest, and keep in memory the number of tens.

IV. Add together all the numbers in the column of tens, remembering to take in (or carry, as it is called) the tens which you were told to recollect in III., and divide this number of tens into tens and hundreds. Thus, if 335 tens be the number obtained, part this into 33 hundreds and 5 tens.

V. Place the number of tens under the tens, and remember the number of hundreds.

VI. Proceed in this way through every column, and at the last column, instead of separating the number you obtain into two parts, write it all down before the rest.

EXAMPLE.—What is

1805+36+19727+3+1474+2008

```
 1805
   36
19727
    3
 1474
 2008
-----
25053
```

The addition of the units' line, or 8+4+3+7+6+5, gives 33, that is, 3 tens and 3 units. Put 3 in the units' place, and add together the line of tens, taking in at the beginning the 3 tens which were created by the addition of the units' line. That is, find 3+0+7+2+3+0, which gives 15 for the number of tens; that is, 1 hundred and 5 tens. Add the line of hundreds together, taking care to add the 1 hundred which arose in the addition of the line of tens; that is, find 1+0+4+7+8, which gives exactly 20 hundreds, or 2 thousands and no hundreds. Put a cipher in the hundreds' place (because, if you do not, the next figure will be taken for hundreds instead of thousands), and add the figures in the thousands' line together, remembering the 2 thousands which arose from the hundreds' line; that is, find 2+2+1+9+1, which gives 15 thousands, or 1 ten thousand and 5 thousand. Write 5 under the line of thousands, and collect the figures in the line of tens of thousands, remembering the ten thousand which arose out of the thousands' line; that is, find 1+1, or 2 ten thousands. Write 2 under the ten thousands' line, and the operation is completed.

34. As an exercise in addition, you may satisfy yourself that what I now say of the following square is correct. The numbers in every row, whether reckoned upright, or from right to left, or from corner to corner, when added together give the number 24156.

2016	4212	1656	3852	1296	3492	936	3132	576	2772	216
252	2052	4248	1692	3888	1332	3528	972	3168	612	2412
2448	288	2088	4284	1728	3924	1368	3564	1008	2808	648
684	2484	324	2124	4320	1764	3960	1404	3204	1044	2844
2880	720	2520	360	2160	4356	1800	3600	1440	3240	1080
1116	2916	756	2556	396	2196	3996	1836	3636	1476	3276
3312	1152	2952	792	2592	36	2232	4032	1872	3672	1512
1548	3348	1188	2988	432	2628	72	2268	4068	1908	3708
3744	1584	3384	828	3024	468	2664	108	2304	4104	1944
1980	3780	1224	3420	864	3060	504	2700	144	2340	4140
4176	1620	3816	1260	3456	900	3096	540	2736	180	2376

35. If two numbers must be added together, it will not alter the sum if you take away a part of one, provided you put on as much to the other. It is plain that you will not alter the whole number of a collection of pebbles in two baskets by taking any number out of one, and putting them into the other. Thus, 15+7 is the same as 12+10, since 12 is 3 less than 15, and 10 is three more than 7. This was the principle upon which the whole of the process in (29) was conducted.

36. Let a and b stand for two numbers, as in (24). It is impossible to tell what their sum will be until the numbers themselves are known. In the mean while $a+b$ stands for this sum. To say, in algebraical

language, that the sum of a and b is not altered by adding c to a, provided we take away c from b, we have the following equation:

$$(a+c)+(b-c)=a+b\,;$$

which may be written without brackets, thus,

$$a+c+b-c=a+b.$$

For the meaning of these two equations will appear to be the same, on consideration.

37. If a be taken twice, three times, &c., the results are represented in algebra by $2a$, $3a$, $4a$, &c. The sum of any two of this series may be expressed in a shorter form than by writing the sign + between them; for though we do not know what number a stands for, we know that, be it what it may, $2a+2a=4a$, $3a+2a=5a$, $4a+9a=13a$; and generally, if a taken m times be added to a taken n times, the result is a taken $m+n$ times, or

$$ma+na=(m+n)a.$$

38. The use of the brackets must here be noticed. They mean, that the expression contained inside them must be used exactly as a single letter would be used in the same place. Thus, pa signifies that a is taken p times, and $(m+n)a$, that a is taken $m+n$ times. It is, therefore, a different thing from $m+na$, which means that a, after being taken n times, is added to m. Thus $(3+4)\times 2$ is 7×2 or 14; while $3+4\times 2$ is $3+8$, or 11.

39. When one number is taken away from another, the number which is left is called the *difference* or *remainder*. The process of finding the difference is called SUBTRACTION. The number which is to be taken away must be of course the lesser of the two.

40. The process of subtraction depends upon these two principles.

I. The difference of two numbers is not altered by adding a number to the first, if you add the same number to the second; or by subtracting a number from the first, if you subtract the same number from the second. Conceive two baskets with pebbles in them, in the first of which are 100 pebbles more than in the second. If I put 50 more pebbies into each of them, there are still only 100 more in the first than in the second, and the same if I take 50 from each. Therefore, in finding the difference of two numbers, if it should be convenient, I may add any number I please to both of them, because, though I alter the numbers themselves by so doing, I do not alter their difference.

II. Since 6 exceeds 4 by 2,
and 3 exceeds 2 by 1,
and 12 exceeds 5 by 7,

6, 3, and 12 together, or 21, exceed 4, 2, and 5 together, or 11, by 2, 1, and 7 together, or 10: the same thing may be said of any other numbers.

41. If a, b, and c be three numbers, of which a is greater than b (40), I. leads to the following,

$$(a+c)-(b+c)=a-b.$$

Again, if c be less than a and b,

$$(a-c)-(b-c)=a-b.$$

The brackets cannot be here removed as in (36). That is, $p-(q-r)$ is not the same thing as $p-q-r$. For, in the first, the difference of q and r is subtracted from p; but in the second, first q and then r are subtracted from p, which is the same as subtracting as much as q and r together, or $q+r$. Therefore $p-q-r$ is $p-(q+r)$. In order to shew how to remove the brackets from $p-(q-r)$ without altering the value of the result, let us take the simple instance 12—(8—5). If we subtract 8 from 12, or form 12—8, we subtract too much; because it is not 8 which is to be taken away, but as much of 8 as is left after diminishing it by 5. In forming 12—8 we have therefore subtracted 5 too much. This must be set right by adding 5 to the result, which gives 12—8+5 for the value of 12—(8—5). The same reasoning applies to every case, and we have therefore,

$$p-(q+r)=p-q-r.$$
$$p-(q-r)=p-q+r.$$

By the same kind of reasoning,

$$a-(b+c-d-e)=a-b-c+d+e.$$

$$2a+3b-(a-2b)=2a+3b-a+2b=a+5b.$$

$$4x+y-(17x-9y)=4x+y-17x+9y=10y-13x.$$

42. I want to find the difference of the numbers 57762 and 34631. Take these to pieces as in (29) and

57762 is 5 ten-th. 7 th. 7 hund. 6 tens and 2 units.

34631 is 3 ten-th. 4 th. 6 hund. 3 tens and 1 unit.

Now 2 units exceed . . .	1 unit	by 1 unit.
6 tens	3 tens	3 tens.
7 hundreds	6 hundreds . . .	1 hundred.
7 thousands	4 thousands . . .	3 thousands.
5 ten-thousands . .	3 ten-thous. . . .	2 ten-thous.

Therefore, by (40, Principle II.) all the first column *together* exceeds all the second column by all the third column, that is, by

2 ten-th. 3 th. 1 hund. 3 tens and 1 unit,

which is 23131. Therefore the difference of 57762 and 34631 is 23131, or $57762-34631=23131$.

43. Suppose I want to find the difference between 61274 and 39628. Write them at length, and

61274 is 6 ten-th. 1 th. 2 hund. 7 tens and 4 units.

39628 is 3 ten-th. 9 th. 6 hund. 2 tens and 8 units.

If we attempt to do the same as in the last article, there is a difficulty immediately, since 8, being greater than 4, cannot be taken from it. But from (40) it appears that we shall not alter the difference of two numbers if we add the same number to *both* of them. Add ten to the first number, that is, let there be 14 units instead of four, and add ten also to the second number, but instead of adding ten to the number of units, add one to the number of tens, which is the same thing. The numbers will then stand thus,

6 ten-thous. 1 thous. 2 hund. 7 tens and 14 *units*.*

3 ten-thous. 9 thous. 6 hund. 3 *tens* and 8 units.

* Those numbers which have been altered are put in italics.

You now see that the units and tens in the lower can be subtracted from those in the upper line, but that the hundreds cannot. To remedy this, add one thousand or 10 hundred to both numbers, which will not alter their difference, and remember to increase the hundreds in the upper line by 10, and the thousands in the lower line by 1, which are the same things. And since the thousands in the lower cannot be subtracted from the thousands in the upper line, add 1 ten thousand or 10 thousand to both numbers, and increase the thousands in the upper line by 10, and the ten thousands in the lower line by 1, which are the same things; and at the close the numbers which we get will be,

6 ten-thous. 11 *thous.* 12 *hund.* 7 tens and 14 *units.*

4 *ten-thous.* 10 *thous.* 6 hund. 3 *tens* and 8 units.

These numbers are not, it is true, the same as those given at the beginning of this article, but their difference is the same, by (40). With the last-mentioned numbers proceed in the same way as in (42), which will give, as their difference,

2 ten-thous. 1 thous. 6 hund. 4 tens, and 6 units, which is 21646.

44. From this we deduce the following rules for subtraction:

I. Write the number which is *to be subtracted* (which is, of course, the lesser of the two, and is called the *subtrahend*) under the other, so that its units shall fall under the units of the other, and so on.

II. Subtract each figure of the lower line from the one above it, if that can be done. Where that cannot be done, add ten to the upper figure, and then subtract the lower figure; but recollect in this case always to increase the next figure in the lower line by 1, before you begin to subtract it from the upper one.

45. If there should not be as many figures in the lower line as in the upper one, proceed as if there were as many ciphers at the beginning of the lower line as will make the number of figures equal. You do not alter a number by placing ciphers at the beginning of it. For example, 00818 is the same number as 818, for it means

0 ten-thous. 0 thous. 8 hunds. 1 ten and 8 units;

the first two signs are nothing, and the rest is

8 hundreds, 1 ten, and 8 units, or 818.

The second does not differ from the first, except in its being said that there are no thousands and no tens of thousands in the number, which may be known without their being mentioned at all. You may ask, perhaps, why this does not apply to a cipher placed in the middle of a number, or at the right of it, as, for example, in 28007 and 39700? But you must recollect, that if it were not for the two ciphers in the first, the 8 would be taken for 8 tens, instead of 8 thousands; and if it were not for the ciphers in the second, the 7 would be taken for 7 units, instead of 7 hundreds.

46. **EXAMPLE.**

What is the difference between	3708291640030174
and	30813649276188
Difference	3677477990753986

EXERCISES.

I. What is 18337+149263200—6472902?—*Answer* 142808635.

What is 1000—464+3279—646?—*Ans.* 3169.

II. Subtract

64+76+144—18 from 33—2+100037.—*Ans.* 99802.

III. What shorter rule might be made for subtraction when all the figures in the upper line are ciphers except the first? for example, in finding

10000000—2731634.

IV. Find 18362+2469 and 18362—2469, add the second result to the first, and then subtract 18362; subtract the second from the first, and then subtract 2469.—*Answer* 18362 and 2469.

V. There are four places on the same line in the order A, B, C, and D. From A to D it is 1463 miles; from A to C it is 728 miles; and from B to D it is 1317 miles. How far is it from A to B, from B to C, and from C to D?—*Answer.* From A to B 146, from B to C 582, and from C to D 735 miles.

VI. In the following table subtract B from A, and B from the remainder, and so on until B can be no longer subtracted. Find how many times B can be subtracted from A, and what is the last remainder.

A	B	No. of times.	Remainder.
23604	9999	2	3606
209961	37173	5	24096
74712	6792	11	0
4802469	654321	7	222222
18849747	3141592	6	195
987654321	123456789	8	9

SECTION III.

MULTIPLICATION.

47. I have said that all questions in arithmetic require nothing but addition and subtraction. I do not mean by this that no rule should ever be used except those given in the last section, but that all other rules only shew shorter ways of finding what might be found, if we pleased, by the methods there deduced. Even the last two rules themselves are only short and convenient ways of doing what may be done with a number of pebbles or counters.

48. I want to know the sum of five seventeens, or I ask the following question: There are five heaps of pebbles, and seventeen pebbles in each heap; how many are there in all? Write five seventeens in a column, and make the addition, which gives 85. In this case 85 is called the *product* of 5 and 17, and the process of finding the product is called MULTIPLICATION, which gives nothing more than the addition of a number of the same quantities. Here 17 is called the *multiplicand*, and 5 is called the *multiplier*.

17
17
17
17
17
——
85

49. If no question harder than this were ever proposed, there would be no occasion for a shorter way than the one here followed. But if

there were 1367 heaps of pebbles, and 429 in each heap, the whole number is then 1367 times 429, or 429 multiplied by 1367. I should have to write 429 1367 times, and then to make an addition of enormous length. To avoid this, a shorter rule is necessary, which I now proceed to explain.

50. The student must first make himself acquainted with the products of all numbers as far as 10 times 10 by means of the following table,* which must be committed to memory.

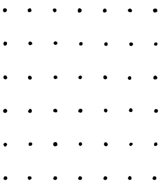
1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

If from this table you wish to know what is 7 times 6, look in the first upright column on the left for either of them; 6 for example. Proceed to the right until you come into the column marked 7 at the top. You there find 42, which is the product of 6 and 7.

51. You may find, in this way, either 6 times 7, or 7 times 6, and for both you find 42. That is, six sevens is the same number as seven

* As it is usual to learn the product of numbers up to 12 times 12, I have extended the table thus far. In my opinion, all pupils who shew a tolerable capacity should slowly commit the products to memory as far as 20 times 20, in the course of their progress through this work.

sixes. This may be shewn as follows: Place seven counters in a line, and repeat that line in all six times. The number of counters in the whole is 6 times 7, or six sevens, if I reckon the rows from the top to the bottom; but if I count the rows that stand side by side, I find seven of them, and six in each row, the whole number of which is 7 times 6, or seven sixes. And the whole number is 42, whichever way I count. The same method may be applied to any other two numbers. If the signs of (23) were used, it would be said that $7\times6=6\times7$.



52. To take any quantity a number of times, it will be enough to take every one of its parts the same number of times. Thus, a sack of corn will be increased fifty-fold, if each bushel which it contains be replaced by 50 bushels. A country will be doubled by doubling every acre of land, or every county, which it contains. Simple as this may appear, it is necessary to state it, because it is one of the principles on which the rule of multiplication depends.

53. In order to multiply by any number, you may multiply separately by any parts into which you choose to divide that number, and add the results. For example, 4 and 2 make 6. To multiply 7 by 6 first multiply 7 by 4, and then by 2, and add the products. This will give 42, which is the product of 7 and 6. Again, since 57 is made up of 32 and 25, 57 times 50 is made up of 32 times 50 and 25 times 50, and so on. If the signs were used, these would be written thus:

$$7\times6 = 7\times4 + 7\times2.$$

$$50\times57 = 50\times32+50\times25.$$

54. The principles in the last two articles may be expressed thus: If a be made up of the parts x, y, and z, ma is made up of mx, my, and mz; or,

$$\text{if} \qquad a = x + y + z.$$

$$ma = mx+my+mz.$$

$$\text{or,} \qquad m(x+y+z) = mx+my+mz.$$

A similar result may be obtained if a, instead of being made up of

x, y, and z, is made by combined additions and subtractions, such as $x+y-z$, $x-y+z$, $x-y-z$, &c. To take the first as an instance:

$$\text{Let} \qquad a = x + y - z.$$
$$\text{then} \qquad ma = mx + my - mz.$$

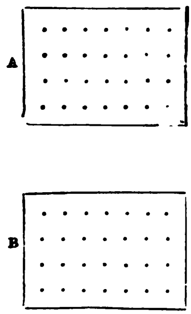
For, if a had been $x+y$, ma would have been $mx+my$. But since a is less than $x+y$ by z, too much by z has been repeated every time that $x+y$ has been repeated;—that is, mz too much has been taken; consequently, ma is not $mx+my$, but $mx+my-mz$. Similar reasoning may be applied to other cases, and the following results may be obtained:

$$m(a+b+c-d) = ma+mb+mc-md.$$

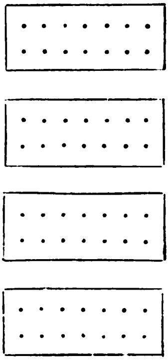
$a(a-b) = aa-ab.$	$7a(7+2b) = 49a+14ab.$
$b(a-b) = ba-bb.$	$(aa+a+1)a = aaa+aa+a.$
$3(2a-4b) = 6a-12b.$	$(3ab-2c)4abc = 12aabbc-8abcc.$

55. There is another way in which two numbers may be multiplied together. Since 8 is 4 times 2, 7 times 8 may be made by multiplying 7 and 4, and then multiplying that *product* by 2. To shew this, place 7 counters in a line, and repeat that line in all 8 times, as in figures I. and II.

I.



II.



The number of counters in all is 8 times 7, or 56. But (as in fig. I.) enclose each four rows in oblong figures, such as A and B. The number in each oblong is 4 times 7, or 28, and there are two of those oblongs; so that in the whole the number of counters is twice 28, or 28×2, or 7 first multipled by 4, and that product multiplied by 2. In figure II. it is shewn that 7 multiplied by 8 is also 7 first multiplied by 2, and that product multiplied by 4. The same method may be applied to other numbers. Thus, since 80 is 8 times 10, 256 times 80 is 256 multiplied by 8, and that product multiplied by 10. If we use the signs, the foregoing assertions are made thus:

$$7\times8 = 7\times4\times2 = 7\times2\times4.$$

$$256\times80 = 256\times8\times10 = 256\times10\times8.$$

EXERCISES.

Shew that $2\times3\times4\times5 = 2\times4\times3\times5 = 5\times4\times2\times3$, &c.

Shew that $18\times100 = 18\times57+18\times43$.

56. Articles (51) and (55) may be expressed in the following way, where by ab we mean a taken b times; by abc, a taken b times, and the result taken c times.

$$ab = ba.$$

$$abc = acb = bca = bac, \text{ \&c.}$$

$$abc = a\times(bc) = b\times(ca) = c\times(ab).$$

If we would say that the same results are produced by multiplying by b, c, and d, one after the other, and by the product bcd at once, we write the following:

$$a\times b\times c\times d = a\times bcd.$$

The fact is, that if any numbers are to be multiplied together, the product of any two or more may be formed, and substituted instead of those two or more; thus, the product $abcdef$ may be formed by multiplying

ab	cde	f	
abf	de	c	
abc	def		&c.

57. In order to multiply by 10, annex a cipher to the right hand of the multiplicand. Thus, 10 times 2356 is 23560. To shew this, write 2356 at length which is

2 thousands, 3 hundreds, 5 tens, and 6 units.

Take each of these parts ten times, which, by (52), is the same as multiplying the whole number by 10, and it will then become

2 tens of thou. 3 tens of hun. 5 tens of tens, and 6 tens,

which is 2 ten-thou. 3 thous. 5 hun. and 6 tens.

This must be written 23560, because 6 is not to be 6 units, but 6 tens. Therefore 2356×10 = 23560.

In the same way you may shew, that in order to multiply by 100 you must affix two ciphers to the right; to multiply by 1000 you must affix three ciphers, and so on. The rule will be best caught from the following table:

13×	10 =	130	142×	1000 =	142000
13×	100 =	1300	23700×	10 =	237000
13×	1000 =	13000	3040×	1000 =	3040000
13×	10000 =	130000	10000×	100000 =	1000000000

58. I now shew how to multiply by one of the numbers, 2, 3, 4, 5, 6, 7, 8, or 9. I do not include 1, because multiplying by 1, or taking the number once, is what is meant by simply writing down the number. I want to multiply 1368 by 8. Write the first number at full length, which is

1 thousand, 3 hundreds, 6 tens, and 8 units.

To multiply this by 8, multiply each of these parts by 8 (50) and (52), which will give

8 thousands, 24 hundreds, 48 tens, and 64 units.

Now 64 units are written thus . . .	64
48 tens	480
24 hundreds	2400
8 thousands	8000

Add these together, which gives 10944 as the product of 1368 and 8, or 1368×8 = 10944. By working a few examples in this way you will see for following rule.

59. I. Multiply the first figure of the multiplicand by the multiplier, write down the units' figure, and reserve the tens.

II. Do the same with the second figure of the multiplicand, and add to the product the number of tens from the first; put down the units' figure of this, and reserve the tens.

III. Proceed in this way till you come to the last figure, and then write down the whole number obtained from that figure.

IV. If there be a cipher in the multiplicand, treat it as if it were a number, observing that $0\times1=0$, $0\times2=0$, &c.

60. In a similar way a number can be multiplied by a figure which is accompanied by ciphers, as, for example, 8000. For 8000 is 8×1000, and therefore (55) you must first multiply by 8 and then by 1000, which last operation (57) is done by placing 3 ciphers on the right. Hence the rule in this case is, Multiply by the simple number, and place the number of ciphers which follow it at the right of the product.

EXAMPLE.

Multiply	1679423800872
by	60000
	100765428052320000

61. EXERCISES.

What is 1007360×7? *Answer*, 7051520.

$123456789\times9+10$ and $123\times9+4$?—*Ans.* 1111111111 and 1111.

What is $136\times3+129\times4+147\times8+27\times3000$?—*Ans.* 83100.

An army is made up of 33 regiments of infantry, each containing 800 men; 14 of cavalry, each containing 600 men; and 2 of artillery, each containing 300 men. The enemy has 6 more regiments of infantry, each containing 100 more men; 3 more regiments of cavalry, each containing 100 men less; and 4 corps of artillery of the same magnitude as those of the first: two regiments of cavalry and one of infantry desert from the former to the latter. How many men has the second army more than the first?—*Answer*, 13400.

62. Suppose it required to multiply 23707 by 4567. Since 4567 is

made up of 4000, 500, 60, and 7, by (53) we must multiply 23707 by each of these, and add the products.

Now (58)	23707× 7	is	165949
(60)	23707× 60	is	1422420
	23707× 500	is	11853500
	23707×4000	is	94828000
	The sum of these	is	108269869

which is the product required.

It will do as well if, instead of writing the ciphers at the end of each line, we keep the other figures in their places without them. If we take away the ciphers, the second line is one place to the left of the first, the third one place to the left of the second, and so on. Write the multiplier and the multiplicand over these lines, and the process will stand thus:

```
    23707
     4567
   ------
   165949
  142242
 118535
 94828
---------
108269869
```

63. There is one more case to be noticed; that is, where there is a cipher in the middle of the multiplier. The following example will shew that in this case nothing more is necessary than to keep the first figure of each line in the column under the figure of the multiplier from which that line arises. Suppose it required to multiply 365 by 101001. The multiplier is made up of 100000, 1000 and 1. Proceed as before, and

	365×1	is	365
(57)	365×1000	is	365000
	365×100000	is	36500000
	The sum of which	is	36865365

and the whole process with the ciphers struck off is:

```
     365
  101001
  ------
     365
   365
 365
--------
36865365
```

64. The following is the rule in all cases:

I. Place the multiplier under the multiplicand, so that the units of one may be under those of the other.

II. Multiply the whole multiplicand by each figure of the multiplier (59), and place the unit of each line in the column under the figure of the multiplier from which it came.

III. Add together the lines obtained by II. column by column.

65. When the multiplier or multiplicand, or both, have ciphers on the right hand, multiply the two together without the ciphers, and then place on the right of the product all the ciphers that are on the right both of the multiplier and multiplicand. For example, what is 3200 ×13000? First, 3200 is 32×100, or one hundred times as great as 32. Again, 32×13000 is 32×13, with three ciphers affixed, that is 416, with three ciphers affixed, or 416000. But the product required must be 100 times as great as this, or must have two ciphers affixed. It is therefore 41600000, having as many ciphers as are in both multiplier and multiplicand.

66. When any number is multiplied by itself any number of times, the result is called a *power* of that number. Thus:

6 is called the first		power of 6
6×6 . .	second	power of 6
6×6×6 . .	third	power of 6
6×6×6×6 .	fourth	power of 6
&c.		&c.

The second and third powers are usually called the *square* and *cube*, which are incorrect names, derived from certain connexions of the second and third power with the square and cube in geometry. As exercises in multiplication, the following powers are to be found.

Number proposed.	Square.	Cube.
972	944784	918330048
1008	1016064	1024192512
3142	9872164	31018339288
3163	10004569	31644451747
5555	30858025	171416328875
6789	46090521	312908547069

The fifth	power of	36	is	60466176
... fourth	. .	50	...	6250000
... fourth	. .	108	...	136048896
... fourth	. .	277	...	5887339441

67. It is required to multiply $a+b$ by $c+d$, that is, to take $a+b$ as many times as there are units in $c+d$. By (53) $a+b$ must be taken c times, and d times, or the product required is $(a+b)c+(a+b)d$. But (52) $(a+b)c$ is $ac+bc$, and $(a+b)d$ is $ad+bd$; whence the product required is $ac+bc+ad+bd$; or,

$$(a+b)(c+d) = ac+bc+ad+bd.$$

By similar reasoning $(a-b)(c+d)$ is $(a-b)c+(a-b)d$, or,

$$(a-b)(c+d) = ac-bc+ad-bd.$$

To multiply $a-b$ by $c-d$, first take $a-b$ c times, which gives $ac-bc$. This is not correct; for in taking it c times instead of $c-d$ times, we have taken it d times too many; or have made a result which is $(a-b)d$ too great. The real result is therefore $ac-bc-(a-b)d$. But $(a-b)d$ is $ad-bd$, and therefore

$$\begin{aligned}(a-b)(c-d) &= ac-bc-(ad-bd)\\ &= ac-bc-ad+bd \qquad (41)\end{aligned}$$

From these three examples may be collected the following rule for the multiplication of algebraic quantities: Multiply each term of the multiplicand by each term of the multiplier; when the two terms have both + or both − before them, put + before their product; when one has + and the other −, put − before their product. In using the first terms, which have no sign, apply the rule as if they had the sign +.

68. For example, $(a+b)(a+b)$ gives $aa+ab+ab+bb$. But $ab+ab$ is $2ab$; hence the *square* of $a+b$ is $aa+2ab+bb$. Again $(a-b)(a-b)$ gives $aa-ab-ab+bb$. But two subtractions of ab are equivalent to subtracting $2ab$; hence the *square* of $a-b$ is $aa-2ab+bb$. Again, $(a+b)(a-b)$ gives $aa+ab-ab-bb$. But the addition and subtraction of ab makes no change; hence the product of $a+b$ and $a-b$ is $aa-bb$.

Again, the square of $a+b+c+d$ or $(a+b+c+d)(a+b+c+d)$ will be found to be $aa+2ab+2ac+2ad+bb+2bc+2bd+cc+2cd+dd$; or the rule for squaring such a quantity is: Square the first term, and multiply all that come *after* by twice that term; do the same with the second, and so on to the end.

SECTION IV.

DIVISION.

69. Suppose I ask whether 156 can be divided into a number of parts each of which is 13, or how many thirteens 156 contains; I propose a question, the solution of which is called DIVISION. In this case, 156 is called the *dividend*, 13 the *divisor*, and the number of parts required is the *quotient;* and when I find the quotient, I am said to divide 156 by 13.

70. The simplest method of doing this is to subtract 13 from 156, and then to subtract 13 from the remainder, and so on; or, in common language, to *tell off* 156 by thirteens. A similar process has already occurred in the exercises on subtraction, Art. (46). Do this, and mark one for every subtraction that is made, to remind you that each subtraction takes 13 once from 156, which operations will stand as follows:

156	
13	1
143	
13	1
130	
13	1
117	
13	1
104	
13	1
91	
13	1
78	
13	1
65	
13	1
52	
13	1
39	
13	1
26	
13	1
13	
13	1
0	

Begin by subtracting 13 from 156, which leaves 143. Subtract 13 from 143, which leaves 130; and so on. At last 13 only remains, from which when 13 is subtracted, there remains nothing. Upon counting the number of times which you have subtracted 13, you find that this number is 12; or 156 contains twelve thirteens, or contains 13 twelve times.

This method is the most simple possible, and might be done with pebbles. Of these you would first count 156. You would then take 13 from the heap, and put them into one heap by themselves. You would then take another 13 from the heap, and place them in another heap by themselves; and so on until there were none left. You would then count the number of heaps, which you would find to be 12.

71. Division is the opposite of multiplication. In multiplication you have a number of heaps, with the same number of pebbles in each, and you want to know how many *pebbles* there are in all. In division you know how many there are

in all, and how many there are to be in each heap, and you want to know how many *heaps* there are.

72. In the last example a number was taken which contains an exact number of thirteens. But this does not happen with every number. Take, for example, 159. Follow the process of (70), and it will appear that after having subtracted 13 twelve times, there remains 3, from which 13 cannot be subtracted. We may say then that 159 contains twelve thirteens and 3 *over*; or that 159, when divided by 13, gives a *quotient* 12, and a *remainder* 3. If we use signs,

$$159 = 13 \times 12 + 3.$$

EXERCISES.

$146 = 24 \times 6 + 2$, or 146 contains six twenty-fours and 2 over.

$146 = 6 \times 24 + 2$, or 146 contains twenty-four sixes and 2 over.

$300 = 42 \times 7 + 6$, or 300 contains seven forty-twos and 6 over.

$39624 = 7277 \times 5 + 3239$.

73. If a contain b q times with a remainder r, a must be greater than bq by r; that is,

$$a = bq + r.$$

If there be no remainder, $a = bq$. Here a is the dividend, b the divisor, q the quotient, and r the remainder. In order to say that a contains b q times, we write,

$$\frac{a}{b} = q, \text{ or } a : b = q,$$

which in old books is often found written thus:

$$a \div b = q.$$

74. If I divide 156 into several parts, and find how often 13 is contained in each of them, it is plain that 156 contains 13 as often as all its parts together. For example, 156 is made up of 91, 39, and 26. Of these

91 contains 13 7 times,

39 contains 13 3 times,

26 contains 13 2 times;

therefore 91+39+26 contains 13 7+3+2 times, or 12 times.

Again, 156 is made up of 100, 50, and 6.

Now 100 contains 13 7 times and 9 over,

50 contains 13 3 times and 11 over

6 contains 13 0 times* and 6 over.

Therefore 100+50+6 contains 13 7+3+0 times and 9+11+6 over; or 156 contains 13 10 times and 26 over. But 26 is itself 2 thirteens; therefore 156 contains 10 thirteens and 2 thirteens, or 12 thirteens.

75. The result of the last article is expressed by saying, that if

$$a=b+c+d, \text{ then } \frac{a}{m}=\frac{b}{m}+\frac{c}{m}+\frac{d}{m}.$$

76. In the first example I did not take away 13 more than once at a time, in order that the method might be as simple as possible. But if I know what is twice 13, 3 times 13, &c., I can take away as many thirteens at a time as I please, if I take care to mark at each step how many I take away. For example, take away 13 ten times at once from 156, that is, take away 130, and afterwards take away 13 twice, or take away 26, and the process is as follows:

```
156
130  10 times 13.
---
 26
 26  2 times 13.
 --
  0
```

Therefore 156 contains 13 10+2, or 12 times.

Again, to divide 3096 by 18.

```
3096
1800  100 times 18.
----
1296
 900   50 times 18.
 ---
 396
 360   20 times 18.
 ---
  36
  36    2 times 18.
  --
   0
```

Therefore 3096 contains 18 100+50+20+2, or 172 times.

77. You will now understand the following sentences, and be able to make similar assertions of other numbers.

450 is 75×6; it therefore contains any number, as 5, 6 times as often as 75 contains it.

* To speak always in the same way, instead of saying that 6 does not contain 13, I say that it contains it 0 times and 6 over, which is merely saying that 6 is 6 more than nothing.

135		3		26 times; therefore,	
Twice 135	contains	3	more than	52 or twice 26	times.
10 times 135		3		260 or 10 times 26	
50 times 135		3		1300 or 50 times 26	

472 contains 18 more than 21 times; therefore,
4720 contains 18 more than 210 times,
47200 contains 18 more than 2100 times,
472000 contains 18 more than 21000 times,

32		12		2		3	
320	contains	12	more than	20	times, and less than	30	times.
3200		12		200		300	
32000		12		2000		3000	
&c.				&c.		&c.	

78. The foregoing articles contain the principles of division. The question now is, to apply them in the shortest and most convenient way Suppose it required to divide 4068 by 18, or to find $\frac{4068}{18}$ (23).

If we divide 4068 into any number of parts, we may, by the process followed in (74), find how many times 18 is contained in each of these parts, and from thence how many times it is contained in the whole. Now, what separation of 4068 into parts will be most convenient? Observe that 4, the first figure of 4068, does not contain 18; but that 40, the first and second figures together, *does contain* 18 *more than twice, but less than three times.** But 4068 (20) is made up of 40 hundreds, and 68; of which, 40 hundreds (77) contains 18 more than 200 times, and less than 300 times. Therefore, 4068 also contains more than 200 times 18, since it must contain 18 more times than 4000 does. It also contains 18 less than 300 times, because 300 times 18 is 5400, a greater number than 4068. Subtract 18 200 times from 4068; that is, subtract 3600, and there remains 468. Therefore, 4068 contains 18 200 times, and as many more times as 468 contains 18.

It remains, then, to find how many times 468 contains 18. Proceed

* If you have any doubt as to this expression, recollect that it means "contains more than two eighteens, but not so much as three."

exactly as before. Observe that 46 contains 18 more than twice, and less than 3 times; therefore, 460 contains it more than 20, and less than 30 times (77); as does also 468. Subtract 18 20 times from 468, that is, subtract 360; the remainder is 108. Therefore, 468 contains 18 20 times, and as many more as 108 contains it. Now, 108 is found to contain 18 6 times exactly; therefore, 468 contains it 20+6 times, and 4068 contains it 200+20+6 times, or 226 times. If we write down the process that has been followed, without any explanation, putting the divisor, dividend, and quotient, in a line separated by parentheses, it will stand, as in example (A).

Let it be required to divide 36326599 by 1342 (B).

```
          B.
1342)36326599(20000+7000+60+9
     26840000
     --------                        A.
      9486599               18)4068(200+20+6
      9394000                  3600
      -------                  ----
        92599                   468
        80520                   360
        -----                   ---
        12079                   108
        12078                   108
        -----                   ---
            1                     0
```

As in the previous example, 36326599 is separated into 36320000 and 6599; the first four figures 3632 being separated from the rest, because it takes four figures from the left of the dividend to make a number which is greater than the divisor. Again, 36320000 is found to contain 1342 more than 20000, and less than 30000 times; and 1342× 20000 is subtracted from the dividend, after which the remainder is 9486599. The same operation is repeated again and again, and the result is found to be, that there is a quotient 20000+7000+60+9, or 27069, and a remainder 1.

Before you proceed, you should now repeat the foregoing article at length in the solution of the following questions. What are

$$\frac{10093874}{3207}, \quad \frac{66779922}{114433}, \quad \frac{2718218}{13352}?$$

the quotients of which are 3147, 583, 203; and the remainders 1445, 65483, 7762.

79. In the examples of the last article, observe, 1st, that it is useless to write down the ciphers which are on the right of each subtrahend, provided that without them you keep each of the other figures in its proper place: 2d, that it is useless to put down the right-hand figures of the dividend so long as they fall over ciphers, because they do not begin to have any share in the making of the quotient until, by continuing the process, they cease to have ciphers under them: 3d, that the quotient is only a number written at length, instead of the usual way. For example, the first quotient is 200+20+6, or 226; the second is 20000+7000+60+9, or 27069. Strike out, therefore, all the ciphers and the numbers which come above them, except those in the first line, and put the quotient in one line; and the two examples of the last article will stand thus:

```
18)4068(226          1342)36326599(27069
   36                     2684
   --                     ----
    46                     9486
    36                     9394
    ---                    -----
    108                      9259
    108                      8052
    ---                      -----
      0                      12079
                             12078
                             -----
                                 1
```

80. Hence the following rule is deduced:

I. Write the divisor and dividend in one line, and place parentheses on each side of the dividend.

II. Take off from the left hand of the dividend the least number of figures which make a number greater than the divisor; find what number of times the divisor is contained in these, and write this number as the first figure of the quotient.

III. Multiply the divisor by the last-mentioned figure, and subtract the product from the number which was taken off at the left of the dividend.

IV. On the right of the remainder place the figure of the dividend which comes next after those already separated in II.: if the remainder thus increased be greater than the divisor, find how many times the divisor is contained in it; put this number at the right of the first figure of the quotient, and repeat the process: if not, on the right place the next figure of the dividend, and the next, and so on until it is greater; but remember to place a cipher in the quotient for every figure of the dividend which you are obliged to take, except the first.

V. Proceed in this way until all the figures of the dividend are exhausted.

In judging how often one large number is contained in another, a first and rough guess may be made by striking off the same number of figures from both, and using the results instead of the numbers themselves. Thus, 4,732 is contained in 14,379 about the same number of times that 4 is contained in 14, or about 3 times. The reason is, that 4 being contained in 14 as often as 4000 is in 14000, and these last only differing from the proposed numbers by lower denominations, viz. hundreds, &c. we may expect that there will not be much difference between the number of times which 14000 contains 4000, and that which 14379 contains 4732: and it generally happens so. But if the second figure of the divisor be 5, or greater than 5, it will be more accurate to increase the first figure of the divisor by 1, before trying the method just explained. Nothing but practice can give facility in this sort of guess-work.

81. This process may be made more simple when the divisor is not greater than 12, if you have sufficient knowledge of the multiplication table (50). For example, I want to divide 132976 by 4. At full length the process stands thus:

```
4)132976(33244
  12
  --
   12
   12
   --
     9
     8
     --
     17
     16
     --
      16
      16
      --
       0
```

But you will recollect, without the necessity of writing it down, that 13 contains 4 three times with a remainder 1; this 1 you will place before 2, the next figure of the dividend, and you know that 12 contains 4 3 times exactly, and so on. It will be more convenient to write down the quotient thus:

$$4)\underline{132976}$$
$$33244$$

While on this part of the subject, we may mention, that the shortest way to multiply by 5 is to annex a cipher and divide by 2, which is equivalent to taking the half of 10 times, or 5 times. To divide by 5, multiply by 2 and strike off the last figure, which leaves the quotient; half the last figure is the remainder. To multiply by 25, annex two ciphers and divide by 4. To divide by 25, multiply by 4 and strike off the last two figures, which leaves the quotient; one fourth of the last two figures, taken as one number, is the remainder. To multiply a number by 9, annex a cipher, and subtract the number, which is equivalent to taking the number ten times, and then subtracting it once. To multiply by 99, annex two ciphers and subtract the number, &c.

In order that a number may be divisible by 2 without remainder, its units' figure must be an even number.* That it may be divisible by 4, its last two figures must be divisible by 4. Take the example 1236: this is composed of 12 hundreds and 36, the first part of which, being hundreds, is divisible by 4, and gives 12 twenty-fives; it depends then upon 36, the last two figures, whether 1236 is divisible by 4 or not. A number is divisible by 8 if the last three figures are divisible by 8; for every digit, except the last three, is a number of thousands, and 1000 is divisible by 8; whether therefore the whole shall be divisible by 8 or not depends on the last three figures: thus, 127946 is not divisible by 8, since 946 is not so. A number is divisible by 3 or 9 only when the sum of its digits is divisible by 3 or 9. Take for example 1234; this is

* Among the even figures we include 0.

	1 thousand,	or 999 and 1
	2 hundred,	or twice 99 and 2
	3 tens,	or three times 9 and 3
and 4		or 4

Now 9, 99, 999, &c. are all obviously divisible by 9 and by 3, and so will be any number made by the repetition of all or any of them any number of times. It therefore depends on 1+2+3+4, or the sum of the digits, whether 1234 shall be divisible by 9 or 3, or not. From the above we gather, that a number is divisible by 6 when it is even, and when the sum of its digits is divisible by 3. Lastly, a number is divisible by 5 only when the last figure is 0 or 5.

82. Where the divisor is unity followed by ciphers, the rule becomes extremely simple, as you will see by the following examples:

```
100)33429(334
    300
    ---
     342
     300
     ---
      429
      400
      ---
       29

10)2717316
   -------
   271731 and rem. 6.
```

This is, then, the rule: Cut off as many figures from the right hand of the dividend as there are ciphers. These figures will be the remainder, and the rest of the dividend will be the quotient.

Or we may prove these results thus: from (20), 2717316 is 271731 tens and 6; of which the first contains 10 271731 times, and the second not at all; the quotient is therefore 271731, and the remainder 6 (72). Again (20), 33429 is 334 hundreds and 29; of which the first contains 100 334 times, and the second not at all; the quotient is therefore 334, and the remainder 29.

83. The following examples will shew how the rule may be shortened when there are ciphers in the divisor. With each example is placed another containing the same process, all unnecessary figures being removed; and from the comparison of the two, the rule at the end of this article is derived.

```
I.  1782000)6424700000(3605          1782)6424700(3605
            5346000                       5346
            -------                       ----
            10787000                      10787
            10692000                      10692
            --------                      -----
               9500000                     9500
               8910000                     8910
               -------                     ----
                590000                      590000

II. 12300000)42176189300(3428        123)421761(3428
             36900000                    369
             --------                    ---
             52761893                    527
             49200000                    492
             --------                    ---
              35618930                    356
              24600000                    246
              --------                    ---
              110189300                   1101
               98400000                    984
               --------                    -----------
               11789300                     11789300
```

The rule, then, is: Strike out as many *figures** from the right of the dividend as there are *ciphers* at the right of the divisor. Strike out all the ciphers from the divisor, and divide in the usual way; but at the end of the process place on the right of the remainder all those figures which were struck out of the dividend.

84. **EXERCISES.**

Dividend.	Divisor.	Quotient.	Remainder.
9694	47	206	12
175618	3136	56	2
23796484	130000	183	6484
14002564	1871	7484	0
310314420	7878	39390	0
3939040647	6889	571787	4
22876792454961	43046721	531441	0

* Including both ciphers and others.

Shew that

I. $\frac{100\times100\times100-43\times43\times43}{100-43}=100\times100+100\times43+43\times43.$

II. $\frac{100\times100\times100+43\times43\times43}{100+43}=100\times100-100\times43+43\times43.$

III. $\frac{76\times76+2\times76\times52+52\times52}{76+52}=76+52.$

IV. $1+12+12\times12+12\times12\times12=\frac{12\times12\times12\times12-1}{12-1}.$

What is the nearest number to 1376429 which can be divided by 36300 without remainder?—*Answer*, 1379400.

If 36 oxen can eat 216 acres of grass in one year, and if a sheep eat half as much as an ox, how long will it take 49 oxen and 136 sheep together to eat 17550 acres?—*Answer*, 25 years.

85. Take any two numbers, one of which divides the other without remainder; for example, 32 and 4. Multiply both these numbers by any other number; for example, 6. The products will be 192 and 24. Now, 192 contains 24 just as often as 32 contains 4. Suppose 6 baskets, each containing 32 pebbles, the whole number of which will be 192. Take 4 from one basket, time after time, until that basket is empty. It is plain that if, instead of taking 4 from that basket, I take 4 from each, the whole 6 will be emptied together: that is, 6 times 32 contains 6 times 4 just as often as 32 contains 4. The same reasoning applies to other numbers, and therefore *we do not alter the quotient if we multiply the dividend and divisor by the same number.*

86. Again, suppose that 200 is to be divided by 50. Divide both the dividend and divisor by the same number; for example, 5. Then, 200 is 5 times 40, and 50 is 5 times 10. But by (85), 40 divided by 10 gives the same quotient as 5 times 40 divided by 5 times 10, and therefore *the quotient of two numbers is not altered by dividing both the dividend and divisor by the same number.*

87. From (55), if a number be multiplied successively by two others, it is multiplied by their product. Thus, 27, first multiplied by 5, and the product multiplied by 3, is the same as 27 multiplied by 5 times 3, or 15. Also, if a number be divided by any number, and the quotient

be divided by another, it is the same as if the first number had been divided by the product of the other two. For example, divide 60 by 4, which gives 15, and the quotient by 3, which gives 5. It is plain, that if each of the four fifteens of which 60 is composed be divided into three equal parts, there are twelve equal parts in all; or, a division by 4, and then by 3, is equivalent to a division by 4×3, or 12.

88. The following rules will be better understood by stating them in an example. If 32 be multiplied by 24 and divided by 6, the result is the same as if 32 had been multiplied by the quotient of 24 divided by 6, that is, by 4; for the sixth part of 24 being 4, the sixth part of any number repeated 24 times is that number repeated 4 times; or, multiplying by 24 and dividing by 6 is equivalent to multiplying by 4.

89. Again, if 48 be multiplied by 4, and that product be divided by 24, it is the same thing as if 48 were divided at once by the quotient of 24 divided by 4, that is, by 6. For, every unit which is repeated 6 times in 48 is repeated 4 times as often, or 24 times, in 4 times 48, or the quotient of 48 and 6 is the same as the quotient of 48×4 and 6×4.

90. The results of the last five articles may be algebraically expressed thus:

$$\frac{ma}{mb} = \frac{a}{b} \qquad (85)$$

If n divide a and b without remainder,

$$\frac{\frac{a}{n}}{\frac{b}{n}} = \frac{a}{b} \qquad (86) \qquad\qquad \frac{\frac{a}{b}}{c} = \frac{a}{bc} \qquad (87)$$

$$\frac{ab}{c} = a \times \frac{b}{c} \qquad (88) \qquad\qquad \frac{ac}{b} = \frac{a}{\frac{b}{c}} \qquad (89)$$

It must be recollected, however, that these have only been proved in the case where all the divisions are without remainder.

91. When one number divides another without leaving any remainder, or is contained an exact number of times in it, it is said to be a *measure* of that number, or to *measure* it. Thus, 4 is a measure of 136, or measures 136; but it does not measure 137. The reason for

using the word measure is this: Suppose you have a rod 4 feet long, with nothing marked upon it, with which you want to measure some length; for example, the length of a street. If that street should happen to be 136 feet in length, you will be able to *measure* it with the rod, because, since 136 contains 4 34 times, you will find that the street is exactly 34 times the length of the rod. But if the street should happen to be 137 feet long, you cannot measure it with the rod; for when you have measured 34 of the rods, you will find a remainder, whose length you cannot tell without some shorter measure. Hence 4 is said to measure 136, but not to measure 137. A measure, then, is a divisor which leaves no remainder.

92. When one number is a measure of two others, it is called a *common measure* of the two. Thus, 15 is a common measure of 180 and 75. Two numbers may have several common measures. For example, 360 and 168 have the common measures 2, 3, 4, 6, 24, and several others. Now, this question may be asked: Of all the common measures of 360 and 168, which is the greatest? The answer to this question is derived from a rule of arithmetic, called the rule for finding the GREATEST COMMON MEASURE, which we proceed to consider.

93. If one quantity measure two others, it measures their sum and difference. Thus, 7 measures 21 and 56. It therefore measures 56+21 and 56—21, or 77 and 35. This is only another way of saying what was said in (74).

94. If one number measure a second, it measures every number which the second measures. Thus, 5 measures 15, and 15 measures 30, 45, 60, 75, &c.; all which numbers are measured by 5. It is plain that if

15 contains 5 3 times,
30, or 15+15 contains 5 3+3 times, or 6 times,
45, or 15+15+15 contains 5 3+3+3 or 9 times;

and so on.

95. Every number which measures both the dividend and divisor measures the remainder also. To shew this, divide 360 by 112. The quotient is 3, and the remainder 24, that is (72) 360 is three times 112

and 24, or 360 = 112×3+24. From this it follows, that 24 is the difference between 360 and 3 times 112, or 24 = 360−112×3. Take any number which measures both 360 and 112; for example, 4. Then

4 measures 360,

4 measures 112, and therefore (94) measures 112×3,

or 112+112+112.

Therefore (93) it measures 360−112×3, which is the remainder 24. The same reasoning may be applied to all other measures of 360 and 112; and the result is, that every quantity which measures both the dividend and divisor also measures the remainder. Hence, every *common measure* of a dividend and divisor is also a *common measure* of the divisor and remainder.

96. Every common measure of the divisor and remainder is also a common measure of the dividend and divisor. Take the same example, and recollect that 360 = 112×3+24. Take any common measure of the remainder 24 and the divisor 112; for example, 8. Then

8 measures 24;

and 8 measures 112, and therefore (94) measures 112×3.

Therefore (93) 8 measures 112×3+24, or measures the dividend 360. Then every common measure of the remainder and divisor is also a common measure of the divisor and dividend, or there is no common measure of the remainder and divisor which is not also a common measure of the divisor and dividend.

97. I. It is proved in (95) that the remainder and divisor have all the common measures which are in the dividend and divisor.

II. It is proved in (96) that they have no others.

It therefore follows, that the greatest of the common measures of the first two is the greatest of those of the second two, which shews how to find the greatest common measure of any two numbers,* as follows:

98. Take the preceding example, and let it be required to find the g. c. m. of 360 and 112, and observe that

* For shortness, I abbreviate the words *greatest common measure* into their initial letters, g. c. m.

360 divided by 112 gives the remainder 24,
112 divided by 24 gives the remainder 16,
24 divided by 16 gives the remainder 8,
16 divided by 8 gives no remainder.

Now, since 8 divides 16 without remainder, and since it also divides itself without remainder, 8 is the g. c. m. of 8 and 16, because it is impossible to divide 8 by any number greater than 8; so that, even if 16 had a greater measure than 8, it could not be *common* to 16 and 8.

Therefore 8 is g. c. m. of 16 and 8,
(97) g. c. m. of 16 and 8 is g. c. m. of 24 and 16,
g. c. m. of 24 and 16 is g. c. m. of 112 and 24,
g. c. m. of 112 and 24 is g. c. m. of 360 and 112,
Therefore 8 is g. c. m. of 360 and 112.

The process carried on may be written down in either of the following ways:

```
112)360(3
    336
    ---
     24)112(4
         96
         --
         16)24(1
            16
            --
             8)16(2
               16
               --
                0
```

The rule for finding the greatest common measure of two numbers is,

I. Divide the greater of the two by the less.

II. Make the remainder a divisor, and the divisor a dividend, and find another remainder.

III. Proceed in this way until there is no remainder, and the last divisor is the greatest common measure required.

112	360	3
96	336	4
16	24	1
16	16	2
0	8	

99. You may perhaps ask how the rule is to shew when the two numbers have no common measure. The fact is, that there are, strictly speaking, no such numbers, because all numbers are measured by 1; that is, contain an exact number of units, and therefore 1 is a common measure of every two numbers. If they have no other common measure, the last divisor will be 1, as in the following example, where the greatest common measure of 87 and 25 is found.

```
25)87(3
   75
   --
   12)25(2
      24
      --
       1)12(12
         12
         --
          0
```

EXERCISES.

Numbers.		g. c. m.
6197	9521	1
58363	2602	1
5547	147008443	1849
6281	326041	571
28915	31495	5
1509	300309	3

What are $36\times36+2\times36\times72+72\times72$

and $36\times36\times36+72\times72\times72$;

and what is their greatest common measure?—*Answer*, 11664.

100. If two numbers be divisible by a third, and if the quotients be again divisible by a fourth, that third is not the greatest common measure. For example, 360 and 504 are both divisible by 4. The quotients are 90 and 126. Now 90 and 126 are both divisible by 9, the quotients of which division are 10 and 14. By (87), dividing a number by 4, and then dividing the quotient by 9, is the same thing as dividing the number itself by 4×9, or by 36. Then, since 36 is a common measure of 360 and 504, and is greater than 4, 4 is not the greatest common measure. Again, since 10 and 14 are both divisible by 2, 36 is not the greatest common measure. It therefore follows, that when two numbers are divided by their greatest common measure, the quotients have no common measure except 1 (99). Otherwise, the number which was called the greatest common measure in the last sentence is not so in reality.

101. To find the greatest common measure of three numbers, find the g. c. m. of the first and second, and of this and the third. For since all common divisors of the first and second are contained in their g. c. m., and no others, whatever is common to the first, second, and third, is common also to the third and the g. c. m. of the first and second, and no others. Similarly, to find the g. c. m. of four numbers, find the g. c. m. of the first, second, and third, and of that and the fourth.

102. When a first number contains a second, or is divisible by it without remainder, the first is called a multiple of the second. The words *multiple* and *measure* are thus connected: Since 4 is a measure

of 24, 24 is a multiple of 4. The number 96 is a multiple of 8, 12, 24, 48, and several others. It is therefore called a *common multiple* of 8, 12, 24, 48, &c. The product of any two numbers is evidently a common multiple of both. Thus, 36×8, or 288, is a common multiple of 36 and 8. But there are common multiples of 36 and 8 less than 288; and because it is convenient, when a common multiple of two quantities is wanted, to use the least of them, I now shew how to find the least common multiple of two numbers.

103. Take, for example, 36 and 8. Find their greatest common measure, which is 4, and observe that 36 is 9×4, and 8 is 2×4. The quotients of 36 and 8, when divided by their greatest common measure, are therefore 9 and 2. Multiply these quotients together, and multiply the product by the greatest common measure, 4, which gives 9×2×4, or 72. This is a multiple of 8, or of 4×2 by (55); and also of 36 or of 4×9. It is also the least common multiple; but this cannot be proved to you, because the demonstration cannot be thoroughly understood without more practice in the use of letters to stand for numbers. But you may satisfy yourself that it is the least in this case, and that the same process will give the least common multiple in any other case which you may take. It is not even necessary that you should know it is the least. Whenever a common multiple is to be used, any one will do as well as the least. It is only to avoid large numbers that the least is used in preference to any other.

When the greatest common measure is 1, the least common multiple of the two numbers is their product.

The rule then is: To find the least common multiple of two numbers, find their greatest common measure, and multiply one of the numbers by the quotient which the other gives when divided by the greatest common measure. To find the least common multiple of three numbers, find the least common multiple of the first two, and find the least common multiple of that multiple and the third, and so on.

EXERCISES.

Numbers proposed.	Least common multiple.
14, 21	42
16, 5, 24	240
1, 2, 3, 4, 5, 6, 7, 8, 9, 10	2520
6, 8, 11, 16, 20	2640
876, 864	63072
868, 854	52948

A convenient mode of finding the least common multiple of several numbers is as follows, when the common measures are easily visible: Pick out a number of common measures of two or more, which have themselves no divisors greater than unity. Write them as divisors, and divide every number which will divide by one or more of them. Bring down the quotients, and also the numbers which will not divide by any of them. Repeat the process with the results, and so on until the numbers brought down have no two of them any common measure except unity. Then, for the least common multiple, multiply all the divisors by all the numbers last brought down. For instance, let it be required to find the least common multiple of all the numbers from 11 to 21.

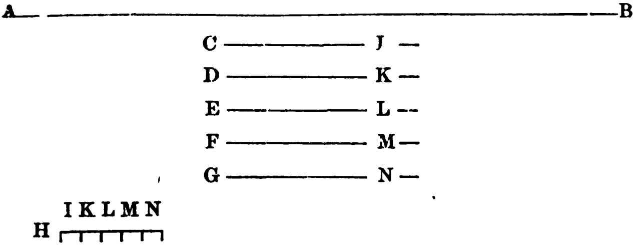
$$\begin{array}{r|ccccccccccc} 2,\ 2,\ 3,\ 5,\ 7 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 \\ \hline & 11 & 1 & 13 & 1 & 1 & 4 & 17 & 3 & 19 & 1 & 1 \end{array}$$

There are now no common measures left in the row, and the least common multiple required is the product of 2, 2, 3, 5, 7, 11, 13, 4, 17, 3, and 19; or 232792560.

SECTION V.

FRACTIONS.

104. Suppose it required to divide 49 yards into five equal parts, or, as it is called, to find the fifth part of 49 yards. If we divide 45 by 5, the quotient is 9, and the remainder is 4; that is (72), 49 is made up of 5 times 9 and 4. Let the line A B represent 49 yards:



Take 5 lines, C, D, E, F, and G, each 9 yards in length, and the line H, 4 yards in length. Then, since 49 is 5 nines and 4, C, D, E, F, G, and H, are together equal to A B. Divide H, which is 4 yards, into five equal parts, I, K, L, M, and N, and place one of these parts opposite to each of the lines, C, D, E, F, and G. It follows that the ten lines, C, D, E, F, G, I, K, L, M, N, are together equal to A B, or 49 yards. Now D and K together are of the same length as C and I together, and so are E and L, F and M, and G and N. Therefore, C and I together, repeated 5 times, will be 49 yards; that is, C and I together make up the fifth part of 49 yards.

105. C is a certain number of yards, viz. 9; but I is a new sort of quantity, to which hitherto we have never come. It is not an exact number of yards, for it arises from dividing 4 yards into 5 parts, and taking one of those parts. It is the fifth part of 4 yards, and is called a FRACTION of a yard. It is written thus, $\frac{4}{5}$ (23), and is what we must add to 9 yards in order to make up the fifth part of 49 yards.

The same reasoning would apply to dividing 49 bushels of corn, or 49 acres of land, into 5 equal parts. We should find for the fifth part of the first, 9 bushels and the fifth part of 4 bushels; and for the second, 9 acres and the fifth part of 4 acres.

We say, then, once for all, that the fifth part of 49 is 9 and $\frac{4}{5}$, or $9+\frac{4}{5}$; which is usually written $9\frac{4}{5}$, or if we use signs, $\frac{49}{5} = 9\frac{4}{5}$.

EXERCISES.

What is the seventeenth part of 1237?—*Answer*, $72\frac{13}{17}$.

What are $\frac{10032}{1974}$, $\frac{663819}{23710}$, and $\frac{22773399}{2424}$?

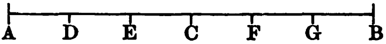
Answer, $5\frac{162}{1974}$, $27\frac{23649}{23710}$, $9394\frac{2343}{2424}$.

106. By the term fraction is understood a part of any number, or the sum of any of the equal parts into which a number is divided. Thus, $\frac{49}{5}$, $\frac{4}{5}$, $\frac{20}{7}$, are fractions. The term fraction even includes whole numbers:* for example, 17 is $\frac{17}{1}$, $\frac{34}{2}$, $\frac{51}{3}$, &c.

The upper number is called the ***numerator***, the lower number is called the ***denominator***, and both of these are called ***terms*** of the fraction. As long as the numerator is less than the denominator, the fraction is less than a unit: thus, $\frac{6}{17}$ is less than a unit; for 6 divided into 6 parts gives 1 for each part, and must give less when divided into 17 parts. Similarly, the fraction is equal to a unit when the numerator and denominator are equal, and greater than a unit when the numerator is greater than the denominator.

107. By $\frac{2}{3}$ is meant the third part of 2. This is the same as twice the third part of 1.

To prove this, let A B be two yards, and divide each of the yards A C and C B into three equal parts.



Then, because A E, E F, and F B, are all equal to one another, A E is the third part of 2. It is therefore $\frac{2}{3}$. But A E is twice A D, and A D is the third part of one yard, or $\frac{1}{3}$; therefore $\frac{2}{3}$ is twice $\frac{1}{3}$; that is, in order to get the length $\frac{2}{3}$, it makes no difference whether we divide *two* yards at once into three parts, and take *one* of them, or whether we divide *one* yard into three parts, and take *two* of them. By the same reasoning, $\frac{5}{8}$ may be found either by dividing 5 into 8 parts, and taking one of them, or by dividing 1 into 8 parts, and taking five of them. In future, of these two meanings I shall use that which is most convenient at the time, as it is proved that they are the same thing. This prin-

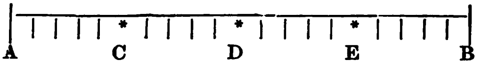
* Numbers which contain an exact number of units, such as 5, 7, 100, &c., are called ***whole numbers*** or ***integers***, when we wish to distinguish them from fractions.

ciple is the same as the following: The third part of any number may be obtained by adding together the thirds of all the units of which it consists. Thus, the third part of 2, or of two units, is made by taking one-third out of each of the units, that is,

$$\frac{2}{3} = \frac{1}{3} \times 2.$$

This meaning appears ambiguous when the numerator is greater than the denominator: thus, $\frac{15}{7}$ would mean that 1 is to be divided into 7 parts, and 15 of them are to be taken. We should here let as many units be each divided into 7 parts as will give more than 15 of those parts, and take 15 of them.

108. The value of a fraction is not altered by multiplying the numerator and denominator by the same quantity. Take the fraction $\frac{3}{4}$, multiply its numerator and denominator by 5, and it becomes $\frac{15}{20}$, which is the same thing as $\frac{3}{4}$; that is, one-twentieth part of 15 yards is the same thing as one-fourth of 3 yards: or, if our second meaning of the word fraction be used, you get the same length by dividing a yard into 20 parts and taking 15 of them, as you get by dividing it into 4 parts and taking 3 of them. To prove this,



let A B represent a yard; divide it into 4 equal parts, A C, C D, D E, and E B, and divide each of these parts into 5 equal parts. Then A E is $\frac{3}{4}$. But the second division cuts the line into 20 equal parts, of which A E contains 15. It is therefore $\frac{15}{20}$. Therefore, $\frac{15}{20}$ and $\frac{3}{4}$ are the same thing.

Again, since $\frac{3}{4}$ is made from $\frac{15}{20}$ by dividing both the numerator and denominator by 5, the value of a fraction is not altered by dividing both its numerator and denominator by the same quantity. This principle, which is of so much importance in every part of arithmetic, is often used in common language, as when we say that 14 out of 21 is 2 out of 3, &c.

109. Though the two fractions $\frac{3}{4}$ and $\frac{15}{20}$ are the same in value, and

either of them may be used for the other without error, yet the first is more convenient than the second, not only because you have a clearer idea of the fourth of three yards than of the twentieth part of fifteen yards, but because the numbers in the first being smaller, are more convenient for multiplication and division. It is therefore useful, when a fraction is given, to find out whether its numerator and denominator have any common divisors or common measures. In (98) was given a rule for finding the greatest common measure of any two numbers; and it was shewn that when the two numbers are divided by their greatest common measure, the quotients have no common measure except 1. Find the greatest common measure of the terms of the fraction, and divide them by that number. The fraction is then said to be *reduced to its lowest terms*, and is in the state in which the best notion can be formed of its magnitude.

EXERCISES.

With each fraction is written the same reduced to its lowest terms.

$$\frac{2794}{2921} = \frac{22\times127}{23\times127} = \frac{22}{23}$$

$$\frac{2788}{4920} = \frac{17\times164}{30\times164} = \frac{17}{30}$$

$$\frac{93208}{13786} = \frac{764\times122}{113\times122} = \frac{764}{113}$$

$$\frac{888800}{40359600} = \frac{22\times40400}{999\times40400} = \frac{22}{999}$$

$$\frac{95469}{359784} = \frac{121\times789}{456\times789} = \frac{121}{456}$$

110. When the terms of the fraction given are already in factors,* any one factor in the numerator may be divided by a number, provided some one factor in the denominator is divided by the same. This follows from (88) and (108). In the following examples the figures altered by division are accented.

* A factor of a number is a number which divides it without remainder: thus, 4, 6, 8, are factors of 24, and 6×4, 8×3, 2×2×2×3, are several ways of decomposing 24 into factors.

$$\frac{12\times11\times10}{2\times3\times4} = \frac{3'\times11\times10}{2\times3\times1'} = \frac{1'\times11\times5'}{1'\times1'\times1'} = 55.$$

$$\frac{18\times15\times13}{20\times54\times52} = \frac{2'\times3'\times1'}{4'\times6'\times4'} = \frac{1'\times1'\times1'}{2'\times2'\times4'} = \frac{1}{16}.$$

$$\frac{27\times28}{9\times70} = \frac{3'\times4'}{1'\times10'} = \frac{3'\times2'}{1'\times5'} = \frac{6}{5}.$$

111. As we can, by (108), multiply the numerator and denominator of a fraction by any number, without altering its value, we can now readily reduce two fractions to two others, which shall have the same value as the first two, and which shall have the same denominator. Take, for example, $\frac{2}{3}$ and $\frac{4}{7}$; multiply both terms of $\frac{2}{3}$ by 7, and both terms of $\frac{4}{7}$ by 3. It then appears that

$$\frac{2}{3} \text{ is } \frac{2\times7}{3\times7} \text{ or } \frac{14}{21}$$

$$\frac{4}{7} \text{ is } \frac{4\times3}{7\times3} \text{ or } \frac{12}{21}.$$

Here are then two fractions $\frac{14}{21}$ and $\frac{12}{21}$, equal to $\frac{2}{3}$ and $\frac{4}{7}$, and having the same denominator, 21; in this case, $\frac{2}{3}$ and $\frac{4}{7}$ are said to be *reduced to a common denominator.*

It is required to reduce $\frac{1}{10}$, $\frac{5}{6}$, and $\frac{7}{9}$ to a common denominator. Multiply both terms of the first by the product of 6 and 9; of the second by the product of 10 and 9; and of the third by the product of 10 and 6. Then it appears (108) that

$$\frac{1}{10} \text{ is } \frac{1\times6\times9}{10\times6\times9} \text{ or } \frac{54}{540}$$

$$\frac{5}{6} \text{ is } \frac{5\times10\times9}{6\times10\times9} \text{ or } \frac{450}{540}$$

$$\frac{7}{9} \text{ is } \frac{7\times10\times6}{9\times10\times6} \text{ or } \frac{420}{540}.$$

On looking at these last fractions, we see that all the numerators and the common denominator are divisible by 6, and (108) this division will not alter their values. On dividing the numerators and denominators of $\frac{54}{540}$, $\frac{450}{540}$, and $\frac{420}{540}$ by 6, the resulting fractions are, $\frac{9}{90}$, $\frac{75}{90}$, and $\frac{70}{90}$. These are fractions with a common denominator, and which are the same as $\frac{1}{10}$, $\frac{5}{6}$, and $\frac{7}{9}$; and therefore these are a more simple answer to the question than the first fractions. Observe also that 540 is one common multiple of 10, 6, and 9, namely, 10×6×9, but that 90 is *the least* common multiple of 10, 6, and 9 (103). The following process, therefore, is better. To reduce the fractions $\frac{1}{10}$, $\frac{5}{6}$, and $\frac{7}{9}$, to others having the same value and a common denominator, begin by finding the least common multiple of 10, 6, and 9, by the rule in (103), which is 90. Observe that 10, 6, and 9 are contained in 90 9, 15, and 10 times. Multiply both terms of the first by 9, of the second by 15, and of the third by 10, and the fractions thus produced are $\frac{9}{90}$, $\frac{75}{90}$, and $\frac{70}{90}$, the same as before.

If one of the numbers be a whole number, it may be reduced to a fraction having the common denominator of the rest, by (106).

EXERCISES.

Fractions proposed	reduced to a common denominator.
$\frac{2}{3}$ $\frac{1}{5}$ $\frac{1}{6}$	$\frac{20}{30}$ $\frac{6}{30}$ $\frac{5}{30}$
$\frac{1}{3}$ $\frac{2}{7}$ $\frac{3}{14}$ $\frac{12}{21}$ $\frac{3}{4}$	$\frac{28}{84}$ $\frac{24}{84}$ $\frac{18}{84}$ $\frac{48}{84}$ $\frac{63}{84}$
3 $\frac{4}{10}$ $\frac{5}{100}$ $\frac{6}{1000}$	$\frac{3000}{1000}$ $\frac{400}{1000}$ $\frac{50}{1000}$ $\frac{6}{1000}$
$\frac{33}{379}$ $\frac{281}{677}$	$\frac{22341}{256583}$ $\frac{106499}{256583}$

112. By reducing two fractions to a common denominator, we are able to compare them; that is, to tell which is the greater and which the less of the two. For example, take $\frac{1}{2}$ and $\frac{7}{15}$. These fractions reduced, without alteration of their value, to a common denominator, are $\frac{15}{30}$ and $\frac{14}{30}$. Of these the first must be the greater, because (107) it may be obtained by dividing 1 into 30 equal parts and taking 15 of them, but the second is made by taking 14 of those parts.

It is evident that of two fractions which have the same denominator, the greater has the greater numerator; and also that of two fractions which have the same numerator, the greater has the less denominator.

Thus, $\frac{8}{7}$ is greater than $\frac{8}{9}$, since the first is a 7th, and the last only a 9th part of 8. Also, any numerator may be made to belong to as small a fraction as we please, by sufficiently increasing the denominator. Thus, $\frac{10}{100}$ is $\frac{1}{10}$, $\frac{10}{1000}$ is $\frac{1}{100}$, and $\frac{10}{1000000}$ is $\frac{1}{100000}$ (108).

We can now also increase and diminish the first fraction by the second. For the first fraction is made up of 15 of the 30 equal parts into which 1 is divided. The second fraction is 14 of those parts. The sum of the two, therefore, must be 15+14, or 29 of those parts; that is, $\frac{1}{2}+\frac{7}{15}$ is $\frac{29}{30}$. The difference of the two must be 15−14, or 1 of those parts; that is, $\frac{1}{2}-\frac{7}{15}=\frac{1}{30}$.

113. From the last two articles the following rules are obtained:

I. To compare, to add, or to subtract fractions, first reduce them to a common denominator. When this has been done, that is the greatest of the fractions which has the greatest numerator.

Their sum has the sum of the numerators for its numerator, and the common denominator for its denominator.

Their difference has the difference of the numerators for its numerator, and the common denominator for its denominator.

EXERCISES.

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{5}=\frac{53}{60} \qquad \frac{44}{3}-\frac{153}{427}=\frac{18329}{1281}$$

$$1+\frac{8}{10}+\frac{3}{100}+\frac{4}{1000}=\frac{1834}{1000} \qquad 2-\frac{1}{7}+\frac{12}{13}=\frac{253}{91}$$

$$\frac{1}{2}+\frac{8}{16}+\frac{94}{188}=\frac{3}{2} \qquad \frac{163}{521}-\frac{97}{881}=\frac{93066}{459001}$$

114. Suppose it required to add a whole number to a fraction, for example, 6 to $\frac{4}{9}$. By (106) 6 is $\frac{54}{9}$, and $\frac{54}{9}+\frac{4}{9}$ is $\frac{58}{9}$; that is, $6+\frac{4}{9}$, or as it is usually written, $6\frac{4}{9}$, is $\frac{58}{9}$. The rule in this case is: Multiply the whole number by the denominator of the fraction, and to the product add the numerator of the fraction; the sum will be the numerator of the result, and the denominator of the fraction will be its denominator. Thus, $3\frac{1}{4}=\frac{13}{4}$, $22\frac{5}{9}=\frac{203}{9}$, $74\frac{2}{55}=\frac{4072}{55}$. This rule is the opposite of that in (105).

115. From the last rule it appears that $1723\frac{907}{10000}$ is $\frac{17230907}{10000}$, $667\frac{225}{1000}$ is $\frac{667225}{1000}$, and $23\frac{99}{100000}$ is $\frac{2300099}{100000}$. Hence, when a whole number is to be added to a fraction whose denominator is 1 followed by *ciphers*, the number of which is not less than the number of *figures* in the numerator, the rule is: Write the whole number first, and then the numerator of the fraction, with as many ciphers between them as the number of *ciphers* in the denominator exceeds the number of *figures* in the numerator. This is the numerator of the result, and the denominator of the fraction is its denominator. If the number of ciphers in the denominator be equal to the number of figures in the numerator, write no ciphers between the whole number and the numerator.

EXERCISES.

Reduce the following mixed quantities to fractions: $1\frac{23707}{100000}$, $2457\frac{6}{10}$, $1207\frac{299}{10000000}$, and $233\frac{2210}{10000}$.

116. Suppose it required to multiply $\frac{2}{3}$ by 4. This by (48) is taking $\frac{2}{3}$ four times; that is, finding $\frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}$. This by (112) is $\frac{8}{3}$; so that to multiply a fraction by a whole number the rule is: Multiply the numerator by the whole number, and let the denominator remain.

117. If the denominator of the fraction be divisible by the whole number, the rule may be stated thus: Divide the denominator of the fraction by the whole number, and let the numerator remain. For example, multiply $\frac{7}{36}$ by 6. This (116) is $\frac{42}{36}$, which, since the numerator and denominator are now divisible by 6, is (108) the same as $\frac{7}{6}$. It is plain that $\frac{7}{6}$ is made from $\frac{7}{36}$ in the manner stated in the rule.

118. Multiplication has been defined to be the taking as many of one number as there are units in another. Thus, to multiply 12 by 7 is to take as many twelves as there are units in 7, or to take 12 as many times as you must take 1 in order to make 7. Thus, what is done with 1 in order to make 7, is done with 12 to make 7 times 12. For example,

7 is 1 + 1 + 1 ÷ 1 + 1 + 1 + 1.

7 times 12 is 12+12+12+12+12+12+12.

When the same thing is done with two fractions, the result is still called their product, and the process is still called multiplication. There is this difference, that whereas a whole number is made by adding 1 to itself a number of times, a fraction is made by dividing 1 into a number of equal parts, and adding *one of these parts* to itself a number of times. This being the meaning of the word multiplication, as applied to fractions, what is $\frac{3}{4}$ multiplied by $\frac{7}{8}$? Whatever is done with 1 in order to make $\frac{7}{8}$ must now be done with $\frac{3}{4}$; but to make $\frac{7}{8}$, 1 is divided into 8 parts, and 7 of them are taken. Therefore, to make $\frac{3}{4}\times\frac{7}{8}$, $\frac{3}{4}$ must be divided into 8 parts, and 7 of them must be taken. Now $\frac{3}{4}$ is, by (108), the same thing as $\frac{24}{32}$. Since $\frac{24}{32}$ is made by dividing 1 into 32 parts, and taking 24 of them, or, which is the same thing, taking 3 of them 8 times, if $\frac{24}{32}$ be divided into 8 equal parts, each of them is $\frac{3}{32}$; and if 7 of these parts be taken, the result is $\frac{21}{32}$ (116): therefore $\frac{3}{4}$ multiplied by $\frac{7}{8}$ is $\frac{21}{32}$; and the same reasoning may be applied to any other fractions. But $\frac{21}{32}$ is made from $\frac{3}{4}$ and $\frac{7}{8}$ by multiplying the two numerators together for the numerator, and the two denominators for the denominator; which furnishes a rule for the multiplication of fractions.

119. If this product $\frac{21}{32}$ is to be multiplied by a third fraction, for example, by $\frac{5}{9}$, the result is, by the same rule, $\frac{105}{288}$; and so on. The general rule for multiplying any number of fractions together is therefore:

Multiply all the numerators together for the numerator of the product, and all the denominators together for its denominator.

120. Suppose it required to multiply together $\frac{15}{16}$ and $\frac{8}{10}$. The product may be written thus: $\frac{15\times 8}{16\times 10}$, and is $\frac{120}{160}$, which reduced to its lowest terms (109) is $\frac{3}{4}$. This result might have been obtained directly, by observing that 15 and 10 are both measured by 5, and 8 and 16 are both measured by 8, and that the fraction may be written thus: $\frac{3\times 5\times 8}{2\times 8\times 2\times 5}$. Divide both its numerator and denominator by 5×8 (108) and (87), and the result is at once $\frac{3}{4}$; therefore, before proceeding to multiply any number of fractions together, if there be any numerator and any denominator, whether belonging to the same fraction or not, which have a common measure, divide them both by that common measure, and use the quotients instead of the dividends.

A whole number may be considered as a fraction whose denominator is 1; thus, 16 is $\frac{16}{1}$ (106); and the same rule will apply when one or more of the quantities are whole numbers.

EXERCISES

$$\frac{126}{7470}\times\frac{268}{919}=\frac{36448}{6864930}=\frac{18224}{3432465}$$

$$\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\frac{}{5}=\frac{1}{5},\qquad \frac{2}{17}\times\frac{17}{45}=\frac{2}{45}$$

$$\frac{2}{59}\times\frac{13}{7}\times\frac{241}{19}=\frac{6266}{7847},\qquad \frac{13}{461}\times\frac{601}{11}=\frac{7813}{5071}$$

Fraction proposed.	Square.	Cube.
$\frac{701}{158}$	$\frac{491401}{24964}$	$\frac{344472101}{3944312}$
$\frac{140}{141}$	$\frac{19600}{19881}$	$\frac{2744000}{2803221}$
$\frac{355}{113}$	$\frac{126025}{12769}$	$\frac{44738875}{1442897}$

From 100 acres of ground, two-thirds of them are taken away; 50 acres are then added to the result, and $\frac{5}{7}$ of the whole is taken; what number of acres does this produce?—*Answer*, $59\frac{11}{21}$.

121. In dividing one whole number by another, for example, 108 by 9, this question is asked,—Can we, by the addition of any number of nines, produce 108? and if so, how many nines will be sufficient for that purpose?

Suppose we take two fractions, for example, $\frac{2}{3}$ and $\frac{4}{5}$, and ask, Can we, by dividing $\frac{4}{5}$ into some number of equal parts, and adding a number of these parts together, produce $\frac{2}{3}$? if so, into *how many parts* must we divide $\frac{4}{5}$, and *how many of them* must we add together? The solution of this question is still called the division of $\frac{2}{3}$ by $\frac{4}{5}$; and the fraction whose denominator is the number of parts into which $\frac{4}{5}$ is divided, and whose numerator is the number of them which is taken, is called the quotient. The solution of this question is as follows: Reduce both these fractions to a common denominator (111), which does not alter their value (108); they then become $\frac{10}{15}$ and $\frac{12}{15}$. The question now is, to divide $\frac{12}{15}$ into a number of parts, and to produce $\frac{10}{15}$ by taking a number of these parts. Since $\frac{12}{15}$ is made by dividing 1 into 15 parts and taking 12 of them, if we divide $\frac{12}{15}$ into 12 equal parts, each of these parts is $\frac{1}{15}$; if we take 10 of these parts, the result is $\frac{10}{15}$. Therefore, in order to produce $\frac{10}{15}$ or $\frac{2}{3}$ (108), we must divide $\frac{12}{15}$ or $\frac{4}{5}$ into 12 parts, and take 10 of them; that is, the quotient is $\frac{10}{12}$. If we call $\frac{2}{3}$ the dividend, and $\frac{4}{5}$ the divisor, as before, the quotient in this case is derived from the following rule, which the same reasoning will shew to apply to other cases:

The numerator of the quotient is the numerator of the dividend multiplied by the denominator of the divisor. The denominator of the quotient is the denominator of the dividend multiplied by the numerator of the divisor. This rule is the reverse of multiplication, as will be seen by comparing what is required in both cases. In multiplying $\frac{4}{5}$ by $\frac{10}{12}$, I ask, if out of $\frac{4}{5}$ be taken 10 parts out of 12, how much *of a unit* is taken, and the answer is $\frac{40}{60}$, or $\frac{2}{3}$. Again, in dividing $\frac{2}{3}$ by $\frac{4}{5}$, I ask what part of $\frac{4}{5}$ is $\frac{2}{3}$, the answer to which is $\frac{10}{12}$.

122. By taking the following instance, we shall see that this rule can be sometimes simplified. Divide $\frac{16}{33}$ by $\frac{28}{15}$. Observe that 16 is 4×4, and 28 is 4×7; 33 is 3×11, and 15 is 3×5; therefore the two fractions are $\frac{4\times4}{3\times11}$ and $\frac{4\times7}{3\times5}$, and their quotient, according to the rule, is $\frac{4\times4\times3\times5}{3\times11\times4\times7}$, in which 4×3 is found both in the numerator and denominator. The fraction is therefore (108) the same as $\frac{4\times5}{11\times7}$, or $\frac{20}{77}$. The rule of the last article, therefore, admits of this modification: If the two numerators or the two denominators have a common measure, divide by that common measure, and use the quotients instead of the dividends.

123. In dividing a fraction by a whole number, for example, $\frac{2}{3}$ by 15, consider 15 as the fraction $\frac{15}{1}$. The rule gives $\frac{2}{45}$ as the quotient. Therefore, to divide a fraction by a whole number, multiply the denominator by that whole number.

EXERCISES.

Dividend.	Divisor.	Quotient.
$\frac{41}{33}$	$\frac{63}{11}$	$\frac{41}{189}$
$\frac{467}{151}$	$\frac{907}{101}$	$\frac{47167}{136957}$
$\frac{7813}{5071}$	$\frac{601}{11}$	$\frac{13}{461}$

What are $\dfrac{\frac{1}{5}\times\frac{1}{5}\times\frac{1}{5}-\frac{2}{17}\times\frac{2}{17}\times\frac{2}{17}}{\frac{1}{5}-\frac{2}{17}}$, and $\dfrac{\frac{8}{11}\times\frac{8}{11}-\frac{3}{11}\times\frac{3}{11}}{\frac{8}{11}-\frac{3}{11}}$?

Answer, $\frac{559}{7225}$, and 1

A can reap a field in 12 days, B in 6, and C in 4 days; in what time can they all do it together?*—*Answer*, 2 days.

In what time would a cistern be filled by cocks which would separately fill it in 12, 11, 10, and 9 hours?—*Answer*, $2\frac{454}{763}$ hours.

124. The principal results of this section may be exhibited algebraically as follows; let a, b, c, &c. stand for any whole numbers. Then

(107) $\frac{a}{b}=\frac{1}{b}\times a$ (108) $\frac{a}{b}=\frac{ma}{mb}$

(111) $\frac{a}{b}$ and $\frac{c}{d}$ are the same as $\frac{ad}{bd}$ and $\frac{bc}{bd}$

(112) $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$ $\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$

(113) $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$ $\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$

(118) $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$ (121) $\frac{a}{b}$ div^{d}. by $\frac{c}{d}$ or $\dfrac{\frac{a}{b}}{\frac{c}{d}}=\frac{ad}{bc}$

* The method of solving this and the following question may be shewn thus: If the number of days in which each could reap the field is given, the part which each could do in a day by himself can be found, and thence the part which all could do together; this being known, the number of days which it would take all to do the whole can be found.

125. These results are true even when the letters themselves represent fractions. For example, take the fraction $\frac{\frac{a}{b}}{\frac{c}{d}}$, whose numerator and denominator are fractional, and multiply its numerator and denominator by the fraction $\frac{e}{f}$, which gives $\frac{\frac{ae}{bf}}{\frac{ce}{df}}$, which (121) is $\frac{aedf}{bfce}$, which, dividing the numerator and denominator by ef (108), is $\frac{ad}{bc}$. But the original fraction itself is $\frac{ad}{bc}$; hence $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b}\times\frac{e}{f}}{\frac{c}{d}\times\frac{e}{f}}$ which corresponds to the second formula* in (124). In a similar manner it may be shewn, that the other formulæ of the same article are true when the letters there used either represent fractions, or are removed and fractions introduced in their place. All formulæ established throughout this work are equally true when fractions are substituted for whole numbers. For example (54), $(m+n)a = ma+na$. Let m, n, and a be respectively the fractions $\frac{p}{q}$, $\frac{r}{s}$, and $\frac{b}{c}$. Then $m+n$ is $\frac{p}{q}+\frac{r}{s}$, or $\frac{ps+qr}{qs}$, and $(m+n)a$ is $\frac{ps+qr}{qs}\times\frac{b}{c}$, or $\frac{(ps+qr)b}{qsc}$ or $\frac{psb+qrb}{qsc}$. But this (112) is $\frac{psb}{qsc}+\frac{qrb}{qsc}$, which is $\frac{pb}{qc}+\frac{rb}{sc}$, since $\frac{psb}{qsc}=\frac{pb}{qc}$, and $\frac{qrb}{qsc}=\frac{rb}{sc}$ (108) But $\frac{pb}{qc}=\frac{p}{q}\times\frac{b}{c}$, and $\frac{rb}{sc} = \frac{r}{s}\times\frac{b}{c}$. Therefore $(m+n)a$, or $\left(\frac{p}{q}+\frac{r}{s}\right)\frac{b}{c} = \frac{p}{q}\times\frac{b}{c}+\frac{r}{s}\times\frac{b}{c}$. In a similar manner the same may be proved of any other formula.

The following examples may be useful:

$$\frac{\frac{a}{b}\times\frac{c}{d}+\frac{e}{f}\times\frac{g}{h}}{\frac{a}{b}\times\frac{e}{f}+\frac{c}{d}\times\frac{g}{h}} = \frac{acfh+bdeg}{aedh+bcfg}$$

$$\frac{1}{a+\frac{1}{b}} = \frac{b}{ab+1}$$

$$\frac{1}{a+\frac{1}{b+\frac{1}{c}}} = \frac{1}{a+\frac{c}{bc+1}} = \frac{bc+1}{abc+a+c}$$

* A formula is a name given to any algebraical expression which is commonly used.

Thus, $\dfrac{1}{6+\dfrac{1}{7+\dfrac{1}{8}}} = \dfrac{1}{6+\dfrac{8}{57}} = \dfrac{57}{350}$

The rules that have been proved to hold good for all numbers may be applied when the numbers are represented by letters.

SECTION VI.

DECIMAL FRACTIONS.

126. We have seen (112) (121) the necessity of reducing fractions to a common denominator, in order to compare their magnitudes. We have seen also how much more readily operations are performed upon fractions which have the same, than upon those which have different, denominators. On this account it has long been customary, in all those parts of mathematics where fractions are often required, to use none but such as either have, or can be easily reduced to others having, the same denominators. Now, of all numbers, those which can be most easily managed are such as 10, 100, 1000, &c., where 1 is followed by ciphers. These are called DECIMAL NUMBERS; and a fraction whose denominator is any one of them, is called a DECIMAL FRACTION, or more commonly, a DECIMAL.

127. A whole number may be reduced to a decimal fraction, or one decimal fraction to another, with the greatest ease. For example, 94 is $\frac{940}{10}$, or $\frac{9400}{100}$, or $\frac{94000}{1000}$ (106); $\frac{3}{10}$ is $\frac{30}{100}$, or $\frac{300}{1000}$, or $\frac{3000}{10000}$ (108). The placing of a cipher on the right hand of any number is the same thing as multiplying that number by 10 (57), and this may be done as often as we please in the numerator of a fraction, provided it be done as often in the denominator (108).

128. The next question is, How can we reduce a fraction which is not decimal to another which is, without altering its value? Take, for example, the fraction $\frac{7}{16}$, multiply both the numerator and denominator successively by 10, 100, 1000, &c., which will give a series of fractions, each of which is equal to $\frac{7}{16}$ (108), viz. $\frac{70}{160}$, $\frac{700}{160\text{0}}$, $\frac{7000}{16000}$,

$\frac{70000}{160000}$, &c. The denominator of each of these fractions can be divided without remainder by 16, the quotients of which divisions form the series of decimal numbers 10, 100, 1000, 10000, &c. If, therefore, one of the numerators be divisible by 16, the fraction to which that numerator belongs has a numerator and denominator both divisible by 16. When that division has been made, which (108) does not alter the value of the fraction, we shall have a fraction whose denominator is one of the series 10, 100, 1000, &c., and which is equal in value to $\frac{7}{16}$. The question is then reduced to finding the first of the numbers 70, 700, 7000, 70000, &c., which can be divided by 16 without remainder.

Divide these numbers, one after the other, by 16, as follows:

16)70(4	16)700(43	16)7000(437	16)70000(4375
64	64	64	64
6	60	60	60
	48	48	48
	12	120	120
		112	112
		8	80
			80
			0

It appears, then, that 70000 is the first of the numerators which is divisible by 16. But it is not necessary to write down each of these divisions, since it is plain that the last contains all which came before. It will do, then, to proceed at once as if the number of ciphers were without end, to stop when the remainder is nothing, and then count the number of ciphers which have been used. In this case, since 70000 is 16×4375, $\frac{70000}{160000}$, which is $\frac{16 \times 4375}{16 \times 10000}$, or $\frac{4375}{10000}$, gives the fraction required.

Therefore, to reduce a fraction to a decimal fraction, annex ciphers to the numerator, and divide by the denominator until there is no remainder. The quotient will be the numerator of the required fraction, and the denominator will be unity, followed by as many ciphers as were used in obtaining the quotient.

EXERCISES.

Reduce to decimal fractions

$$\frac{1}{2}, \frac{1}{4}, \frac{2}{25}, \frac{1}{50}, \frac{3927}{1250}, \text{ and } \frac{453}{625}.$$

$$\textit{Answer}, \frac{5}{10}, \frac{25}{100}, \frac{8}{100}, \frac{2}{100}, \frac{31416}{10000}, \text{ and } \frac{7248}{10000}.$$

129. It will happen in most cases that the annexing of ciphers to the numerator will never make it divisible by the denominator without remainder. For example, try to reduce $\frac{1}{7}$ to a decimal fraction.

$$7)\underline{10000000000000000000, \&c.}$$
$$142857142857142857, \&c.$$

The quotient here is a continual repetition of the figures 1, 4, 2, 8, 5, 7, in the same order; therefore $\frac{1}{7}$ cannot be reduced to a decimal fraction. But, nevertheless, if we take as a numerator any number of figures from the quotient 142857142857, &c., and as a denominator 1 followed by as many ciphers as were used in making that part of the quotient, we shall get a fraction which differs very little from $\frac{1}{7}$, and which will differ still less from it if we put more figures in the numerator and more ciphers in the denominator.

Thus,	$\frac{1}{10}$	is less than	$\frac{1}{7}$	by	$\frac{3}{70}$	which is not so much as $\frac{1}{10}$
	$\frac{14}{100}$	. . .	$\frac{1}{7}$	. .	$\frac{2}{700}$	. . . . . . $\frac{1}{100}$
	$\frac{142}{1000}$	. . .	$\frac{1}{7}$	. .	$\frac{6}{7000}$	. . . . $\frac{1}{1000}$
	$\frac{1428}{10000}$	. . .	$\frac{1}{7}$	.	$\frac{4}{70000}$	. . . . . . $\frac{1}{10000}$
	$\frac{14285}{100000}$	. . .	$\frac{1}{7}$	.	$\frac{5}{700000}$	. . . . . . $\frac{1}{100000}$
	$\frac{142857}{1000000}$	. . .	$\frac{1}{7}$		$\frac{1}{7000000}$	. . . . . $\frac{1}{1000000}$
	&c.		&c.		&c.	&c.

In the first column is a series of decimal fractions, which come nearer and nearer to $\frac{1}{7}$, as the third column shews. Therefore, though we cannot find a decimal fraction which is exactly $\frac{1}{7}$, we can find one which differs from it as little as we please.

This may also be illustrated thus: It is required to reduce $\frac{1}{7}$ to a decimal fraction without the error of say a millionth of a unit; multiply the numerator and denominator of $\frac{1}{7}$ by a million, and then divide both by 7; we have then

$$\frac{1}{7} = \frac{1000000}{7000000} = \frac{142857\frac{1}{7}}{1000000}$$

If we reject the fraction $\frac{1}{7}$ in the numerator, what we reject is really the 7th part of the millionth part of a unit; or less than the millionth part of a unit. Therefore $\frac{142857}{1000000}$ is the fraction required.

EXERCISES.

Make similar tables with these fractions $\frac{3}{91}$, $\frac{17}{143}$, and $\frac{1}{247}$.

The recurring quotient of $\frac{3}{91}$ is 329670,329670, &c.

. . . . . . $\frac{17}{143}$ ... 118881,118881, &c.

. . . . . . $\frac{1}{247}$ ... 404858299595141700,4048582 &c.

130. The reason for the *recurrence* of the figures of the quotient in the same order is as follows: If 1000, &c. be divided by the number 247, the remainder at each step of the division is less than 247, being either 0, or one of the first 246 numbers. If, then, the remainder never become nothing, by carrying the division far enough, one remainder will occur a second time. If possible, let the first 246 remainders be all different, that is, let them be 1, 2, 3, &c., up to 246, variously distributed. As the 247th remainder cannot be so great as 247, it must be one of these which have preceded. From the step where the remainder becomes the same as a former remainder, it is evident that former figures of the quotient must be repeated in the same order.

131. You will here naturally ask, What is the use of decimal fractions, if the greater number of fractions cannot be reduced at all to decimals? The answer is this: The addition, subtraction, multiplication, and division of decimal fractions are much easier than those of

common fractions; and though we cannot reduce all common fractions to decimals, yet we can find decimal fractions so near to each of them, that the error arising from using the decimal instead of the common fraction will not be perceptible. For example, if we suppose an inch to be divided into ten million of equal parts, one of those parts by itself will not be visible to the eye. Therefore, in finding a length, an error of a ten-millionth part of an inch is of no consequence, even where the finest measurement is necessary. Now, by carrying on the table in (129), we shall see that $\frac{1428571}{10000000}$ does not differ from $\frac{1}{7}$ by $\frac{1}{10000000}$; and if these fractions represented parts of an inch, the first might be used for the second, since the difference is not perceptible. In applying arithmetic to practice, nothing can be measured so accurately as to be represented in numbers without any error whatever, whether it be length, weight, or any other species of magnitude. It is therefore unnecessary to use any other than decimal fractions, since, by means of them, any quantity may be represented with as much correctness as by any other method.

EXERCISES.

Find decimal fractions which do not differ from the following fractions by $\frac{1}{100000000}$.

$\frac{1}{3}$—*Answer*, $\frac{33333333}{100000000}$.

$\frac{4}{7}$ . . . $\frac{57142857}{100000000}$.

$\frac{113}{355}$—*Answer*, $\frac{31830985}{100000000}$.

$\frac{355}{113}$ . . . $\frac{314159292}{100000000}$.

132. Every decimal may be immediately reduced to a quantity consisting either of a whole number and more simple decimals, or of more simple decimals alone, having one figure only in each of the numerators. Take, for example, $\frac{147326}{1000}$. By (115) $\frac{147326}{1000}$ is $147\frac{326}{1000}$; and since 326 is made up of 300, and 20, and 6; by (112) $\frac{326}{1000} = \frac{300}{1000} + \frac{20}{1000} + \frac{6}{1000}$. But (108) $\frac{300}{1000}$ is $\frac{3}{10}$, and $\frac{20}{1000}$ is $\frac{2}{100}$. Therefore, $\frac{147326}{1000}$ is made up of $147+\frac{3}{10}+\frac{2}{100}+\frac{6}{1000}$. Now, take any number, for example, 147326, and form a number of fractions having for their numerators this number, and for their denominators 1, 10, 100, 1000, 10000, &c., and

reduce these fractions into numbers and more simple decimals, in the foregoing manner, which will give the table below.

DECOMPOSITION OF A DECIMAL FRACTION.

$$\frac{147326}{1} = 147326$$

$$\frac{147326}{10} = 14732 + \frac{6}{10}$$

$$\frac{147326}{100} = 1473 + \frac{2}{10} + \frac{6}{100}$$

$$\frac{147326}{1000} = 147 + \frac{3}{10} + \frac{2}{100} + \frac{6}{1000}$$

$$\frac{147326}{10000} = .\ 14 + \frac{7}{10} + \frac{3}{100} + \frac{2}{1000} + \frac{6}{10000}$$

$$\frac{147326}{100000} = .\ .\ 1 + \frac{4}{10} + \frac{7}{100} + \frac{3}{1000} + \frac{2}{10000} + \frac{6}{100000}$$

$$\frac{147326}{1000000} = .\ .\ .\ .\ \frac{1}{10} + \frac{4}{100} + \frac{7}{1000} + \frac{3}{10000} + \frac{2}{100000} + \frac{6}{1000000}$$

$$\frac{147326}{10000000} = .\ .\ .\ .\ .\ \frac{1}{100} + \frac{4}{1000} + \frac{7}{10000} + \frac{3}{100000} + \frac{2}{1000000} + \frac{6}{10000000}$$

N.B. The student should write this table himself, and then proceed to make similar tables from the following exercises.

EXERCISES.

Reduce the following fractions into a series of numbers and more simple fractions:

$$\frac{31415926}{10},\quad \frac{31415926}{100},\quad \&c.$$

$$\frac{2700031}{10},\quad \frac{2700031}{100},\quad \&c.,$$

$$\frac{2073000}{10},\quad \frac{2073000}{100},\quad \&c.$$

$$\frac{3331303}{1000},\quad \frac{3331303}{10000},\quad \&c.$$

133. If, in this table, and others made in the same manner, you look at those fractions which contain a whole number, you will see that they

may be made thus: Mark off, from the right hand of the numerator, as many *figures* as there are *ciphers* in the denominator by a point, or any other convenient mark.

This will give 14732·6 when the fraction is $\frac{147326}{10}$

. . . . 1473·26 . . . . $\frac{147326}{100}$

. . . . 147·326 . . . . $\frac{147326}{1000}$

&c. &c.

The figures on the left of the point by themselves make the whole number which the fraction contains. Of those on its right, the first is the numerator of the fraction whose denominator is 10, the second of that whose denominator is 100, and so on. We now come to those fractions which do not contain a whole number.

134. The first of these is $\frac{147326}{1000000}$, in which the number of *ciphers* in the denominator is the same as the number of *figures* in the numerator. If we still follow the same rule, and mark off all the figures, by placing the point before them all, thus, ·147326, the observation in (133) still holds good; for, on looking at $\frac{147326}{1000000}$ in the table, we find it is

$$\frac{1}{10}+\frac{4}{100}+\frac{7}{1000}+\frac{3}{10000}+\frac{2}{100000}+\frac{6}{1000000}$$

The next fraction is $\frac{147326}{10000000}$, which we find by the table to be

$$\frac{1}{100}+\frac{4}{1000}+\frac{7}{10000}+\frac{3}{100000}+\frac{2}{1000000}+\frac{6}{10000000}.$$

In this, 1 is not divided by 10, but by 100; if, therefore, we put a point before the whole, the rule is not true, for the first figure on the left of the point has the denominator which, according to the rule, the second ought to have, the second that which the third ought to have, and so on. In order to keep the same rule for this case, we must contrive to make 1 the second figure on the right of the point instead of the first. This may be done by placing a cipher between it and the point, thus, ·0147326. Here the rule holds good, for by that rule this fraction is

$$\frac{0}{10}+\frac{1}{100}+\frac{4}{1000}+\frac{7}{10000}+\frac{3}{100000}+\frac{2}{1000000}+\frac{6}{10000000}$$

which is the same as the preceding line, since $\frac{0}{10}$ is 0, and need not be reckoned.

Similarly, when there are two ciphers more in the denominator than there are figures in the numerator, the rule will be true if we place two ciphers between the point and the numerator. The rule, therefore, stated fully, is this :

To reduce a decimal fraction to a whole number and more simple decimals, or to more simple decimals alone if it do not contain a whole number, mark off by a point as many figures from the numerator as there are ciphers in the denominator. If the numerator have not places enough for this, write as many ciphers before it as it wants places, and put the point before these ciphers. Then, if there be any figures before the point, they make the *whole number* which the fraction contains. The first figure after the point with the denominator 10, the second with the denominator 100, and so on, are the *fractions* of which the first fraction is composed.

135. Decimal fractions are not usually written at full length. It is more convenient to write the numerator only, and to cut off from the numerator as many figures as there are ciphers in the denominator, when that is possible, by a point. When there are more ciphers in the denominator than figures in the numerator, as many ciphers are placed before the numerator as will supply the deficiency, and the point is placed before the ciphers. Thus, ·7 will be used in future to denote $\frac{7}{10}$, ·07 for $\frac{7}{100}$, and so on. The following tables will give the whole of this notation at one view, and will shew its connexion with the decimal notation explained in the first section. You will observe that the numbers on the right of the units' place stand for units *divided* by 10, 100, 1000, &c. while those on the left are units *multiplied* by 10, 100, 1000, &c.

The student is recommended always to write the decimal point in a line with the top of the figures or in the middle, as is done here, and never at the bottom. The reason is, that it is usual in the higher branches of mathematics to use a point placed between two numbers or letters which are multiplied together; thus, 15.16, $a.b$, $\overline{a+b}.\overline{c+d}$ stand for the products of those numbers or letters.

I. $123{\cdot}4$ stands for $\dfrac{1234}{10}$ or $123\dfrac{4}{10}$ or $123+\dfrac{4}{10}$

$$12{\cdot}34 \;\ldots\; \frac{1234}{100} \text{ or } 12\frac{34}{100} \text{ or } 12+\frac{3}{10}+\frac{4}{100}$$

$$1{\cdot}234 \;\ldots\; \frac{1234}{1000} \text{ or } 1\frac{234}{1000} \text{ or } 1+\frac{2}{10}+\frac{3}{100}+\frac{4}{1000}$$

$$\cdot1234 \;\ldots\; \frac{1234}{10000} \;\ldots\; \text{or} \;\ldots\; \frac{1}{10}+\frac{2}{100}+\frac{3}{1000}+\frac{4}{10000}$$

$$\cdot01234 \;\ldots\; \frac{1234}{100000} \;\ldots\; \text{or} \;\ldots\; \frac{1}{100}+\frac{2}{1000}+\frac{3}{10000}+\frac{4}{100000}$$

$$\cdot001234 \;\ldots\; \frac{1234}{1000000} \;\ldots\; \text{or} \;\ldots\; \frac{1}{1000}+\frac{2}{10000}+\frac{3}{100000}+\frac{4}{1000000}$$

II.

$$\cdot01003 \text{ is } \frac{1003}{100000} \text{ or } \frac{1}{100}+\frac{3}{100000}$$

$$\cdot1003 \text{ is } \frac{1003}{10000} \text{ or } \frac{1}{10}+\frac{3}{10000}$$

$$10{\cdot}03 \text{ is } \frac{1003}{100} \text{ or } 10+\frac{3}{100}$$

$$100{\cdot}3 \text{ is } \frac{1003}{10} \text{ or } 100+\frac{3}{10}$$

III.

$$\cdot1283 = \frac{1}{10}+\frac{2}{100}+\frac{8}{1000}+\frac{3}{10000}$$

$$= \cdot1 + \cdot02 + \cdot008 + \cdot0003$$

$$= \cdot1 + \cdot0283 = \cdot12 + \cdot0083$$

$$= \cdot128 + \cdot0003 = \cdot108 + \cdot0203$$

$$= \cdot1003 + \cdot028 = \cdot1203 + \cdot008$$

IV. In 1234·56789 inches the

1 is	1000 inches
2 is	200 . .
3 is	30 . .
4 is	4 . .
5 is	$\frac{5}{10}$ of an inch
6 is	$\frac{6}{100}$ . . .
7 is	$\frac{7}{1000}$ . . .
8 is	$\frac{8}{10000}$ . . .
9 is	$\frac{9}{100000}$ . . .

136. The ciphers on the right hand of the decimal point serve the same purpose as the ciphers in (10). They are not counted as any thing themselves, but serve to shew the place in which the accompanying numbers stand. They might be dispensed with by writing the numbers in ruled columns, as in the first section. They are distinguished from the numbers which accompany them by calling the latter *significant figures.* Thus, ·0003747 is a decimal of seven places with four significant figures, ·346 is a decimal of three places with three significant figures, &c.

137. The value of a decimal is not altered by putting any number of ciphers on its right. Take, for example, ·3 and ·300. The first (135) is $\frac{3}{10}$, and the second $\frac{300}{1000}$, which is made from the first by multiplying both its numerator and denominator by 100, and (108) is the same quantity.

138. To reduce two decimals to a common denominator, put as many ciphers on the right of that which has the smaller number of places as will make the number of places in both fractions the same. Take, for example, ·54 and 4·3297. The first is $\frac{54}{100}$, and the second $\frac{43297}{10000}$. Multiply the numerator and denominator of the first by 100 (108), which reduces it to $\frac{5400}{10000}$, which has the same denominator as $\frac{43297}{10000}$. But $\frac{5400}{10000}$ is ·5400 (135). In whole numbers, the decimal point should be placed at the end: thus, 129 should be written 129·. It is, however, usual to omit the point; but you must recollect that 129 and 129·000 are of the same value, since the first is 129 and the second $\frac{129000}{1000}$.

139. The rules which were given in the last chapter for addition, subtraction, multiplication, and division, apply to all fractions, and therefore to decimal fractions among the rest. But the way of writing decimal fractions, which is explained in this chapter, makes the application of these rules more simple. We proceed to the different cases.

Suppose it required to add 42·634, 45·2806, 2·001, and 54. By (112) these must be reduced to a common denominator, which is done (138) by writing them as follows: 42·6340, 45·2806, 2·0010, and 54·0000. These are decimal fractions, whose numerators are 426340, 452806, 20010, and 540000, and whose common denominator is 10000. By (112) their sum is $\frac{426340+452806+20010+540000}{10000}$, which is $\frac{1439156}{10000}$ or 143·9156. The simplest way of doing this is as follows: write the decimals down under one another, so that the decimal points may fall under one another, thus:

$$\begin{array}{r} 42{\cdot}634 \\ 45{\cdot}2806 \\ 2{\cdot}001 \\ 54\phantom{{\cdot}0000} \\ \hline 143{\cdot}9156 \end{array}$$

Add the different columns together as in common addition, and place the decimal point under the other decimal points.

EXERCISES.

What are 1527+64·732094+2·0013+·00001974;

2276·3+·107+·9+26·3172+56732·001;

and 1·11+7·7+·0039+·00142+·8838?

Answer, 1593·73341374, 59035·6252, 9·69912.

140. Suppose it required to subtract 91·07324 from 137·321. These fractions when reduced to a common denominator are 91·07324 and 137·32100 (138). Their difference is therefore $\frac{13732100-9107324}{100000}$, which is $\frac{4624776}{100000}$ or 46·24776. This may be most simply done as fol-

lows: write the less number under the greater, so that its decimal point may fall under that of the greater, thus:

$$\begin{array}{r} 137{\cdot}321 \\ 91{\cdot}07324 \\ \hline 46{\cdot}24776 \end{array}$$

Subtract the lower from the upper line, and wherever there is a figure in one line and not in the other, proceed as if there were a cipher in the vacant place.

EXERCISES.

What is 12362—274·22107+·5;

9976·2073942—·00143976728;

and 1·2+·03+·004—·0005?

Answer, 12088·27893, 9976·20595443272; and 1·2335.

141. The multiplication of a decimal by 10, 100, 1000, &c., is performed by merely moving the decimal point to the right. Suppose, for example, 13·2079 is to be multiplied by 100. The decimal is $\frac{132079}{10000}$, which multiplied by 100 is (117) $\frac{132079}{100}$, or 1320·79. Again, 1·309× 100000 is $\frac{1309}{1000}\times 100000$, or (116) $\frac{130900000}{1000}$ or 130900. From these and other instances we get the following rule: To multiply a decimal fraction by a decimal number (126), move the decimal point as many places to the right as there are ciphers in the decimal number. When this cannot be done, annex ciphers to the right of the decimal (137) until it can.

142. Suppose it required to multiply 17·036 by 4·27. The first of these decimals is $\frac{17036}{1000}$, and the second $\frac{427}{100}$. By (118) the product of these fractions has for its numerator the product of 17036 and 427, and for its denominator the product of 1000 and 100; therefore this product is $\frac{7274372}{100000}$, or 72·74372. This may be done more shortly by multiplying the two numbers 17036 and 427, and cutting off by the decimal point as many places as there are decimal places both in 17·036 and 4·27, because the product of two decimal numbers will contain as many ciphers as there are ciphers in both.

143. This question now arises: What if there should not be as many figures in the product as there are decimal places in the multiplier and multiplicand together? To see what must be done in this case, multiply ·172 by ·101, or $\frac{172}{1000}$ by $\frac{101}{1000}$. The product of these two is $\frac{17372}{1000000}$, or ·017372 (135). Therefore, when the number of places in the product is not sufficient to allow the rule of the last article to be followed, as many ciphers must be placed at the beginning as will make up the deficiency.

ADDITIONAL EXAMPLES.

·001×·01 is ·00001

56×·0001 is ·0056.

EXERCISES.

Shew that

3·002×3·002 = 3×3+2×3×·002+·002×·002

11·5609×5·3191 = 8·44×8·44−3·1209×3·1209

8·217×10·001 = 8×10+8×·001+10×·217+·001×·217.

Fraction.	Square.	Cube.
82·92	6875·7264	570135·233088
·0173	·00029929	·000005177717
1·43	2·0449	2·924207
·009	·000081	·000000729

15·625× 64 = 1000 ·15625× ·64 = ·1

1·5625× ·64 = 1 1562·5×·064 = 100

·015625×·0064 = ·0001 15625000×·064 = 1000000

144. The division of a decimal by a decimal number, such as 10, 100, 1000, &c., is performed by moving the decimal point as many places to the left as there are ciphers in the decimal number. If there are not places enough in the dividend to allow of this, annex ciphers to the beginning of it until there are. For example, divide 1734·229 by 1000: the decimal fraction is $\frac{1734229}{1000}$, which divided by 1000 (123) is $\frac{1734229}{1000000}$, or 1·734229. If, in the same way, 1·2106 be divided by 10000, the result is ·00012106.

145. Before proceeding to shorten the rule for the division of one

decimal fraction by another, it will be necessary to resume what was said in (128) upon the reduction of any fraction to a decimal fraction. It was there shewn that $\frac{7}{16}$ is the same fraction as $\frac{4375}{10000}$ or ·4375. As another example, convert $\frac{3}{128}$ into a decimal fraction. Follow the same process as in (128), thus:

```
128)300000000000(234375
     256
     ---
      440
      384
      ---
       560
       512
       ---
        480
        384
        ---
         960
         896
         ---
          640
          640
          ---
            0
```

Since 7 ciphers are used, it appears that 30000000 is the first of the series 30, 300, &c., which is divisible by 128; and therefore $\frac{3}{128}$, or, which is the same thing (108), $\frac{30000000}{1280000000}$ is equal to $\frac{234375}{10000000}$ or ·0234375 (135).

From these examples the rule for reducing a fraction to a decimal is: Annex ciphers to the numerator; divide by the denominator, and annex a cipher to each remainder after the figures of the numerator are all used, proceeding exactly as if the numerator had an unlimited number of ciphers annexed to it, and was to be divided by the denominator. Continue this process until there is no remainder, and observe how many ciphers have been used. Place the decimal point in the quotient so as to cut off as many figures as you have used ciphers; and if there be not figures enough for this, annex ciphers to the beginning until there are places enough.

146. From what was shewn in (129), it appears that it is not every fraction which can be reduced to a decimal fraction. It was there shewn, however, that there is no fraction to which we may not find a decimal fraction as near as we please. Thus, $\frac{1}{10}$, $\frac{14}{100}$, $\frac{142}{1000}$, $\frac{1428}{10000}$, $\frac{14285}{100000}$, &c., or ·1, ·14, ·142, ·1428, ·14285, were shewn to be fractions which approach nearer and nearer to $\frac{1}{7}$. To find either of these fractions, the rule is the same as that in the last article, with this exception,

that, I. instead of stopping when there is no remainder, which never happens, stop at any part of the process, and make as many decimal places in the quotient as are equal in number to the number of ciphers which have been used, annexing ciphers to the beginning when this cannot be done, as before. II. Instead of obtaining a fraction which is exactly equal to the fraction from which we set out, we get a fraction which is very near to it, and may get one still nearer, by using more of the quotient. Thus, ·1428 is very near to $\frac{1}{7}$, but not so near as ·142857; nor is this last, in its turn, so near as ·142857142857, &c.

147. If there should be ciphers in the numerator of a fraction, these must not be reckoned with the number of ciphers which are necessary in order to follow the rule for changing it into a decimal fraction. Take, for example, $\frac{100}{125}$; annex ciphers to the numerator, and divide by the denominator. It appears that 1000 is divisible by 125, and that the quotient is 8. One cipher only has been annexed to the numerator, and therefore 100 divided by 125 is ·8. Had the fraction been $\frac{1}{125}$, since 1000 divided by 125 gives 8, and three ciphers would have been annexed to the numerator, the fraction would have been ·008.

148. Suppose that the given fraction has ciphers at the right of its denominator; for example, $\frac{31}{2500}$. The annexing a cipher to the numerator is the same thing as taking one away from the denominator; for, (108) $\frac{310}{2500}$ is the same thing as $\frac{31}{250}$, and $\frac{310}{250}$ as $\frac{31}{25}$. The rule, therefore, is in this case: Take away the ciphers from the denominator; annex cyphers to the numerator; proceed as before; and in counting how many cyphers have been used, reckon not only the cyphers which have been annexed to the numerator, but also those which have been taken away from the denominator.

EXERCISES.

Reduce the following fractions to decimal fractions:

$$\frac{1}{800}, \frac{36}{1250}, \frac{297}{64}, \text{ and } \frac{1}{128}.$$

Answer, ·00125, ·0288, 4·640625, and ·0078125.

Find decimals of 6 places very near to the following fractions:

$\frac{27}{49}$, $\frac{156}{33}$, $\frac{22}{37000}$, $\frac{194}{13}$, $\frac{2637}{9907}$, $\frac{1}{2908}$, $\frac{1}{466}$, and $\frac{3}{277}$

Answer, ·551020, 4·727272, ·000594, 14·923076, ·266175, ·000343, ·002145, and ·010830.

149. From (121) it appears, that if two fractions have the same denominator, the first may be divided by the second by dividing the numerator of the first by the numerator of the second. Suppose it required to divide 17·762 by 6·25. These fractions (138), when reduced to a common denominator, are 17·762 and 6·250, or $\frac{17762}{1000}$ and $\frac{6250}{1000}$. Their quotient is therefore $\frac{17762}{6250}$, which must now be reduced to a decimal fraction by the last rule. The process at full length is as follows: Leave out the cipher in the denominator, and annex ciphers to the numerator, or, which will do as well, to the remainders, when it becomes necessary, and divide as in (145).

```
625)17762(284192
    1250
    ----
     5262
     5000
     ----
      2620
      2500
      ----
       1200
        625
       ----
        5750
        5625
        ----
         1250
         1250
         ----
            0
```

Here four ciphers have been annexed to the numerator, and one has been taken from the denominator. Make five decimal places in the quotient, which then becomes 2·84192, and this is the quotient of 17·762 divided by 6·25.

150. The rule for division of one decimal by another is as follows: Equalise the number of decimal places in the dividend and divisor, by annexing ciphers to that which has fewest places. Then, further, annex as many ciphers to the dividend * as it is required to have decimal places, throw away the decimal point, and operate as in common division. Make the required number of decimal places in the quotient.

Thus, to divide 6·7173 by ·014 to three decimal places, I first write 6·7173 and ·0140, with four places in each. Having to provide for three decimal places, I should annex three ciphers to 6·7173; but, observing

* Or remove ciphers from the divisor; or make up the number of ciphers partly by removing from the divisor and annexing to the dividend, if there be not a sufficient number in the divisor.

that the divisor ·0140 has one cipher, I strike that one out and annex two ciphers to 6·7173. Throwing away the decimal points, then divide 6717300 by 014 or 14 in the usual way, which gives the quotient 479807 and the remainder 2. Hence 479·807 is the answer.

The common rule is: Let the quotient contain as many decimal places as there are decimal places in the dividend more than in the divisor. But this rule becomes inoperative except when there are more decimals in the dividend than in the divisor, and a number of ciphers must be annexed to the former. The rule in the text amounts to the same thing, and provides for an assigned number of decimal places. But the student is recommended to make himself familiar with the rule of the *characteristic* given in the Appendix, and also to accustom himself to *reason out* the place of the decimal point. Thus, it should be visible, that 26·119÷7·2436 has one figure before the decimal point, and that 26·119÷724·36 has one cipher after it, preceding all significant figures.

Or the following rule may be used: Expunge the decimal point of the divisor, and move that of the dividend as many places to the right as there were places in the divisor, using ciphers if necessary. Then proceed as in common division, making one decimal place in the quotient for every decimal place of the final dividend which is used. Thus 17·314 divided by 61·2 is 173·14 divided by 612, and the decimal point must precede the first figure of the quotient. But 17·314 divided by 6617·5 is 173·14 by 66175; and since three decimal places of 173·14000 ... must be used before a quotient figure can be found, that quotient figure is the third decimal place, or the quotient is ·002 .....

EXAMPLES.

$$\frac{3{\cdot}1}{{\cdot}0025} = 1240, \quad \frac{{\cdot}00062}{{\cdot}64} = {\cdot}00096875$$

EXERCISES.

Shew that $\frac{15{\cdot}006\times15{\cdot}006-{\cdot}004\times{\cdot}004}{15{\cdot}01} = 15{\cdot}002$, and that $\frac{{\cdot}01\times{\cdot}01\times{\cdot}01+2{\cdot}9\times2{\cdot}9\times2{\cdot}9}{2{\cdot}91} = 2{\cdot}9\times2{\cdot}9-2{\cdot}9\times{\cdot}01+{\cdot}01\times{\cdot}01$

What are $\frac{1}{3{\cdot}14159}$, $\frac{1}{2{\cdot}7182818}$, and $\frac{365}{{\cdot}18349}$, as far as 6 places of decimals?—*Answer*, ·318310, ·367879, and 1989·209221.

Calculate 10 terms of each of the following series, as far as 5 places of decimals.

$$1+\frac{1}{2}+\frac{1}{2\times3}+\frac{1}{2\times3\times4}+\frac{1}{2\times3\times4\times5}+\&c. = 1{\cdot}71824.$$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\&c. = 2{\cdot}92895.$$

$$\frac{80}{81}+\frac{81}{82}+\frac{82}{83}+\frac{83}{84}+\frac{84}{85}+\&c. = 9{\cdot}88286.$$

151. We now enter upon methods by which unnecessary trouble is saved in the computation of decimal quantities. And first, suppose a number of miles has been measured, and found to be 17·846217 miles. If you were asked how many miles there are in this distance, and a rough answer were required which should give miles only, and not parts of miles, you would probably say 17. But this, though the number of whole miles contained in the distance, is not the nearest number of miles; for, since the distance is more than 17 miles and 8 tenths, and therefore more than 17 miles and a half, it is nearer the truth to say, it is 18 miles. This, though too great, is not so much too great as the other was too little, and the error is not so great as half a mile. Again, if the same were required within a tenth of a mile, the correct answer is 17·8; for though this is too little by ·046217, yet it is not so much too little as 17·9 is too great; and the error is less than half a tenth, or $\frac{1}{20}$. Again, the same distance, within a hundredth of a mile, is more correctly 17·85 than 17·84, since the last is too little by ·006217, which is greater than the half of ·01; and therefore 17·84+·01 is nearer the truth than 17·84. Hence this general rule: When a certain number of the decimals given is sufficiently accurate for the purpose, strike off the rest from the right hand, observing, if the first figure struck off be equal to or greater than 5, to increase the last remaining figure by 1.

The following are examples of a decimal abbreviated by one place at a time.

3·14159, 3·1416, 3·142, 3·14, 3·1, 3·0

2·7182818, 2·718282, 2·71828, 2·7183, 2·718, 2·72, 2·7, 3·0

1·9919, 1·992, 1·99, 2·00, 2·0

152. In multiplication and division it is useless to retain more places of decimals in the result than were certainly correct in the multiplier, &c., which gave that result. Suppose, for example, that 9·98 and 8·96 are distances in inches which have been measured correctly to two places of decimals, that is, within half a hundredth of an inch each way. The real value of that which we call 9·98 may be any where between 9·975 and 9·985, and that of 8·96 may be any where between 8·955 and 8·965. The product, therefore, of the numbers which represent the correct distances will lie between 9·975×8·955 and 9·985×8·965, that is, taking three decimal places in the products, between 89·326 and 89·516. The product of the actual numbers given is 89·4208. It appears, then, that in this case no more than the whole number 89 can be depended upon in the product, or, at most, the first place of decimals. The reason is, that the error made in measuring 8·96, though only in the third place of decimals, is in the multiplication increased at least 9·975, or nearly 10 times; and therefore affects the second place. The following simple rule will enable us to judge how far a product is to be depended upon. Let a be the multiplier, and b the multiplicand; if these be true only to the first decimal place, the product is within $\frac{a+b}{20}$* of the truth; if to two decimal places, within $\frac{a+b}{200}$; if to three, within $\frac{a+b}{2000}$; and so on. Thus, in the above example, we have 9·98 and 8·96, which are true to two decimal places: their sum divided by 200 is ·0947, and their product is 89·4208, which is therefore within ·0947 of the truth. If, in fact, we increase and diminish 89·4208 by ·0947, we get 89·5155 and 89·3261, which are very nearly the limits found within which the product must lie. We see, then, that we cannot in this case depend upon the first place of decimals, as (151) an error of ·05 cannot exist if this place be correct; and here is a possible error of ·09 and upwards. It is hardly necessary to say, that if the numbers given be exact, their product

* These are not quite correct, but sufficiently so for every practical purpose.

is exact also, and that this article applies where the numbers given are correct only to a certain number of decimal places. The rule is: Take half the sum of the multiplier and multiplicand, remove the decimal point as many places to the left as there are correct places of decimals in either the multiplier or multiplicand; the result is the quantity within which the product can be depended upon. In division, the rule is: Proceed as in the last rule, putting the dividend and divisor in place of the multiplier and multiplicand, and divide by the *square* of the divisor; the quotient will be the quantity within which the division of the first dividend and divisor may be depended upon. Thus, if 17·324 be divided by 53·809, both being correct to the third place, their half sum will be 35·566, which, by the last rule, is made ·035566, and is to be divided by the square of 53·809, or, which will do as well for our purpose, the square of 50, or 2500. The result is something less than ·00002, so that the quotient of 17·324 and 53·809 can be depended on to four places of decimals.

153. It is required to multiply two decimal fractions together, so as to retain in the product only a given number of decimal places, and dispense with the trouble of finding the rest. First, it is evident that we may write the figures of any multiplier in a contrary order (for example, 4321 instead of 1234), provided that in the operation we move each line one place to the right instead of to the left, as in the following example:

```
   2221              2221
   1234              4321
   ----              ----
   8884              2221
  6663                4442
 4442                  6663
2221                    8884
-------           -------
2740714           2740714
```

Suppose now we wish to multiply 348·8414 by 51·30742, reserving only four decimal places in the product. If we reverse the multiplier, and proceed in the manner just pointed out, we have the following:

```
        3488414
        2470315 |
     -----------|
      .17442070 |
        3488414 |
       1046524|2
         24418|898
          1395|3656
            69|76828
   -----------|--------
   17898·1522|23188
```

Cut off, by a vertical line, the first four places of decimals, and the columns which produced them. It is plain that in forming our abbreviated rule, we have to consider only, I. all that is on the left of the vertical line; II. all that is carried from the first column on the right of the line. On looking at the first column to the left of the line, we see 4, 4, 8, 5, 9, of which the first 4 comes from $4\times1'$,* the second 4 from $1\times3'$, the 8 from $8\times7'$, the 5 from $8\times4'$, and the 9 from $4\times2'$. If, then, we arrange the multiplicand and the reversed multiplier thus,

3488414
2470315

each figure of the multiplier is placed under the first figure of the multiplicand which is used with it in forming the first *four* places of decimals. And here observe, that the units' figure in the multiplier 51·30742, viz. 1, comes under 4, the *fourth* decimal place in the multiplicand. If there had been no carrying from the right of the vertical line, the rule would have been: Reverse the multiplier, and place it under the multiplicand, so that the figure which was the units' figure in the multiplier may stand under the last place of decimals in the multiplicand which is to be preserved; place ciphers over those figures of the multiplier which have none of the multiplicand above them, if there be any: proceed to multiply in the usual way, but begin each figure of the multiplier with the figure of the multiplicand which comes above it, taking no account of those on the right: place the first figures of all the lines under one another. To correct this rule, so as to allow for what is carried from the right of the vertical line, observe that this consists of two parts, 1st, what is carried directly in the formation of the different lines, and 2dly, what is carried from the addition of the first column on the right. The first of these may be taken into account by beginning each figure of the multiplier with the one which comes

* The 1′ here means that the 1 is in the multiplier.

on its right in the multiplicand, and carrying the tens to the next figure as usual, but without writing down the units. But both may be allowed for at once, with sufficient correctness, on the principle of (151), by carrying 1 from 5 up to 15, 2 from 15 up to 25, &c.; that is, by carrying the nearest ten. Thus, for 37, 4 would be carried, 37 being nearer to 40 than to 30. This will not always give the last place quite correctly, but the error may be avoided by setting out so as to keep one more place of decimals in the product than is absolutely required to be correct. The rule, then, is as follows:

154. To multiply two decimals together, retaining only *n* decimal places.

I. Reverse the multiplier, strike out the decimal points, and place the multiplier under the multiplicand, so that what was its units' figure shall fall under the n^{th} decimal place of the multiplicand, placing ciphers, if necessary, so that every place of the multiplier shall have a figure or cipher above it.

II. Proceed to multiply as usual, beginning each figure of the multiplier with the one which is in the place to its right in the multiplicand: do not set down this first figure, but carry its *nearest* ten to the next, and proceed.

III. Place the first figures of all the lines under one another; add as usual; and mark off *n* places from the right for decimals.

It is required to multiply 136·4072 by 1·30609, retaining 7 decimal places.

```
1364072000
    906031
----------
1364072000
 409221600
   8184432
    122766
----------
178·1600798
```

In the following examples the first two lines are the multiplicand and multiplier; and the number of decimals to be retained will be seen from the results.

```
   ·4471618          33·166248          3·4641016
  3·7719214           1·4142136      1732·508
  =========          =========       ===========
  377192 14          033166248         346410160
  8161744            63124141            8052371
  ---------          ---------         ---------
  15087686            3316625          346410160
   1508768            1326650          242487112
    264034              33166           10392305
      3772              13266             692820
      2263                663             173205
        38                 33               2771
        30                 10          ---------
  ---------                 2         6001·58373
  1·6866591          ---------
                     46·90415
```

Exercises may be got from article (143).

155. With regard to division, take any two numbers, for example, 16·80437921 and 3·142, and divide the first by the second, as far as any required number of decimal places, for example, five. This gives the following:

```
                3·142)16·80437921(5·34830
                      15 710
                      ------
                       1 0943
                        9426
                        -----
                        15177
        (A)             12568
       -----            -----
       2609              2609|9
       2514              2513|6
       ----              ----|--
         95                96|32
         94                94|26
         --                --|--
          1                 2|06
```

Now cut off by a vertical line, as in (153), all the figures which come on the right of the first figure 2, in the last remainder 2061. As in multiplication, we may obtain all that is on the left of the vertical line by an abbreviated method, as represented at (A). After what has been said on multiplication, it is useless to go further into the detail; the following rule will be sufficient: To divide one decimal by another, retaining only n places: Proceed one step in the ordinary division, and determine, by (150), in what place is the quotient so obtained; proceed in the ordinary way, until the number of figures remaining to be found in the quotient is less than the number of figures in the divisor: if this should be already the case, proceed no further in the ordinary way. Instead of annexing a figure or cipher to the remainder, cut off a figure from the divisor, and proceed one step with this curtailed divisor as usual, remembering, however, in multiplying this divisor, to carry the *nearest ten*, as in (154), from the figure which was struck off; repeat this, striking off another figure of the divisor, and so on, until no figures are left. Since we know from the beginning in what place the first figure of the quotient is, and also how many decimals are required, we can tell from the beginning how many figures there will be in the whole quotient. If the divisor contain more figures than the quotient, it will be unnecessary to use them: and they may be rejected, the rest being corrected as in (151): if there be ciphers at the beginning of the divisor, if it be, for example, ·003178, since this is $\frac{\cdot 3178}{100}$, divide by ·3178 in the usual way, and afterwards multiply the quotient by 100, or remove the decimal point two places to the right. If, therefore, six decimals be required, eight places must be taken in dividing by ·3178, for an obvious reason. In finding the last figure of the quotient, the nearest should be taken, as in the second of the subjoined examples.

Places required,	2	8
Divisor,	·41432	3·1415927
Dividend,	673·1489	2·71828180
	414 32	2·51327416
	258 828	20500764
	248 592	18849556
	10 237*	1651208
	8 286	1570796
	1 951	80412
	1 657	62832
	294	17580
	290	15708
	4	1872
	4	1571
	0	301
		283
		18
		19
Quotient,	1624·71	·86525596

Examples may be obtained from (143) and (150).

SECTION VII.

ON THE EXTRACTION OF THE SQUARE ROOT.

156. We have already remarked (66), that a number multiplied by itself produces what is called the *square* of that number. Thus, 169, or 13×13, is the square of 13. Conversely, 13 is called the *square root* of 169, and 5 is the square root of 25; and any number is the square root of another, which when multiplied by itself will produce that other. The square root is signified by the sign $\surd$ or $\sqrt{}$; thus, $\surd 25$ means the square root of 25, or 5; $\sqrt{16+9}$ means the square root of 16+9, and is 5, and must not be confounded with $\surd 16+\surd 9$, which is 4+3, or 7.

* This is written 7 instead of 6, because the figure which is abandoned in the dividend is 9 (151).

157. The following equations are evident from the definition:

$$\sqrt{a} \times \sqrt{a} = a$$

$$\sqrt{aa} = a$$

$$\sqrt{ab} \times \sqrt{ab} = ab$$

$$(\sqrt{a} \times \sqrt{b}) \times (\sqrt{a} \times \sqrt{b}) = \sqrt{a} \times \sqrt{a} \times \sqrt{b} \times \sqrt{b} = a\,b$$

whence $$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

158. It does not follow that a number has a square root because it has a square; thus, though 5 can be multiplied by itself, there is no number which multiplied by itself will produce 5. It is proved in algebra, that no fraction* multiplied by itself can produce a whole number, which may be found true in any number of instances; therefore 5 has neither a whole nor a fractional square root; that is, it has no square root at all. Nevertheless, there are methods of finding fractions whose squares shall be as *near* to 5 as we please, though not exactly equal to it. One of these methods gives $\frac{15127}{6765}$, whose square, viz. $\frac{15127}{6765} \times \frac{15127}{6765}$ or $\frac{228826129}{45765225}$, differs from 5 by only $\frac{4}{45765225}$, which is less than ·0000001: hence we are enabled to use $\sqrt{5}$ in arithmetical and algebraical reasoning: but when we come to the practice of any problem, we must substitute for $\sqrt{5}$ one of the fractions whose square is nearly 5, and on the degree of accuracy we want, depends what fraction is to be used. For some purposes, $\frac{123}{55}$ may be sufficient, as its square only differs from 5 by $\frac{4}{3025}$; for others, the fraction first given might be necessary, or one whose square is even nearer to 5. We proceed to shew how to find the square root of a number, when it has one, and from thence how to find fractions whose squares shall be as near as we please to the number, when it has not. We premise, what is sufficiently evident, that of two numbers, the greater has the greater square; and that if one number lie between two others, its square lies between the squares of those others.

159. Let x be a number consisting of any number of parts, for example, four, viz. a, b, c, and d; that is, let

* Meaning, of course, a really fractional number, such as $\frac{7}{8}$ or $\frac{15}{11}$, not one which, though fractional in form, is whole in reality, such as $\frac{10}{5}$ or $\frac{27}{3}$.

$$x = a+b+c+d$$

The square of this number, found as in (68), will be

$$\begin{aligned} &aa+2a(b+c+d)\\ &+bb+2b(c+d)\\ &+cc+2cd\\ &+dd \end{aligned}$$

The rule there found for squaring a number consisting of parts was. Square each part, and multiply all that come after by twice that part, the sum of all the results so obtained will be the square of the whole number. In the expression above obtained, instead of multiplying $2a$ by *each* of the succeeding parts, b, c, and d, and adding the results, we multiplied $2a$ by the *sum of all* the succeeding parts, which (52) is the same thing; and as the parts, however disposed, make up the number, we may reverse their order, putting the last first, &c.; and the rule for squaring will be: Square each part, and multiply all that come before by twice that part. Hence a reverse rule for extracting the square root presents itself with more than usual simplicity. It is: To extract the square root of a number N, choose a number A, and see if N will bear the subtraction of the square of A; if so, take the remainder, choose a second number B, and see if the remainder will bear the subtraction of the square of B, and twice B multiplied by the preceding part A: if it will, there is a second remainder. Choose a third number C, and see if the second remainder will bear the subtraction of the square of C, and twice C multiplied by A+B: go on in this way either until there is no remainder, or else until the remainder will not bear the subtraction arising from any new part, even though that part were the least number, which is 1. In the first case, the square root is the sum of A, B, C, &c.; in the second, there is no square root.

160. For example, I wish to know if 2025 has a square root. I choose 20 as the first part, and find that 400, the square of 20, subtracted from 2025, gives 1625, the first remainder. I again choose 20, whose square, together with twice itself, multiplied by the preceding part, is $20\times20+2\times20\times20$, or 1200; which subtracted from 1625, the

first remainder, gives 425, the second remainder. I choose 7 for the third part, which appears to be too great, since 7×7, increased by 2×7 multiplied by the sum of the preceding parts 20+20, gives 609, which is more than 425. I therefore choose 5, which closes the process, since 5×5, together with 2×5 multiplied by 20+20, gives exactly 425. The square root of 2025 is therefore 20+20+5, or 45, which will be found, by trial, to be correct; since 45×45 = 2025. Again, I ask if 13340 has, or has not, a square root. Let 100 be the first part, whose square is 10000, and the first remainder is 3340. Let 10 be the second part. Here 10×10+2×10×100 is 2100, and the second remainder, or 3340−2100, is 1240. Let 5 be the third part; then 5×5+2×5×(100+10) is 1125, which, subtracted from 1240, leaves 115. There is, then, no square root; for a single additional unit will give a subtraction of 1×1+2×1×(100+10+5), or 231, which is greater than 115. But if the number proposed had been less by 115, each of the remainders would have been 115 less, and the last remainder would have been nothing. Therefore 13340 − 115, or 13225, has the square root 100+10+5, or 115; and the answer is, that 13340 has no square root, and that 13225 is the next number below it which has one, namely, 115.

161. It only remains to put the rule in such a shape as will guide us to those parts which it is most convenient to choose. It is evident (57) that any number which terminates with ciphers, as 4000, has double the number of ciphers in its square. Thus, 4000×4000 = 16000000; therefore, any square number,* as 49, with an even number of ciphers annexed, as 490000, is a square number. The root† of 490000 is 700. This being premised, take any number, for example, 76176; setting out from the right hand towards the left, cut off two figures; then two more, and so on, until one or two figures only are left: thus, 7,61,76. This number is greater than 7,00,00, of which the first figure is not a square number, the nearest square below it being 4. Hence, 4,00,00 is the nearest square number below 7,00,00, which

* By square number I mean, a number which has a square root. Thus, 25 is a square number, but 26 is not.

† The term 'root' is frequently used as an abbreviation of square root.

has four ciphers, and its square root is 200. Let this be the first part chosen: its square subtracted from 76176 leaves 36176, the first remainder; and it is evident that we have obtained the highest number of the highest denomination which is to be found in the square root of 76176; for 300 is too great, its square, 9,00,00, being greater than 76176: and any denomination higher than hundreds has a square still greater. It remains, then, to choose a second part, as in the examples of (160), with the remainder 36176. This part cannot be as great as 100, by what has just been said; its highest denomination is therefore a number of tens. Let N stand for a number of tens, which is one of the simple numbers 1, 2, 3, &c.; that is, let the new part be 10N, whose square is 10N×10N, or 100NN, and whose double multiplied by the former part is 20N×200, or 4000N; the two together are 4000N+100NN. Now, N must be so taken that this may not be greater than 36176: still more 4000N must not be greater than 36176. We may therefore try, for N, the number of times which 36176 contains 4000, or that which 36 contains 4. The remark in (80) applies here. Let us try 9 tens or 90. Then, 2×90×200+90×90, or 44100, is to be subtracted, which is too great, since the whole remainder is 36176. We then try 8 tens or 80, which gives 2×80×200+80×80, or 38400, which is likewise too great. On trying 7 tens, or 70, we find 2×70×200+70×70, or 32900, which subtracted from 36176 gives 3276, the second remainder. The rest of the square root can only be units. As before, let N be this number of units. Then, the sum of the preceding parts being 200+70, or 270, the number to be subtracted is 270×2N+NN, or 540N+NN. Hence, as before, 540N must be less than 3276, or N must not be greater than the number of times which 3276 contains 540, or (80) which 327 contains 54. We therefore try if 6 will do, which gives 2×6×270+6×6, or 3276, to be subtracted. This being exactly the second remainder, the third remainder is nothing, and the process is finished. The square root required is therefore 200+70+6, or 276.

The process of forming the numbers to be subtracted may be shortened thus. Let A be the sum of the parts already found, and N a new part: there must then be subtracted 2AN+NN, or (54) 2A+N multiplied by N. The rule, therefore, for forming it is: Double the sum of all the preceding parts, add the new part, and multiply the result by the new part.

162. The process of the last article is as follows:

```
        7,61,76(200                 7,61,76(276
        4 00 00   70                4
        -------    6                --
400)   3,61,76                   47)361
 70)   3 29 00                      329
       -------                      ---
400)     32 76                  546)3276
140)     32 76                      3276
  6)     -----                      ----
             0                         0
```

In the first of these, the numbers are written at length, as we found them; in the second, as in (79), unnecessary ciphers are struck off, and the periods 61, 76, are not brought down, until, by the continuance of the process, they cease to have ciphers under them. The following is another example, to which the reasoning of the last article may be applied.

```
          34,86,78,44,01(50000         34,86,78,44,01(59049
          25 00 00 00 00   9000        25
          --------------     40        --
100000)   9 86 78 44 01       9     109) 986
  9000)   9 81 00 00 00                  981
          -------------                  ----
100000)      5 78 44 01           11804) 57844
 18000)      4 72 16 00                  47216
    49)      ----------                  -----
100000)      1 06 28 01          118089)1062801
 18000)      1 06 28 01                 1062801
    80)      ----------                 -------
     9)               0                       0
```

163. The rule is as follows: To extract the square root of a number;—

I. Beginning from the right hand, cut off periods of two figures each, until not more than two are left.

II. Find the root of the nearest square number next below the number in the first period. This root is the first figure of the required root; subtract its square from the first period, which gives the first remainder.

III. Annex the second period to the right of the remainder, which gives the first dividend.

IV. Double the first figure of the root; see how often this is contained in the number made by cutting one figure from the right of the first dividend, attending to IX., if necessary; use the quotient as the second figure of the root; annex it to the right of the double of the first figure, and call this the first divisor.

V. Multiply the first divisor by the second figure of the root; if the product be greater than the first dividend, use a lower number for the second figure of the root, and for the last figure of the divisor, until the multiplication just mentioned gives the product less than the first dividend; subtract this from the first dividend, which gives the second remainder.

VI. Annex the third period to the second remainder, which gives the second dividend.

VII. Double the first two figures of the root;* see how often the result is contained in the number made by cutting one figure from the right of the second dividend; use the quotient as the third figure of the root; annex it to the right of the double of the first two figures, and call this the second divisor.

VIII. Get a new remainder, as in V., and repeat the process until all the periods are exhausted; if there be then no remainder, the square root is found; if there be a remainder, the proposed number has no square root, and the number found as its square root is the square root of the proposed number diminished by the remainder.

IX. When it happens that the double of the figures of the root is not contained at all in all the dividend except the last figure, or when, being contained once, 1 is found to give more than the dividend, put a cipher in the square root and in the divisor, and bring down the next period; should the same thing still happen, put another cipher in the root and divisor, and bring down another period; and so on.

* Or, more simply, add the second figure of the root to the first divisor.

EXERCISES.

Numbers proposed.	Square roots.
73441	271
2992900	1730
6414247921	80089
903687890625	950625
42420747482776576	205962976
13422659310152401	115856201

164. Since the square of a fraction is obtained by squaring the numerator and the denominator, the square root of a fraction is found by taking the square root of both. Thus, the square root of $\frac{25}{64}$ is $\frac{5}{8}$, since 5×5 is 25, and 8×8 is 64. If the numerator or denominator, or both, be not square numbers, it does not therefore follow that the fraction has no square root; for it may happen that multiplication or division by the same number may convert both the numerator and denominator into square numbers (108). Thus, $\frac{27}{48}$, which appears at first to have no square root, has one in reality, since it is the same as $\frac{9}{16}$, whose square root is $\frac{3}{4}$.

165. We now proceed from (158), where it was stated that any number or fraction being given, a second may be found, whose square is as near to the first as we please. Thus, though we cannot solve the problem, "Find a fraction whose square is 2," we can solve the following, "Find a fraction whose square shall not differ from 2 by so much as ·00000001." Instead of this last, a still smaller fraction may be substituted; in fact, any one however small: and in this process we are said to approximate to the square root of 2. This can be done to any extent, as follows: Suppose we wish to find the square root of 2 within $\frac{1}{57}$ of the truth; by which I mean, to find a fraction $\frac{a}{b}$ whose square is less than 2, but such that the square of $\frac{a}{b}+\frac{1}{57}$ is greater than 2. Multiply the numerator and denominator of $\frac{2}{1}$ by the square of 57, or 3249, which gives $\frac{6498}{3249}$. On attempting to extract the square root of the numerator, I find (163) that there is a remainder 98, and that the square number next below 6498 is 6400, whose root is 80. Hence, the square of 80 is less than 6498, while that of 81 is greater. The

square root of the denominator is of course 57. Hence, the square of $\frac{80}{57}$ is less than $\frac{6498}{3249}$, or 2, while that of $\frac{81}{57}$ is greater, and these two fractions only differ by $\frac{1}{57}$; which was required to be done.

166. In practice, it is usual to find the square root true to a certain number of places of decimals. Thus, 1·4142 is the square root of 2 true to four places of decimals, since the square of 1·4142, or 1·99996164, is less than 2, while an increase of only 1 in the fourth decimal place, giving 1·4143, gives the square 2·00024449, which is greater than 2. To take a more general case: Suppose it required to find the square root of 1·637 true to four places of decimals. The fraction is $\frac{1637}{1000}$, whose square root is to be found within ·0001, or $\frac{1}{10000}$. Annex ciphers to the numerator and denominator, until the denominator becomes the square of $\frac{1}{10000}$, which gives $\frac{163700000}{100000000}$: extract the square root of the numerator, as in (163), which shews that the square number nearest to it is 163700000 − 13564, whose root is 12794. Hence, $\frac{12794}{10000}$, or 1·2794, gives a square less than 1·637, while 1·2795 gives a square greater. In fact, these two squares are 1·63686436 and 1·63712025.

167. The rule, then, for extracting the square root of a number or decimal to any number of places is: Annex ciphers until there are twice as many places following the units' place as there are to be decimal places in the root; extract the nearest square root of this number, and mark off the given number of decimals. Or, more simply: Divide the number into periods, so that the units' figure shall be the last of a period; proceed in the usual way; and if, when decimals follow the units' place, there is one figure on the right, in a period by itself, annex a cipher in bringing down that period, and afterwards let each new period consist of two ciphers. Place the decimal point after that figure in forming which the period containing the units was used.

168. For example, what is the square root of $1\frac{3}{8}$ to five places of decimals? This is (145) 1·375, and the process is the first example over leaf. The second example is the extraction of the root of ·081 to seven places, the first period being 08, from which the cipher is omitted as useless.

```
         1,37,5(1·17260
         1
        ----
    21) 37
        21
        ------
   227)1650
       1589
       -------
  2342) 6100
         4684
         -------
 23446)141600
        140676
        --------
 23452)    92400
```

```
      8,1(·2846049
      4
     ----
   48)410
      384
      ------
  564) 2600
       2256
       -------
 5686) 34400
       34116
       ----------
569204) 2840000
        2276816
        ---------
569208) 56318400
```

```
·00000241367222 1(·001553599
      1
     ----
 25) 141
     125
     -----
 305) 1636
      1525
      ------
 3103) 11172
         9309
        -------
 31065) 186322
        155325
        --------
310709) 3099710
        2796381
        ---------
         30332900
```

169. When more than half the decimals required have been found, the others may be simply found by dividing the dividend by the divisor, as in (155). The extraction of the square root of 12 to ten places, which will be found in the next page, is an example. It must, however, be observed in this process, as in all others where decimals are obtained by approximation, that the last place cannot always be depended upon: on which account it is advisable to carry the process so far, that one or even two more decimals shall be obtained than are absolutely required to be correct.

A

```
12(3·46410161513
   9
64)   300
      256
686)  4400
      4116
6924)  28400
       27696
69281)   70400
         69281
6928201)  11190000
           6928201
69282026)  4261799|00
           4156921|56
692820321)  104877|4400
             69282|0321
6928203225)  35595|407900
             34641|016125
69282032301)   954|39177500
               692|82032301
692820323023)  261|5714519900
               207|8460969069
                53|7253550831
```

B

```
692820323026)537253550831(77545870549
             484974226118
              52279324713
              48497422611
               3781902102
               3464101615
                317800487
                277128129
                 40672358
                 34641016
                  6031342
                  5542562
                   488780
                   484974
                     3806
                     3464
                      342
                      277
                       65
                       62
                        3
```

If from any remainder we cut off the ciphers, and all figures which would come under or on the right of these ciphers, by a vertical line, we find on the left of that line a contracted division, such as those in (155). Thus, after having found the root as far as 3·464101, we have the remainder 4261799, and the divisor 6928202. The figures on the left of the line are nothing more than the contracted division of this remainder by the divisor, with this difference, however, that we have to begin by striking a figure off the divisor, instead of using the whole divisor once, and then striking off the first figure. By this alone we might have doubled our number of decimal places, and got the additional figures 615137, the last 7 being obtained by carrying the contracted division one step further with the remainder 53. We have, then, this rule: When half the number of decimal places have been obtained, instead of annexing two ciphers to the remainder, strike off a figure from what would be the divisor if the process were continued at length. and divide the remainder by this contracted divisor, as in (155).

As an example, let us double the number of decimal places already obtained, which are contained in 3·46410161513. The remainder is 537253550831, the divisor 692820323026, and the process is as in (B). Hence the square root of 12 is,

3·4641016151377545870549;

which is true to the last figure, and a little too great; but the substitution of 8 instead of 9 on the right hand would make it too small.

EXERCISES.

Numbers.	Square roots.
·001728	·0415692194
64·34	8·02122185
8074	89·8554394
10	3·16227766
1·57	1·2529964086141667788495

SECTION VIII.

ON THE PROPORTION OF NUMBERS.

170. When two numbers are named in any problem, it is usually necessary, in some way or other, to compare the two; that is, by considering the two together, to establish some connexion between them, which may be useful in future operations. The first method which suggests itself, and the most simple, is to observe which is the greater, and by how much it differs from the other. The connexion thus established between two numbers may also hold good of two other numbers; for example, 8 differs from 19 by 11, and 100 differs from 111 by the

same number. In this point of view, 8 stands to 19 in the same situation in which 100 stands to 111, the first of both couples differing in the same degree from the second. The four numbers thus noticed, viz.:

8, 19, 100, 111,

are said to be in *arithmetical* proportion.* When four numbers are thus placed, the first and last are called the *extremes*, and the second and third the *means*. It is obvious that $111+8 = 100+19$, that is, the sum of the extremes is equal to the sum of the means. And this is not accidental, arising from the particular numbers we have taken, but must be the case in every arithmetical proportion; for in $111+8$, by (35), any diminution of 111 will not affect the sum, provided a corresponding increase be given to 8; and, by the definition just given, one mean is as much less than 111 as the other is greater than 8.

171. A set or series of numbers is said to be in *continued* arithmetical proportion, or in arithmetical *progression*, when the difference between every two succeeding terms of the series is the same. This is the case in the following series:

1,	2,	3,	4,	5,	&c.
3,	6,	9,	12,	15,	&c.
$1\frac{1}{2}$,	2,	$2\frac{1}{2}$,	3,	$3\frac{1}{2}$,	&c.

The difference between two succeeding terms is called the common difference. In the three series just given, the common differences are, 1, 3, and $\frac{1}{2}$.

172. If a certain number of terms of any arithmetical series be taken, the sum of the first and last terms is the same as that of any other two terms, provided one is as distant from the beginning of the series as the other is from the end. For example, let there be 7 terms, and let them be,

a b c d e f g.

* This is a very incorrect name, since the term 'arithmetical' applies equally to every notion in this book. It is necessary, however, that the pupil should use words in the sense in which they will be used in his succeeding studies.

Then, since, by the nature of the series, b is as much above a as f is below g (170), $a+g = b+f$. Again, since c is as much above b as e is below f (170), $b+f = c+e$. But $a+g = b+f$; therefore $a+g = c+e$, and so on. Again, twice the middle term, or the term equally distant from the beginning and the end (which exists only when the number of terms is odd), is equal to the sum of the first and last terms; for since c is as much below d as e is above it, we have $c+e = d+d = 2d$. But $c+e = a+g$; therefore, $a+g = 2d$. This will give a short rule for finding the sum of any number of terms of an arithmetical series. Let there be 7, viz. those just given. Since $a+g$, $b+f$, and $c+e$, are the same, their sum is three times $(a+g)$, which with d, the middle term, or half $a+g$, is three times and a half $\overline{a+g}$, or the sum of the first and last terms multiplied by $3\frac{1}{2}$, or $\frac{7}{2}$, or half the number of terms. If there had been an even number of terms, for example, six, viz. a, b, c, d, e, and f, we know now that $a+f$, $b+e$, and $c+d$, are the same, whence the sum is three times $\overline{a+f}$, or the sum of the first and last terms multiplied by half the number of terms, as before. The rule, then, is: To sum any number of terms of an arithmetical progression, multiply the sum of the first and last terms by half the number of terms. For example, what are 99 terms of the series 1, 2, 3, &c.? The 99th term is 99, and the sum is $(99+1)\frac{99}{2}$, or $\frac{100\times 99}{2}$, or 4950. The sum of 50 terms of the series $\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2$, &c. is $\left(\frac{1}{3}+\frac{50}{3}\right)\frac{50}{2}$, or 17×25, or 425.

173. The first term being given, and also the common difference and number of terms, the last term may be found by adding to the first term the common difference multiplied by one less than the number of terms. For it is evident that the second term differs from the first by the common difference, the *third* term by *twice*, the *fourth* term by *three* times the common difference; and so on. Or, the passage from the first to the nth term is made by $n-1$ steps, at each of which the common difference is added.

EXERCISES.

Given.					To find.	
Series.				No. of terms.	Last term.	Sum.
4,	$6\frac{1}{2}$,	9,	&c.	33	84	1452
1,	3,	5,	&c.	28	55	784
2,	20,	38,	&c.	100,000	1799984	89999300000

174. The sum being given, the number of terms, and the first term, we can thence find the common difference. Suppose, for example, the first term of a series to be one, the number of terms 100, and the sum 10,000. Since 10,000 was made by multiplying the sum of the first and last terms by $\frac{100}{2}$, if we divide by this, we shall recover the sum of the first and last terms. Now, $\frac{10,000}{1}$ divided by $\frac{100}{2}$ is (122) 200, and the first term being 1, the last term is 199. We have then to pass from 1 to 199, or through 198, by 99 equal steps. Each step is, therefore, $\frac{198}{99}$, or 2, which is the common difference; or the series is 1, 3, 5, &c., up to 199.

Given.			To find.	
Sum.	No. of terms.	First term.	Last term.	Common diff.
1809025	1345	1	2689	2
44	10	3	$\frac{29}{5}$	$\frac{14}{45}$
7075600	1330	4	10636	8

175. We now return to (170), in which we compared two numbers together by their difference. This, however, is not the method of comparison which we employ in common life, as any single familiar instance will shew. For example, we say of A, who has 10 thousand pounds, that he is much richer than B, who has only 3 thousand; but we do not say that C, who has 107 thousand pounds, is much richer than D, who has 100 thousand, though the difference of fortune is the same in both cases, viz. 7 thousand pounds. In comparing numbers we take into our reckoning not only the differences, but the numbers themselves. Thus, if B and D both received 7 thousand pounds, B would receive 233 pounds and a third for every 100 pounds which he had before, while D for every 100 pounds would receive only 7 pounds.

And though, in the view taken in (170), 3 is as near to 10 as 100 is to 107, yet, in the light in which we now regard them, 3 is not so near to 10 as 100 is to 107, for 3 differs from 10 by more than twice itself, while 100 does not differ from 107 by so much as one-fifth of itself. This is expressed in mathematical language by saying, that the *ratio* or *proportion* of 10 to 3 is greater than the *ratio* or *proportion* of 107 to 100. We proceed to define these terms more accurately.

176. When we use the term *part* of a number or fraction in the remainder of this section, we mean, one of the various sets of *equal* parts into which it may be divided, either the half, the third, the fourth, &c.: the term multiple has been already explained (102). By the term *multiple-part* of a number we mean, the abbreviation of the words *multiple of a part*. Thus, 1, 2, 3, 4, and 6, are parts of 12; $\frac{1}{2}$ is also a part of 12, being contained in it 24 times; 12, 24, 36, &c., are multiples of 12; and 8, 9, $\frac{5}{2}$, &c. are multiple parts of 12, being multiples of some of its parts. And when multiple-parts generally are spoken of, the parts themselves are supposed to be included, on the same principle that 12 is counted among the multiples of 12, the multiplier being 1. The multiples themselves are also included in this term; for 24 is also 48 halves, and is therefore among the multiple parts of 12. Each part is also in various ways a multiple-part; for one-fourth is two-eighths, and three-twelfths, &c.

177. Every number or fraction is a multiple-part of every other number or fraction. If, for example, we ask what part 12 is of 7, we see that on dividing 7 into 7 parts, and repeating one of these parts 12 times, we obtain 12; or, on dividing 7 into 14 parts, each of which is one-half, and repeating one of these parts 24 times, we obtain 24 halves, or 12. Hence, 12 is $\frac{12}{7}$, or $\frac{24}{14}$, or $\frac{36}{21}$ of 7; and so on. Generally, when a and b are two whole numbers, $\frac{a}{b}$ expresses the multiple-part which a is of b, and $\frac{b}{a}$ that which b is of a. Again, suppose it required to determine what multiple-part $2\frac{1}{7}$ is of $3\frac{1}{5}$, or $\frac{15}{7}$ of $\frac{16}{5}$. These fractions, reduced to a common denominator, are $\frac{75}{35}$ and $\frac{112}{35}$, of which the second, divided into 112 parts, gives $\frac{1}{35}$, which repeated 75 times

gives $\frac{75}{35}$, the first. Hence, the multiple-part which the first is of the second is $\frac{75}{112}$, which being obtained by the rule given in (121), shews that $\frac{a}{b}$, or a divided by b, according to the notion of division there given, expresses the multiple-part which a is of b in every case.

178. When the first of four numbers is the same multiple-part of the second which the third is of the fourth, the four are said to be *geometrically* proportional*, or simply *proportional*. This is a word in common use; and it remains to shew that our mathematical definition of it, just given, is, in fact, the common notion attached to it. For example, suppose a picture is copied on a smaller scale, so that a line of two inches long in the original is represented by a line of one inch and a half in the copy; we say that the copy is not correct unless all the parts of the original are reduced in the same proportion, namely, that of 2 to $1\frac{1}{2}$. Since, on dividing two inches into 4 parts, and taking 3 of them, we get $1\frac{1}{2}$, the same must be done with all the lines in the original, that is, the length of any line in the copy must be three parts out of four of its length in the original. Again, interest being at 5 per cent, that is, £5 being given for the use of £100, a similar proportion of every other sum would be given; the interest of £70, for example, would be just such a part of £70 as £5 is of £100.

Since, then, the part which a is of b is expressed by the fraction $\frac{a}{b}$, or any other fraction which is equivalent to it, and that which c is of d by $\frac{c}{d}$, it follows, that when a, b, c, and d, are proportional, $\frac{a}{b} = \frac{c}{d}$. This equation will be the foundation of all our reasoning on proportional quantities; and in considering proportionals, it is necessary to observe not only the quantities themselves, but also the order in which they come. Thus, a, b, c, and d, being proportionals, that is, a being the same multiple-part of b which c is of d, it does not follow that a, d, b, and c are proportionals, that is, that a is the same multiple-part of d

* The same remark may be made here as was made in the note on the term arithmetical proportion,' page 101. The word 'geometrical' is, generally speaking, dropped, except when we wish to distinguish between this kind of proportion and that which has been called arithmetical.

which b is of c. It is plain that a is greater than, equal to, or less than b, according as c is greater than, equal to, or less than d.

179. Four numbers, a, b, c, and d, being proportional in the order written, a and d are called the *extremes*, and b and c the *means*, of the proportion. For convenience, we will call the two extremes, or the two means, *similar* terms, and an extreme and a mean, *dissimilar* terms. Thus, a and d are similar, and so are b and c; while a and b, a and c, d and b, d and c, are dissimilar. It is customary to express the proportion by placing dots between the numbers, thus:

$$a : b :: c : d$$

180. Equal numbers will still remain equal when they have been increased, diminished, multiplied, or divided, by equal quantities. This amounts to saying that if $a = b$ and $p = q$, $a+p = b+q$, $a-p = b-q$, $ap = bq$, and $\frac{a}{p} = \frac{b}{q}$. It is also evident, that $a+p-p$, $a-p+p$, $\frac{ap}{p}$, and $\frac{a}{p}\times p$, are all equal to a.

181. The product of the extremes is equal to the product of the means. Let $\frac{a}{b} = \frac{c}{d}$, and multiply these equal numbers by the product bd. Then, $\frac{a}{b}\times bd = \frac{abd}{b}$ (116) $= ad$, and $\frac{c}{d}\times bd = \frac{cbd}{d} = cb$: hence (180), $ad = bc$. Thus, 6, 8, 21, and 28, are proportional, since $\frac{6}{8} = \frac{3}{4} = \frac{3\times 7}{4\times 7} = \frac{21}{28}$ (180); and it appears that $6\times 28 = 8\times 21$, since both products are 168.

182. If the product of two numbers be equal to the product of two others, these numbers are proportional in any order whatever, provided the numbers in the same product are so placed as to be similar terms; that is, if $ab = pq$, we have the following proportions:—

$a : p :: q : b$	$p : a :: b : q$
$a : q :: p : b$	$p : b :: a : q$
$b : p :: q : a$	$q : a :: b : p$
$b : q :: p : a$	$q : b :: a : p$

To prove any one of these, divide both ab and pq by the product of its second and fourth terms; for example, to shew the truth of $a : q :: p : b$, divide both ab and pq by bq. Then, $\frac{ab}{bq} = \frac{a}{q}$, and $\frac{pq}{bq} = \frac{p}{b}$; hence (180),

$\frac{a}{q} = \frac{p}{b}$, or $a : q :: p : b$. The pupil should not fail to prove every one of the eight cases, and to verify them by some simple examples, such as $1 \times 6 = 2 \times 3$, which gives $1 : 2 :: 3 : 6$, $3 : 1 :: 6 : 2$, &c

183. Hence, if four numbers be proportional, they are also proportional in any other order, provided it be such that similar terms still remain similar. For since, when $\frac{a}{b} = \frac{c}{d}$, it follows (181) that $ad = bc$, all the proportions which follow from $ad = bc$, by the last article, follow also from $\frac{a}{b} = \frac{c}{d}$.

184. From (114) it follows that $1 + \frac{a}{b} = \frac{b+a}{b}$, and if $\frac{a}{b}$ be less than 1, $1 - \frac{a}{b} = \frac{b-a}{b}$, while if $\frac{a}{b}$ be greater than 1, $\frac{a}{b} - 1 = \frac{a-b}{b}$. Also (122), if $\frac{a+b}{b}$ be divided by $\frac{a-b}{b}$ the result is $\frac{a+b}{a-b}$. Hence, a, b, c, and d, being proportionals, we may obtain other proportions, thus:

Let $\frac{a}{b} = \frac{c}{d}$

Then (114) $1 + \frac{a}{b} = 1 + \frac{c}{d}$

or $\frac{a+b}{b} = \frac{c+d}{d}$

or $a+b : b :: c+d : d$

That is, the sum of the first and second is to the second as the sum of the third and fourth is to the fourth. For brevity, we shall not state in words any more of these proportions, since the pupil will easily supply what is wanting.

Resuming the proportion $a : b :: c : d$

or $\frac{a}{b} = \frac{c}{d}$

$1 - \frac{a}{b} = 1 - \frac{c}{d}$, if $\frac{a}{b}$ be less than 1

or $\frac{b-a}{b} = \frac{d-c}{d}$

that is, $b-a : b :: d-c : d$

or. $a-b : b :: c-d : d$, if $\frac{a}{b}$ be greater than 1.

Again, since $\frac{a+b}{b}=\frac{c+d}{d}$ and $\frac{a-b}{b}=\frac{c-d}{d}$ ($\frac{a}{b}$ being greater than 1) dividing the first by the second we have $\frac{a+b}{a-b}=\frac{c+d}{c-d}$,

$$\text{or} \quad a+b : a-b :: c+d : c-d$$

and also $a+b : b-a :: c+d : d-c$, if $\frac{a}{b}$ be less than 1.

185. Many other proportions might be obtained in the same manner. We will, however, content ourselves with writing down a few which can be obtained by combining the preceding articles.

$$a+b : a :: c+d : c$$
$$a : a-b :: c : c-d$$
$$a+c : a-c :: b+d : b-d.$$

In these and all others it must be observed, that when such expressions as $a-b$ and $c-d$ occur, it is supposed that a is greater than b, and c greater than d.

186. If four numbers be proportional, and any two dissimilar terms be both multiplied, or both divided by the same quantity, the results are proportional. Thus, if $a : b :: c : d$, and m and n be any two numbers, we have also the following:

$$ma : b :: mc : d \qquad ma : nb :: mc : nd$$
$$a : mb :: c : md \qquad \frac{a}{m} : \frac{b}{m} :: \frac{c}{m} : \frac{d}{m}$$
$$\frac{a}{n} : mb :: \frac{c}{n} : md \qquad \frac{a}{m} : \frac{b}{m} :: \frac{c}{n} : \frac{d}{n}$$

and various others. To prove any one of these, recollect that nothing more is necessary to make four numbers proportional except that the product of the extremes should be equal to that of the means. Take the third of those just given; the product of its extremes is $\frac{a}{n}\times md$, or $\frac{mad}{n}$, while that of the means is $mb\times\frac{c}{n}$, or $\frac{mbc}{n}$. But since $a : b :: c : d$, by (181) $ad = bc$, whence, by (180), $mad = mbc$, and $\frac{mad}{n}=\frac{mbc}{n}$. Hence, $\frac{a}{n}$, mb, $\frac{c}{n}$, and md, are proportionals.

187. If the terms of one proportion be multiplied by the terms of a second, the products are proportional; that is, if $a : b :: c : d$, and

$p : q :: r : s$, it follows that $ap : bq :: cr : ds$. For, since $ad = bc$, and $ps = qr$, by (180) $adps = bcqr$, or $ap \times ds = bq \times cr$, whence (182) $ap : bq :: cr : ds$.

188. If four numbers be proportional, any similar powers of these numbers are also proportional; that is, if

$$a : b :: c : d$$

Then $$aa : bb :: cc : dd$$

$$aaa : bbb :: ccc : ddd$$

&c. &c.

For, if we write the proportion twice, thus,

$$a : b :: c : d$$

$$a : b :: c : d$$

by (187) $$aa : bb :: cc : dd$$

But $$a : b :: c : d$$

Whence (187) $aaa : bbb :: ccc : ddd$; and so on.

189. An expression is said to be homogeneous with respect to any two or more letters, for instance, a, b, and c, when every term of it contains the same number of letters, counting a, b, and c only. Thus, $maab+nabc+rccc$ is homogeneous with respect to a, b, and c; and of the third degree, since in each term there is either a, b, and c, or one of these repeated alone, or with another, so as to make three in all. Thus, $8aaabc$, $12abccc$, $maaaaa$, $naabbc$, are all homogeneous, and of the fifth degree, with respect to a, b, and c only; and any expression made by adding or subtracting these from one another, will be homogeneous and of the fifth degree. Again $ma+mnb$ is homogeneous with respect to a and b, and of the first degree; but it is not homogeneous with respect to m and n, though it is so with respect to a and n. This being premised, we proceed to a theorem,* which will contain all the results of (184), (185), and (188).

190. If any four numbers be proportional, and if from the first two,

* A theorem is a general mathematical fact: thus, that every number is divisible by four when its last two figures are divisible by four, is a theorem; that in every proportion the product of the extremes is equal to the product of the means, is another.

a and b, any two homogeneous expressions of the same degree be formed; and if from the last two, two other expressions be formed, in precisely the same manner, the four results will be proportional. For example, if $a : b :: c : d$, and if $2aaa+3aab$ and $bbb+abb$ be chosen, which are both homogeneous with respect to a and b, and both of the third degree; and if the corresponding expressions $2ccc+3ccd$ and $ddd+cdd$ be formed, which are made from c and d precisely in the same manner as the two former ones from a and b, then will

$$2aaa+3aab : bbb+abb :: 2ccc+3ccd : ddd+cdd$$

To prove this, let $\frac{a}{b}$ be called x. Then, since $\frac{a}{b} = x$, and $\frac{a}{b} = \frac{c}{d}$, it follows that $\frac{c}{d} = x$. But since a divided by b gives x, x multiplied by b will give a, or $a = bx$. For a similar reason, $c = dx$. Put bx and dx instead of a and c in the four expressions just given, recollecting that when quantities are multiplied together, the result is the same in whatever order the multiplications are made; that, for example, $bxbxbx$ is the same as $bbbxxx$.

Hence, $2aaa+3aab = 2bxbxbx+3bxbxb$

$= 2bbbxxx+3bbbxx$

which is bbb multiplied by $2xxx+3xx$

or $bbb\,(2xxx+3xx)$*

Similarly, $2ccc+3ccd = ddd\,(2xxx+3xx)$

Also, $bbb+abb = bbb+bxbb$

$= bbb$ multiplied by $1+x$

or $bbb\,(1+x)$

Similarly, $ddd+cdd = ddd\,(1+x)$

Now, $bbb : bbb :: ddd : ddd$

Whence (186), $bbb(2xxx+3xx) : bbb(1+x) :: ddd(2xxx+3xx) : ddd(1+x)$, which, when instead of these expressions their equals just found are substituted, becomes $2aaa+3aab : bbb+abb :: 2ccc+3ccd : ddd+cdd$.

* If bx be substituted for a in any expression which is homogeneous with respect to a and b, the pupil may easily see that b must occur in every term as often as there are units in the degree of the expression: thus, $aa+ab$ becomes $bxbx+bxb$ r $bb(xx+x)$; $aaa+bbb$ becomes $bxbxbx+bbb$ or $bbb(xxx+1)$; and so on.

The same reasoning may be applied to any other case, and the pupil may in this way prove the following theorems:

If $a : b :: c : d$

$$2a+3b : b :: 2c+3d : d$$

$$aa+bb : aa-bb :: cc+dd : cc-dd$$

$$mab : 2aa+bb :: mcd : 2cc+dd$$

191. If the two means of a proportion be the same, that is, if $a : b :: b : c$, the three numbers, a, b, and c, are said to be in *continued* proportion, or in *geometrical progression.* The same terms are applied to a series of numbers, of which any three that follow one another are in continued proportion, such as

$$1 \quad 2 \quad 4 \quad 8 \quad 16 \quad 32 \quad 64 \quad \&c.$$

$$2 \quad \frac{2}{3} \quad \frac{2}{9} \quad \frac{2}{27} \quad \frac{2}{81} \quad \frac{2}{243} \quad \frac{2}{729} \quad \&c.$$

Which are in continued proportion, since

$$1 : 2 :: 2 : 4 \qquad 2 : \frac{2}{3} :: \frac{2}{3} : \frac{2}{9}$$

$$2 : 4 :: 4 : 8 \qquad \frac{2}{3} : \frac{2}{9} :: \frac{2}{9} : \frac{2}{27}$$

&c. &c.

192. Let a, b, c, d, &c. be in continued proportion; we have then

$$a : b :: b : c \quad \text{or} \quad \frac{a}{b}=\frac{b}{c} \quad \text{or} \quad ac = bb$$

$$b : c :: c : d \quad \ldots \quad \frac{b}{c}=\frac{c}{d} \quad \ldots \quad bd = cc$$

$$c : d :: d : e \quad \ldots \quad \frac{c}{d}=\frac{d}{e} \quad \ldots \quad ce = dd$$

Each term is formed from the preceding, by multiplying it by the same number. Thus, $b = \frac{b}{a}\times a$ (180); $c = \frac{c}{b}\times b$; and since $\frac{a}{b} = \frac{b}{c}$, $\frac{b}{a} = \frac{c}{b}$ or $c = \frac{b}{a}\times b$. Again, $d = \frac{d}{c}\times c$, but $\frac{d}{c} = \frac{c}{b}$, which is $= \frac{b}{a}$; therefore, $d = \frac{b}{a}\times c$, and so on. If, then, $\frac{b}{a}$ (which is called the *common ratio* of the series) be denoted by r, we have

$$b = ar \qquad c = br = arr \qquad d = cr = arrr$$

and so on; whence the series

	a	b	c	d	&c.
is	a	ar	arr	$arrr$	&c.

Hence $a : c :: a : arr$

(186) $:: aa : aarr$

$:: aa : bb$

because, b being ar, bb is $arar$ or $aarr$. Again,

$a : d :: a : arrr$

(186) $:: aaa : aaarrr$

$:: aaa : bbb$

Also $a : e :: aaaa : bbbb$, and so on; that is, the first bears to the n^{th} term from the first the same proportion as the n^{th} power of the first to the n^{th} power of the second.

193. A short rule may be found for adding together any number of terms of a continued proportion. Let it be first required to add together the terms 1, r, rr, &c. where r is greater than unity. It is evident that we do not alter any expression by adding or subtracting any numbers, provided we afterwards subtract or add the same. For example,

$$p = p-q+q-r+r-s+s$$

Let us take four terms of the series, 1, r, rr, &c. or,

$$1+r+rr+rrr$$

It is plain that

$$rrrr-1 = rrrr-rrr+rrr-rr+rr-r+r-1$$

Now (54), $rr-r = r(r-1)$, $rrr-rr = rr(r-1)$, $rrrr-rrr = rrr\ (r-1)$, and the above equation becomes $rrrr-1 = rrr\,(r-1) + rr\,(r-1) + r\,(r-1) +r-1$; which is (54) $rrr+rr+r+1$ taken $r-1$ times. Hence, $rrrr-1$ divided by $r-1$ will give $1+r+rr+rrr$, the sum of the terms required. In this way may be proved the following series of equations:

$$1+r = \frac{rr-1}{r-1}$$

$$1+r+rr = \frac{rrr-1}{r-1}$$

$$1+r+rr+rrr = \frac{rrrr-1}{r-1}$$

$$1+r+rr+rrr+rrrr = \frac{rrrrr-1}{r-1}$$

If r be less than unity, in order to find $1+r+rr+rrr$, observe that

$$1-rrrr = 1-r+r-rr+rr-rrr+rrr-rrrr$$
$$= 1-r+r(1-r)+rr(1-r)+rrr(1-r);$$

whence, by similar reasoning, $1+r+rr+rrr$ is found by dividing $1-rrrr$ by $1-r$; and equations similar to these just given may be found. which are,

$$1+r = \frac{1-rr}{1-r}$$

$$1+r+rr = \frac{1-rrr}{1-r}$$

$$1+r+rr+rrr = \frac{1-rrrr}{1-r}$$

$$1+r+rr+rrr+rrrr = \frac{1-rrrrr}{1-r}$$

The rule is: To find the sum of n terms of the series, $1+r+rr+$&c., divide the difference between 1 and the $(n+1)^{th}$ term by the difference between 1 and r.

194. This may be applied to finding the sum of any number of terms of a continued proportion. Let a, b, c, &c. be the terms of which it is required to sum four, that is, to find $a+b+c+d$, or (192) $a+ar+arr+arrr$, or (54) $a(1+r+rr+rrr)$, which (193) is $\frac{rrrr-1}{r-1}\times a$, or $\frac{1-rrrr}{1-r}\times a$, according as r is greater or less than unity. The first fraction is $\frac{arrrr-a}{r-1}$, or (192) $\frac{e-a}{r-1}$. Similarly, the second is $\frac{a-e}{1-r}$. The rule, therefore, is: To sum n terms of a continued proportion, divide the difference of the $\overline{n+1}^{th}$ and first terms by the difference between unity and the common measure. For example, the sum of 10 terms of the series $1+3+9+27+$&c. is required. The eleventh term is 59049, and $\frac{59049-1}{3-1}$ is 29524. Again, the sum of 18 terms of the series $2+1+\frac{1}{2}+\frac{1}{4}+$&c. of which the nineteenth term is $\frac{1}{131072}$, is $\dfrac{2-\frac{1}{131072}}{1-\frac{1}{2}} = 3\frac{131070}{131072}$.

EXAMPLES.

9 terms of $1+4+16+$&c. are 87381

10 terms of $3+\frac{6}{7}+\frac{12}{49}+$&c. are $\frac{847422675}{201768035}$

20 ...... $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+$&c. ... $\frac{1048575}{1048576}$

195. The powers of a number or fraction greater than unity increase; for since $2\frac{1}{2}$ is greater than 1, $2\frac{1}{2}\times 2\frac{1}{2}$ is $2\frac{1}{2}$ taken more than once, that is, is greater than $2\frac{1}{2}$, and so on. This increase goes on without limit; that is, there is no quantity so great but that some power of $2\frac{1}{2}$ is greater. To prove this, observe that every power of $2\frac{1}{2}$ is made by multiplying the preceding power by $2\frac{1}{2}$, or by $1+1\frac{1}{2}$, that is, by adding to the former power that power itself and its half. There will, therefore, be more added to the 10th power to form the 11th, than was added to the 9th power to form the 10th. But it is evident that if any given quantity, however small, be continually added to $2\frac{1}{2}$, the result will come in time to exceed any other quantity that was also given, however great; much more, then, will it do so if the quantity added to $2\frac{1}{2}$ be increased at each step, which is the case when the successive powers of $2\frac{1}{2}$ are formed. It is evident, also, that the powers of 1 never increase, being always 1; thus, $1\times 1 = 1$, &c. Also, if a be greater than m times b, the square of a is greater than mm times the square of b. Thus, if $a = 2b+c$, where a is greater than $2b$, the square of a, or aa, which is (68) $4bb+4bc+cc$ is greater than $4bb$, and so on.

196. The powers of a fraction less than unity continually decrease; thus, the square of $\frac{2}{5}$, or $\frac{2}{5}\times\frac{2}{5}$, is less than $\frac{2}{5}$, being only two-fifths of it. This decrease continues without limit; that is, there is no quantity so small but that some power of $\frac{2}{5}$ is less. For if $\frac{5}{2} = x$, $\frac{2}{5} = \frac{1}{x}$, and the powers of $\frac{2}{5}$ are $\frac{1}{xx}$, $\frac{1}{xxx}$, and so on. Since x is greater than 1 (195), some power of x may be found which shall be greater than a given quantity. Let this be called m; then $\frac{1}{m}$ is the corresponding power of $\frac{2}{5}$; and a fraction whose denominator can be made as great as we please, can itself be made as small as we please (112).

197. We have, then, in the series

$$1 \qquad r \qquad rr \qquad rrr \qquad rrrr \qquad \text{\&c.}$$

I. A series of increasing terms, if r be greater than 1. II. Of terms having the same value, if r be equal to 1. III. A series of decreasing terms, if r be less than 1. In the first two cases, the sum

$$1+r+rr+rrr+\&c.$$

may evidently be made as great as we please, by sufficiently increasing the number of terms. But in the third this may or may not be the case; for though something is added at each step, yet, as that augmentation diminishes at every step, we may not certainly say that we can, by any number of such augmentations, make the result as great as we please. To shew the contrary in a simple instance, consider the series,

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\&c.$$

Carry this series to what extent we may, it will always be necessary to add the last term in order to make as much as 2. Thus,

$$\left(1+\frac{1}{2}+\frac{1}{4}\right)+\frac{1}{4}=1+\frac{1}{2}+\frac{1}{2}=1+1=2$$

$$\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)+\frac{1}{8}=2.$$

$$\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\right)+\frac{1}{16}=2, \&c.$$

But in the series, every term is only the half of the preceding; consequently no number of terms, however great, can be made as great as 2 by adding one more. The sum, therefore, of 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c. continually approaches to 2, diminishing its distance from 2 at every step, but never reaching it. Hence, 2 is called the *limit* of $1+\frac{1}{2}+\frac{1}{4}+\&c.$ We are not, therefore, to conclude that *every* series of decreasing terms has a limit. The contrary may be shewn in the very simple series, $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\&c.$ which may be written thus:

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\ldots\text{up to }\frac{1}{8}\right)+\left(\frac{1}{9}+\ldots\text{up to }\frac{1}{16}\right)+\left(\frac{1}{17}+\ldots\text{up to }\frac{1}{32}\right)+\&c.$$

We have thus divided all the series, except the first two terms, into lots, each containing half as many terms as there are units in the denominator of its last term. Thus, the fourth lot contains 16 or $\frac{32}{2}$ terms. Each of these lots may be shewn to be greater than $\frac{1}{2}$. Take the third, for example, consisting of $\frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}, \frac{1}{15}$, and $\frac{1}{16}$. All except $\frac{1}{16}$, the last, are greater than $\frac{1}{16}$; consequently, by substituting $\frac{1}{16}$ for each of them, the amount of the whole lot would be lessened; and as it would then become $\frac{8}{16}$, or $\frac{1}{2}$, the lot itself is greater than $\frac{1}{2}$. Now, if to $1+\frac{1}{2}$, $\frac{1}{2}$ be continually added, the result will in time exceed any given number. Still more will this be the case if, instead of $\frac{1}{2}$, the several lots written above be added one after the other. But it is thus that the series $1+\frac{1}{2}+\frac{1}{3}$, &c. is composed, which proves what was said, that this series has no limit.

198. The series $1+r+rr+rrr+$&c. always has a limit when r is less than 1. To prove this, let the term succeeding that at which we stop be a, whence (194) the sum is $\frac{1-a}{1-r}$, or (112) $\frac{1}{1-r}-\frac{a}{1-r}$. The terms decrease without limit (196), whence we may take a term so far distant from the beginning, that a, and therefore $\frac{a}{1-r}$, shall be as small as we please. But it is evident that in this case $\frac{1}{1-r}-\frac{a}{1-r}$, though always less than $\frac{1}{1-r}$, may be brought as near to $\frac{1}{1-r}$ as we please; that is, the series $1+r+rr+$&c. continually approaches to the limit $\frac{1}{1-r}$. Thus $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+$&c. where $r=\frac{1}{2}$, continually approaches to $\frac{1}{1-\frac{1}{2}}$ or 2, as was shewn in the last article.

EXERCISES.

The limit of $2+\frac{2}{3}+\frac{2}{9}+$&c.

or $2\left(1+\frac{1}{3}+\frac{1}{9}+\text{\&c.}\right)$ is 3

.... $1+\frac{9}{10}+\frac{81}{100}+$&c. ... 10

.... $5+\frac{15}{7}+\frac{45}{49}+$&c. ... $8\frac{3}{4}$

199. When the fraction $\frac{a}{b}$ is not equal to $\frac{c}{d}$, but greater, a is said to have to b a greater ratio than c has to d; and when $\frac{a}{b}$ is less than $\frac{c}{d}$, a to have to b a less ratio than c has to d. We propose the fol-

lowing questions as exercises, since they follow very simply from this definition.

I. If a be greater than b, and c less than or equal to d, a will have a greater ratio to b than c has to d.

II. If a be less than b, and c greater than or equal to d, a has a less ratio to b than c has to d.

III. If a be to b as c is to d, and if a have a greater ratio to b than c has to x, d is less than x; and if a have a less ratio to b than c to x, d is greater than x.

IV. a has to b a greater ratio than ax to $bx+y$, and a less ratio than ax to $bx-y$.

200. If a have to b a greater ratio than c has to d, $a+c$ has to $b+d$ a less ratio than a has to b, but a greater ratio than c has to d; or, in other words, if $\frac{a}{b}$ be the greater of the two fractions $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a+c}{b+d}$ will be greater than $\frac{c}{d}$, but less than $\frac{a}{b}$. To shew this, observe that $\frac{mx+ny}{m+n}$ must lie between x and y, if x and y be unequal: for if x be the less of the two, it is certainly greater than $\frac{mx+nx}{m+n}$ or than x; and if y be the greater of the two, it is certainly less than $\frac{my+ny}{m+n}$, or than y. It therefore lies between x and y. Now let $\frac{a}{b}$ be x, and let $\frac{c}{d}$ be y: then $a = bx$, $c = dy$. Now $\frac{bx+dy}{b+d}$ is something between x and y, as was just proved; therefore $\frac{a+c}{b+d}$ is something between $\frac{a}{b}$ and $\frac{c}{d}$. Again, since $\frac{a}{b}$ and $\frac{c}{d}$ are respectively equal to $\frac{ap}{bp}$ and $\frac{cq}{dq}$, and since, as has just been proved, $\frac{ap+cq}{bp+dq}$ lies between the two last, it also lies between the two first; that is, if p and q be any numbers or fractions whatsoever, $\frac{ap+cq}{bp+dq}$ lies between $\frac{a}{b}$ and $\frac{c}{d}$.

201. By the last article we may often form some notion of the value of an expression too complicated to be easily calculated. Thus, $\frac{1+x}{1+xx}$ lies between $\frac{1}{1}$ and $\frac{x}{xx}$, or 1 and $\frac{1}{x}$; $\frac{ax+by}{axx+bbyy}$ lies between $\frac{ax}{axx}$ and $\frac{by}{bbyy}$, that is, between $\frac{1}{x}$ and $\frac{1}{by}$. And it has been shewn that $\frac{a+b}{2}$ lies between a and b, the denominator being considered as $1+1$.

202. It may also be proved that a fraction such as $\frac{a+b+c+d}{p+q+r+s}$ always lies among $\frac{a}{p}$, $\frac{b}{q}$, $\frac{c}{r}$, and $\frac{d}{s}$, that is, is less than the greatest of them, and greater than the least. Let these fractions be arranged in order of

magnitude; that is, let $\frac{a}{p}$ be greater than $\frac{b}{q}$, $\frac{b}{q}$ be greater than $\frac{c}{r}$, and $\frac{c}{r}$ greater than $\frac{d}{s}$. Then by (200)

$$\begin{array}{lcccl} \frac{a+b}{p+q} & & \frac{a}{p} & & \frac{b}{q} \text{ and } \frac{c}{r} \\ \frac{a+b+c}{p+q+r} & \text{is less than} & \frac{a+b}{p+q} \text{ and } \frac{a}{p} & \text{and greater than} & \frac{c}{r} \text{ and } \frac{d}{s} \\ \frac{a+b+c+d}{p+q+r+s} & & \frac{a+b+c}{p+q+r} \text{ and } \frac{a}{p} & & \frac{d}{s} \end{array}$$

whence the proposition is evident.

203. It is usual to signify "a is greater than b" by $a > b$, and "a is less than b" by $a < b$; the opening of $\wedge$ being turned towards the greater quantity. The pupil is recommended to make himself familiar with these signs.

SECTION IX.

ON PERMUTATIONS AND COMBINATIONS.

204. If a number of counters, distinguished by different letters, be placed on the table, and any number of them, say four, be taken away, the question is, to determine in how many different ways this can be done. Each way of doing it gives what is called a *combination* of four, but which might with more propriety be called a *selection* of four. Two combinations or selections are called different, which differ in any way whatever; thus, *abcd* and *abce* are different, *d* being in one and *e* in the other, the remaining parts being the same. Let there be six counters, *a*, *b*, *c*, *d*, *e*, and *f*; the combinations of three which can be made out of them are twenty in number, as follow:

abc	*ace*	*bcd*	*bef*
abd	*acf*	*bce*	*cde*
abe	*ade*	*bcf*	*cdf*
abf	*adf*	*bde*	*cef*
acd	*aef*	*bdf*	*def*

The combinations of four are fifteen in number, namely,

abcd	*abde*	*acde*	*adef*	*bcef*
abce	*abdf*	*acdf*	*bcde*	*bdef*
abcf	*abef*	*acef*	*bcdf*	*cdef*

and so on.

205. Each of these combinations may be written in several different orders; thus, *abcd* may be disposed in any of the following ways:

abcd	*acbd*	*acdb*	*abdc*	*adbc*	*adcb*
bacd	*cabd*	*cadb*	*badc*	*dabc*	*dacb*
bcad	*cbad*	*cdab*	*bdac*	*dbac*	*dcab*
bcda	*cbda*	*cdba*	*bdca*	*dbca*	*dcba*

of which no two are entirely in the same order. Each of these is said to be a distinct *permutation* of *abcd.* Considered as a *combination*, they are all the same, as each contains *a*, *b*, *c*, and *d.*

206. We now proceed to find how many *permutations*, each containing one given number, can be made from the counters in another given number, six, for example. If we knew how to find all the permutations containing four counters, we might make those which contain five thus: Take any one which contains four, for example, *abcf*, in which *d* and *e* are omitted; write *d* and *e* successively at the end, which gives *abcfd*, *abcfe*, and repeat the same process with every other permutation of four; thus, *dabc* gives *dabce* and *dabcf.* No permutation of five can escape us if we proceed in this manner, provided only we know those of four; for any given permutation of five, as *dbfea*, will arise in the course of the process from *dbfe*, which, according to our rule, furnishes *dbfea.* Neither will any permutation be repeated twice, for *dbfea*, if the rule be followed, can only arise from the permutation *dbfe.* If we begin in this way to find the permutations of two out of the six,

a *b* *c* *d* *e* *f*

each of these gives five; thus,

a gives *ab* *ac* *ad* *ae* *af*
b ...... *ba* *bc* *bd* *be* *bf*

and the whole number is 6×5, or 30.

Again, *ab* gives *abc abd abe abf*

ac ...... *acb acd ace acf*

and here are 30, or 6×5 permutations of 2, each of which gives 4 permutations of 3; the whole number of the last is therefore 6×5×4, or 120.

Again *abc* gives *abcd abce abcf*

abd ...... *abdc abde abdf*

and here are 120, or 6×5×4, permutations of three, each of which gives 3 permutations of four; the whole number of the last is therefore 6×5×4×3, or 360.

In the same way, the number of permutations of 5 is 6×5×4×3×2, and the number of permutations of six, or the number of different ways in which the whole six can be arranged, is 6×5×4×3×2×1. The last two results are the same, which must be; for since a permutation of five only omits one, it can only furnish one permutation of six. If instead of six we choose any other number, x, the number of permutations of two will be $x(x-1)$, that of three will be $x(x-1)(x-2)$, that of four $x(x-1)(x-2)(x-3)$, the rule being: Multiply the whole number of counters by the next less number, and the result by the next less, and so on, until as many numbers have been multiplied together as there are to be counters in each permutation: the product will be the whole number of permutations of the sort required. Thus, out of 12 counters, permutations of four may be made to the number of 12×11×10×9, or 11880.

EXERCISES.

207. In how many different ways can eight persons be arranged on eight seats? *Answer*, 40320.

In how many ways can eight persons be seated at a round table, so that all shall not have the same neighbours in any two arrangements?*

Answer, 5040.

If the hundredth part of a farthing be given for every different

* The difference between this problem and the last is left to the ingenuity of the pupil.

arrangement which can be made of fifteen persons, to how much will the whole amount? *Answer*, £13621608.

Out of seventeen consonants and five vowels, how many words can be made, having two consonants and one vowel in each? *Answer*, 4080.

208. If two or more of the counters have the same letter upon them, the number of distinct permutations is less than that given by the last rule. Let there be a, a, a, b, c, d, and, for a moment, let us distinguish between the three *a*s thus, a, a', a''. Then, $abca'a''d$, and $a''bcaa'd$ are reckoned as distinct permutations in the rule, whereas they would not have been so, had it not been for the accents. To compute the number of distinct permutations, let us make one with b, c, and d, leaving places for the *a*s, thus, ( ) bc ( ) ( ) d. If the *a*s had been distinguished as a, a', a'', we might have made $3\times2\times1$ distinct permutations, by filling up the vacant places in the above, all which six are the same when the *a*s are not distinguished. Hence, to deduce the number of permutations of a, a, a, b, c, d, from that of $aa'a''bcd$, we must divide the latter by $3\times2\times1$, or 6, which gives $\frac{6\times5\times4\times3\times2\times1}{3\times2\times1}$ or 120. Similarly, the number of permutations of $aaaabbbcc$ is $\frac{9\times8\times7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1\times3\times2\times1\times2\times1}$.

EXERCISE.

How many variations can be made of the order of the letters in the word antitrinitarian? *Answer*, 126126000.

209. From the number of permutations we can easily deduce the number of combinations. But, in order to form these combinations independently, we will shew a method similar to that in (206). If we know the combinations of two which can be made out of a, b, c, d, e, we can find the combinations of three, by writing successively at the end of each combination of two, the letters which come after the last contained in it. Thus, ab gives abc, abd, abe; ad gives ade only. No combination of three can escape us if we proceed in this manner, provided only we know the combinations of two; for any given combination of three, as acd, will arise in the course of the process from ac, which, according to our rule, furnishes acd. Neither will any combination be repeated

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twice, for *acd*, if the rule be followed, can only arise from *ac*, since neither *ad* nor *cd* furnishes it. If we begin in this way to find the combinations of the five,

a *b* *c* *d* *e*

a	gives	*ab*	*ac*	*ad*	*ae*
b	......		*bc*	*bd*	*be*
c	......			*cd*	*ce*
d	......				*de*

Of these,

ab	gives	*abc*	*abd*	*abe*
ac	......		*acd*	*ace*
ad	......			*ade*
bc	......		*bcd*	*bce*
bd	......			*bde*
cd	......			*cde*

ae *be* *ce* and *de* give none.

Of these,

abc	gives	*abcd*	*abce*
abd	......		*abde*
acd	......		*acde*
bcd	......		*bcde*

Those which contain *e* give none, as before.

Of the last, *abcd* gives *abcde*, and the others none, which is evidently true, since only one selection of five can be made out of five things.

210. The rule for calculating the number of combinations is derived directly from that for the number of permutations. Take 7 counters; then, since the number of permutations of two is 7×6, and since two permutations, *ba* and *ab*, are in any combination *ab*, the number of combinations is half that of the permutations, or $\frac{7\times6}{2}$. Since the number of permutations of three is $7\times6\times5$, and as each combination *abc* has $3\times2\times1$ permutations, the number of combinations of three is $\frac{7\times6\times5}{1\times2\times3}$. Also, since any combination of four, *abcd*, contains $4\times3\times2\times1$ permutations, the number of combinations of four is $\frac{7\times6\times5\times4}{1\times2\times3\times4}$, and so on. The rule is: To find the number of combinations, each containing *n* counters, divide the corresponding number

of permutations by the product of 1, 2, 3, &c. up to n. If x be the whole number, the number of combinations of two is $\frac{x(x-1)}{1\times 2}$; that of three is $\frac{x(x-1)(x-2)}{1\times 2\times 3}$; that of four is $\frac{x(x-1)(x-2)(x-3)}{1\times 2\times 3\times 4}$; and so on.

211. The rule may in half the cases be simplified, as follows. Out of ten counters, for every distinct selection of seven which is taken, a distinct combination of 3 is left. Hence, the number of combinations of seven is as many as that of three. We may, therefore, find the combinations of three instead of those of seven; and we must moreover expect, and may even assert, that the two formulæ for finding these two numbers of combinations are the same in result, though different in form. And so it proves; for the number of combinations of seven out of ten is $\frac{10\times 9\times 8\times 7\times 6\times 5\times 4}{1\times 2\times 3\times 4\times 5\times 6\times 7}$, in which the product $7\times 6\times 5\times 4$ occurs in both terms, and therefore may be removed from both (108), leaving $\frac{10\times 9\times 8}{1\times 2\times 3}$, which is the number of combinations of three out of ten. The same may be shewn in other cases.

EXERCISES.

How many combinations of four can be made out of twelve things?

Answer, 495.

What number of combinations can be made of $\left\{\begin{matrix} 6 \\ 4 \\ 26 \\ 6 \end{matrix}\right\}$ out of $\left\{\begin{matrix} 8 \\ 11 \\ 28 \\ 15 \end{matrix}\right\}$ *Answer*, $\left\{\begin{matrix} 28 \\ 330 \\ 378 \\ 5005 \end{matrix}\right.$

How many combinations can be made of 13 out of 52; or how many different hands may a person hold at the game of whist?

Answer, 635013559600.

BOOK II.

COMMERCIAL ARITHMETIC.

SECTION I.

WEIGHTS, MEASURES, &c.

212. In making the calculations which are necessary in commercial affairs, no more processes are required than those which have been explained in the preceding book. But there is still one thing wanted —not to insure the accuracy of our calculations, but to enable us to compare and judge of their results. We have hitherto made use of a single unit (15), and have treated of other quantities which are made up of a number of units, in Sections II., III., and IV., and of those which contain parts of that unit in Sections V. and VI. Thus, if we are talking of distances, and take a mile as the unit, any other length may be represented,* either by a certain number of miles, or a certain number of parts of a mile, and (1 meaning one mile) may be expressed

* It is not true, that if we choose any quantity as a unit, *any* other quantity of the same kind can be exactly represented either by a certain number of units, or of parts of a unit. To understand how this is proved, the pupil would require more knowledge than he can be supposed to have; but we can shew him that, for any thing he knows to the contrary, there may be quantities which are neither units nor parts of the unit. Take a mathematical line of one foot in length, divide it into ten parts, each of those parts into ten parts, and so on continually. If a point A be taken at hazard in the line, it does not appear self-evident that if the decimal division be continued ever so far, one of the points of division must at last fall exactly on A: neither would the same appear necessarily true if the division were made into sevenths, or elevenths, or in any other way. There may then possibly be a part of a foot which is no exact numerical fraction whatever of the foot; and this, in a higher branch of

either by a whole number or a fraction. But we can easily see that in many cases inconveniences would arise. Suppose, for example, I say, that the length of one room is $\frac{1}{180}$ of a mile, and of another $\frac{1}{174}$ of a mile, what idea can we form as to how much the second is longer than the first? It is necessary to have some smaller measure; and if we divide a mile into 1760 equal parts, and call each of these parts a yard, we shall find that the length of the first room is 9 yards and $\frac{7}{9}$ of a yard, and that of the second 10 yards and $\frac{10}{87}$ of a yard. From this we form a much better notion of these different lengths, but still not a very perfect one, on account of the fractions $\frac{7}{9}$ and $\frac{10}{87}$. To get a clearer idea of these, suppose the yard to be divided into three equal parts, and each of these parts to be called a foot; then $\frac{7}{9}$ of a yard contains $2\frac{1}{3}$ feet, and $\frac{10}{87}$ of a yard contains $\frac{30}{87}$ of a foot, or a little more than $\frac{1}{3}$ of a foot. Therefore the length of the first room is now 9 yards, 2 feet, and $\frac{1}{3}$ of a foot; that of the second is 10 yards and a little more than $\frac{1}{3}$ of a foot. We see, then, the convenience of having large measures for large quantities, and smaller measures for small ones; but this is done for convenience only, for it is *possible* to perform calculations upon any sort of quantity, with one measure alone, as certainly as with more than one; and not only possible, but more convenient, as far as the mere calculation is concerned.

The measures which are used in this country are not those which would have been chosen had they been made all at one time, and by a people well acquainted with arithmetic and natural philosophy. We proceed to shew how the results of the latter science are made useful in our system of measures. Whether the circumstances introduced are sufficiently well known to render the following methods exact enough for the recovery of *astronomical* standards, may be matter of opinion; but no doubt can be entertained of their being amply correct for commercial purposes.

mathematics, is found to be the case times without number. What is meant in the words on which this note is written, is, that any part of a foot can be represented as nearly as we please by a numerical fraction of it; and this is sufficient for practical purposes.

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It is evidently desirable that weights and measures should always continue the same, and that posterity should be able to replace any one of them when the original measure is lost. It is true that a yard, which is now exact, is kept by the public authorities; but if this were burnt by accident,* how are those who shall live 500 years hence to know what was the length which their ancestors called a yard? To ensure them this knowledge, the measure must be derived from something which cannot be altered by man, either from design or accident. We find such a quantity in the time of the daily revolution of the earth, and also in the length of the year, both of which, as is shewn in astronomy, will remain the same, at least for an enormous number of centuries, unless some great and totally unknown change take place in the solar system. So long as astronomy is cultivated, it is impossible to suppose that either of these will be lost, and it is known that the latter is 365·24224 mean solar days, or about $365\frac{1}{4}$ of the average interval which elapses between noon and noon, that is, between the times when the sun is highest in the heavens. Our year is made to consist of 365 days, and the odd quarter is allowed for by adding one day to every fourth year, which gives what we call leap-year. This is the same as adding $\frac{1}{4}$ of a day to each year, and is rather too much, since the excess of the year above 365 days is not ·25 but ·24224 of a day. The difference is ·00776 of a day, which is the quantity by which our average year is too long. This amounts to a day in about 128 years, or to about 3 days in 4 centuries. The error is corrected by allowing only one out of four of the years which close the centuries to be leap-years. Thus, A.D. 1800 and 1900 are not leap-years, but 2000 is so.

213. The day is therefore the first measure obtained, and is divided into 24 parts or hours, each of which is divided into 60 parts or minutes, and each of these again into 60 parts or seconds. One second, marked thus, 1^s,† is therefore the 86400th part of a day, and the following is the

* Since this was first written, the accident has happened. The *standard yard* was so injured as to be rendered useless by the fire at the Houses of Parliament.

† The minute and second are often marked thus, $1'$, $1''$: but this notation is now almost entirely appropriated to the minute and second of *angular* measure.

MEASURE OF TIME.*

60 *seconds*	are	1 *minute*	1 m.
60 *minutes*	. . .	1 *hour*	1 h.
24 *hours*	. . . .	1 *day*	1 d.
7 *days*	. . . .	1 *week*	1 wk.
365 *days*	. . . .	1 *year*	1 yr.

214. The *second* having been obtained, a pendulum can be constructed which shall, when put in motion, perform one vibration in exactly one second, in the latitude of Greenwich.† If we were inventing measures, it would be convenient to call the length of this pendulum a yard, and make it the standard of all our measures of length. But as there is a yard already established, it will do equally well to tell the length of the pendulum in yards. It was found by commissioners appointed for the purpose, that this pendulum in London was 39·1393 inches, or about one yard, three inches, and $\frac{5}{36}$ of an inch. The following is the division of the yard.

MEASURES OF LENGTH.

The lowest measure is a barleycorn.‡

3 *barleycorns*	are	1 *inch*	1 in.
12 *inches*	. . . .	1 *foot*	1 ft.
3 *feet*	. . . .	1 *yard*	1 yd.
$5\frac{1}{2}$ *yards*	. . . .	1 *pole*	1 po.
40 *poles* or 220 *yards*	. .	1 *furlong*	1 fur.
8 *furlongs* or 1760 *yards*	.	1 *mile*	1 mi.

* The measures in italics are those which it is most necessary that the student should learn by heart.

† The lengths of the pendulums which will vibrate in one second are slightly different in different latitudes. Greenwich is chosen as the station of the Royal Observatory. We may add, that much doubt is now entertained as to the system of standards derived from nature being capable of that extreme accuracy which was once attributed to it.

‡ The inch is said to have been originally obtained by putting together three grains of barley.

Also 6 feet	. . . .	1 fathom .	1 fth.
$69\frac{1}{3}$ miles	. . . .	1 degree .	1 deg. or 1°.

A geographical mile is $\frac{1}{60}$th of a degree, and three such miles are one nautical league.

In the measurement of cloth or linen the following are also used :

$2\frac{1}{4}$ inches	are	1 nail	1 nl.
4 nails	. . . .	1 quarter (of a yard) .	1 qr.
3 quarters	. . .	1 Flemish ell . . .	1 Fl. e.
5 quarters	. . .	1 English ell . . .	1 E. e.
6 quarters	. . .	1 French ell . . .	1 Fr. e.

215. MEASURES OF SURFACE, OR SUPERFICIES.

All surfaces are measured by square inches, square feet, &c.; the square inch being a square whose side is an inch in length, and so on. The following measures may be deduced from the last, as will afterwards appear.

144 *square inches*	are	1 *square foot* .	1 sq. ft.
9 *square feet*	. . .	1 *square yard* .	1 sq. yd.
$30\frac{1}{4}$ square yards	. . .	1 square pole .	1 sq. p.
40 square poles	. .	1 rood . . .	1 rd.
4 roods	. . . . . .	1 acre . . .	1 ac.

Thus, the acre contains 4840 square yards, which is ten times a square of 22 yards in length and breadth. This 22 yards is the length which land-surveyors' chains are made to have, and the chain is divided into 100 links, each ·22 of a yard or 7·92 inches. An acre is then 10 square chains. It may also be noticed that a square whose side is $69\frac{4}{7}$ yards is nearly an acre, not exceeding it by $\frac{1}{5}$ of a square foot.

216. MEASURES OF SOLIDITY OR CAPACITY.*

Cubes are solids having the figure of dice. A cubic inch is a cube each of whose sides is an inch, and so on.

* 'Capacity' is a term which cannot be better explained than by its use. When one measure holds more than another, it is said to be more capacious, or to have a greater capacity.

1728 cubic inches	are 1 cubic foot	. .	1 c. ft.
27 cubic feet . . .	1 cubic yard	. .	1 c. yd.

This measure is not much used, except in purely mathematical questions. In the measurements of different commodities various measures were used, which are now reduced, by act of parliament, to one. This is commonly called the imperial measure, and is as follows:

MEASURE OF LIQUIDS AND OF ALL DRY GOODS.

4 *gills* are	1 *pint* . . .	1 pt.
2 *pints* . . .	1 *quart* . . .	1 qt.
4 *quarts* . .	1 *gallon* . .	1 gall.
2 *gallons* . .	1 *peck** . .	1 pk.
4 *pecks* . .	1 *bushel* . .	1 bu.
8 *bushels* . .	1 *quarter* . .	1 qr.
5 *quarters* .	1 *load* . . .	1 ld.

The gallon in this measure is about 277·274 cubic inches; that is, very nearly $277\frac{1}{4}$ cubic inches.†

217. The smallest weight in use is the grain, which is thus determined. A vessel whose interior is a cubic inch, when filled with water,‡ has its weight increased by 252·458 grains. Of the grains so determined, 7000 are a pound *averdupois*, and 5760 a pound *troy*. The

* This measure, and those which follow, are used for dry goods only.

† Since the publication of the third edition, the *heaped* measure, which was part of the new system, has been abolished. The following paragraph from the third edition will serve for reference to it:

"The other imperial measure is applied to goods which it is customary to sell by *heaped measure*, and is as follows:

2 gallons	1 peck
4 pecks	1 bushel
3 bushels	1 sack
12 sacks	1 chaldron

The gallon and bushel in this measure hold the same when only just filled, as in the last. The bushel, however, heaped up as directed by the act of parliament, is a little more than one-fourth greater than before."

‡ Pure water, cleared from foreign substances by distillation, at a temperature of 62° Fahr.

first pound is always used, except in weighing precious metals and stones, and also medicines. It is divided as follows:

AVERDUPOIS WEIGHT.

$27\frac{11}{32}$ grains	are	1 dram	1 dr.
16 *drams*, or *drachms*		1 *ounce**	1 oz.
16 *ounces*		1 *pound*	1 lb.
28 *pounds*		1 *quarter*	1 qr.
4 *quarters*		1 *hundred-weight*	1 cwt.
20 *hundred-weight*		1 *ton*	1 ton.

The pound averdupois contains 7000 grains. A cubic foot of water weighs 62·3210606 pounds averdupois, or 997·1369691 ounces.

For the precious metals and for medicines, the pound troy, containing 5760 grains, is used, but is differently divided in the two cases. The measures are as follow:

TROY WEIGHT.

24 *grains*	are	1 *pennyweight*	1 dwt.
20 *pennyweights*		1 *ounce*	1 oz.
12 *ounces*		1 *pound*	1 lb.

The pound troy contains 5760 grains. A cubic foot of water weighs 75·7374 pounds troy, or 908·8488 ounces.

APOTHECARIES' WEIGHT.

20 *grains*	are	1 *scruple*	℈
3 *scruples*		1 *dram*	ʒ
8 *drams*		1 *ounce*	℥
12 *ounces*		1 *pound*	℔

218. The standard coins of copper, silver, and gold, are,—the penny, which is $10\frac{2}{3}$ drams of copper; the shilling, which weighs 3 pennyweights 15 grains, of which 3 parts out of 40 are alloy, and the rest pure silver; and the sovereign, weighing 5 pennyweights and $3\frac{1}{4}$ grains, of which 1 part out of 12 is copper, and the rest pure gold.

* It is more common to divide the ounce into four quarters than into sixteen drams.

MEASURES OF MONEY.

The lowest coin is a farthing, which is marked thus, $\frac{1}{4}$, being one fourth of a penny.

2 *farthings*	are	1 *halfpenny*	$\frac{1}{2}d.$
2 *halfpence*	. . . . .	1 *penny*	1*d.*
12 *pence*	. . . . . .	1 *shilling*	1*s.*
20 *shillings*	. . . . .	1 *pound** or *sovereign*	£1
21 *shillings*	. . . . .	1 *guinea.*†	

219. When any quantity is made up of several others, expressed in different units, such as £1 . 14 . 6, or 2cwt. 1qr. 3lbs., it is called a *compound quantity*. From these tables it is evident that any compound quantity of any substance can be measured in several different ways. For example, the sum of money which we call five pounds four shillings is also 104 shillings, or 1248 pence, or 4992 farthings. It is easy to reduce any quantity from one of these measurements to another; and the following examples will be sufficient to shew how to apply the same process, usually called Reduction, to all sorts of quantities.

I. How many farthings are there in £18 . 12 . $6\frac{3}{4}$?‡

Since there are 20 shillings in a pound, there are, in £18, 18×20, or 360 shillings; therefore, £18 . 12 is 360+12, or 372 shillings. Since there are 12 pence in a shilling, in 372 shillings there are 372×12, or 4464 pence; and, therefore, in £18 . 12 . 6 there are 4464+6, or 4470 pence.

Since there are 4 farthings in a penny, in 4470 pence there are

* The English pound is generally called a *pound sterling*, which distinguishes it from the weight called a pound, and also from foreign coins.

† The coin called a guinea is now no longer in use, but the name is still given, from custom, to 21 shillings. The pound, which was not a coin, but a note promising to pay 20 shillings to the bearer, is also disused for the present, and the sovereign supplies its place; but the name pound is still given to 20 shillings.

‡ Farthings are never written but as parts of a penny. Thus, three farthings being $\frac{3}{4}$ of a penny, is written $\frac{3}{4}$, or ¾. One halfpenny may be written either as $\frac{2}{4}$ or $\frac{1}{2}$; the latter is most common.

4470×4, or 17880 farthings; and, therefore, in £18 . 12 . $6\frac{3}{4}$ there are 17880+3, or 17883 farthings. The whole of this process may be written as follows:

$$
\begin{array}{l}
£18 \,.\, 12 \,.\, 6\frac{3}{4} \\
\;\;\underline{20} \\
360+12 = 372 \\
\qquad\qquad\;\; \underline{12} \\
\qquad\qquad 4464+6 = 4470 \\
\qquad\qquad\qquad\qquad\quad\; \underline{4} \\
\qquad\qquad\qquad\qquad 17880+3 = 17883
\end{array}
$$

II. In 17883 farthings, how many pounds, shillings, pence, and farthings are there?

Since 17883, divided by 4, gives the quotient 4470, and the remainder 3, 17883 farthings are 4470 pence and 3 farthings (218).

Since 4470, divided by 12, gives the quotient 372, and the remainder 6, 4470 pence is 372 shillings and 6 pence.

Since 372, divided by 20, gives the quotient 18, and the remainder 12, 372 shillings is 18 pounds and 12 shillings.

Therefore, 17883 farthings is $4470\frac{3}{4}$*d.*, which is 372*s.* $6\frac{3}{4}$*d.*, which is £18 . 12 . $6\frac{3}{4}$.

The process may be written as follows:

$$
\begin{array}{r}
4)\underline{17883} \\
12)\underline{4470} \;\ldots\; 3 \\
20)\underline{372} \;\ldots\; 6 \\
£18 \,.\, 12 \,.\, 6\frac{3}{4}
\end{array}
$$

EXERCISES.

A has £100 . 4 . $11\frac{1}{2}$, and B has 64392 farthings. If A receive 1492 farthings, and B £1 . 2 . $3\frac{1}{2}$, which will then have the most, and by how much?—*Answer*, A will have £33 . 12 . 3 more than B.

In the following table the quantities written opposite to each other are the same: each line furnishes two exercises.

£15 . 18 . $9\frac{1}{2}$	15302 farthings.
115^{lbs} 1^{oz} 8^{dwt}	663072 grains.
3^{lbs} 14^{oz} 9^{dr}	1001 drams.
3^{m} 149^{yds} 2^{ft} 9^{in}	195477 inches.
19^{bu} 2^{pks} 1^{gall} 2^{qts}	1260 pints.
16^{h} 23^{m} 47^{s}	59027 seconds.

220. The same may be done where the number first expressed is fractional. For example, how many shillings and pence are there in $\frac{4}{15}$ of a pound? Now, $\frac{4}{15}$ of a pound is $\frac{4}{15}$ of 20 shillings; $\frac{4}{15}$ of 20 is $\frac{4\times20}{15}$, or $\frac{4\times4}{3}$ (110), or $\frac{16}{3}$, or (105) $5\frac{1}{3}$ of a shilling. Again, $\frac{1}{3}$ of a shilling is $\frac{1}{3}$ of 12 pence, or 4 pence. Therefore, £$\frac{4}{15}$ = 5*s*. 4*d*.

Also, ·23 of a day is ·23×24 in hours, or $5^{h}\cdot52$; and ·52 of an hour is ·52×60 in minutes, or $31^{m}\cdot2$; and ·2 of a minute is ·2×60 in seconds, or 12^{s}; whence ·23 of a day is 5^{h} 31^{m} 12^{s}.

Again, suppose it required to find what part of a pound 6*s*. 8*d*. is. Since 6*s*. 8*d*. is 80 pence, and since the whole pound contains 20×12 or 240 pence, 6*s*. 8*d*. is made by dividing the pound into 240 parts, and taking 80 of them. It is therefore £$\frac{80}{240}$ (107), but $\frac{80}{240} = \frac{1}{3}$ (108); therefore, 6*s*. 8*d*. = £$\frac{1}{3}$.

EXERCISES.

$\frac{2}{5}$ of a day	is	9^{h} 36^{m}
·12841 of a day	. . .	3^{h} 4^{m} $54^{s}\cdot624$*
·257 of a cwt.	. . .	28^{lbs} 12^{oz} $8^{dr}\cdot704$
£·14936	. . . .	2^{s} 11^{d} $3^{f}\cdot3856$

221, 222. I have thought it best to refer the mode of converting shillings, pence, and farthings into decimals of a pound to the Appendix (See Appendix *On Decimal Money*). I should strongly recommend the reader to make himself perfectly familiar with the modes given in

* When a decimal follows a whole number, the decimal is always of the same unit as the whole number. Thus, $5^{s}\cdot5$ is five *seconds* and five-tenths of a *second*. Thus, $0^{s}\cdot5$ means five-tenths of a second; $0^{h}\cdot3$, three-tenths of an hour.

that Appendix. To prevent the subsequent sections from being altered in their numbering, I have numbered this paragraph as above.

223. The rule of addition* of two compound quantities of the same sort will be evident from the following example. Suppose it required to add £192 . 14 . $2\frac{1}{2}$ to £64 . 13 . $11\frac{3}{4}$. The sum of these two is the whole of that which arises from adding their several parts. Now

$\frac{3}{4}d.$+	$\frac{1}{2}d.$ =	$\frac{5}{4}d.$ =	£0 . 0 . $1\frac{1}{4}$	(219)
11$d.$+	2$d.$ =	13$d.$ =	0 . 1 . 1	
13$s.$+	14$s.$ =	27$s.$ =	1 . 7 . 0	
£64 + £192	=		256 . 0 . 0	

The sum of all of which is £257 . 8 . $2\frac{1}{4}$

This may be done at once, and written as follows:

$$
\begin{array}{r}
£192 \;.\; 14 \;.\; 2\frac{1}{2} \\
64 \;.\; 13 \;.\; 11\frac{3}{4} \\
\hline
£257 \;.\; 8 \;.\; 2\frac{1}{4}
\end{array}
$$

Begin by adding together the farthings, and reduce the result to pence and farthings. Set down the last only, carry the first to the line of pence, and add the pence in both lines to it. Reduce the sum to shillings and pence; set down the last only, and carry the first to the line of shillings, and so on. The same method must be followed when the quantities are of any other sort; and if the tables be kept in memory, the process will be easy.

224. Subtraction is performed on the same principle as in (40), namely, that the difference of two quantities is not altered by adding the same quantity to both. Suppose it required to subtract £19 . 13 . $10\frac{3}{4}$ from £24 . 5 . $7\frac{1}{2}$. Write these quantities under one another thus:

* Before reading this article and the next, articles (29) and (42) should be read again carefully.

$$\begin{array}{r} £24 \;.\; 5 \;.\; 7\frac{1}{2} \\ 19 \;.\; 13 \;.\; 10\frac{3}{4} \end{array}$$

Since $\frac{3}{4}$ cannot be taken from $\frac{1}{2}$ or $\frac{2}{4}$, add 1*d.* to both quantities, which will not alter their difference; or, which is the same thing, add 4 farthings to the first, and 1*d.* to the second. The pence and farthings in the two lines then stand thus: $7\frac{6}{4}d.$ and $11\frac{3}{4}d.$ Now subtract $\frac{3}{4}$ from $\frac{6}{4}$, and the difference is $\frac{3}{4}$, which must be written under the farthings. Again, since 11*d.* cannot be subtracted from 7*d.*, add 1*s.* to both quantities by adding 12*d.* to the first, and 1*s.* to the second. The pence in the first line are then 19, and in the second 11, and the difference is 8, which write under the pence. Since the shillings in the lower line were increased by 1, there are now 14*s.* in the lower, and 5*s.* in the upper one. Add 20*s.* to the upper and £1 to the lower line, and the subtraction of the shillings in the second from those in the first leaves 11*s.* Again, there are now £20 in the lower, and £24 in the upper line, the difference of which is £4; therefore the whole difference of the two sums is £4 . 11 . $8\frac{3}{4}$. If we write down the two sums with all the additions which have been made, the process will stand thus:

$$\begin{array}{lr} & £24 \;.\; 25 \;.\; 19\frac{6}{4} \\ & 20 \;.\; 14 \;.\; 11\frac{3}{4} \\ \hline \text{Difference} & £4 \;.\; 11 \;.\; 8\frac{3}{4} \end{array}$$

225. The same method may be applied to any of the quantities in the tables. The following is another example:

From 7 cwt. 2 qrs. 21 lbs. 14 oz.
Subtract 2 cwt. 3 qrs. 27 lbs. 12 oz.

After alterations have been made similar to those in the last article, the question becomes:

From 7 cwt. 6 qrs. 49 lbs. 14 oz.
Subtract 3 cwt. 4 qrs. 27 lbs. 12 oz.

The difference is 4 cwt. 2 qrs. 22 lbs. 2 oz.

In this example, and almost every other, the process may be a little shortened in the following way. Here we do not subtract 27 lbs. from 21 lbs., which is impossible, but we increase 21 lbs. by 1 qr. or 28 lbs. and then subtract 27 lbs. from the sum. It would be shorter, and lead to the same result, first to subtract 27 lbs. from 1 qr. or 28 lbs. and add the difference to 21 lbs.

226. EXERCISES.

A man has the following sums to receive: £193 . 14 . $11\frac{1}{4}$, £22 . 0 . $6\frac{3}{4}$, £6473 . 0 . 0, and £49 . 14 . $4\frac{1}{2}$; and the following debts to pay: £200 . 19 . $6\frac{1}{4}$, £305 . 16 . 11. £22, and £19 . 6 . $0\frac{1}{2}$. How much will remain after paying the debts? *Answer*, £6190 . 7 . $4\frac{3}{4}$.

There are four towns, in the order A, B, C, and D. If a man can go from A to B in $5^h\ 20^m\ 33^s$, from B to C in $6^h\ 49^m\ 2^s$, and from A to D in $19^h\ 0^m\ 17^s$, how long will he be in going from B to D, and from C to D? *Answer*, $13^h\ 39^m\ 44^s$, and $6^h\ 50^m\ 42^s$.

227. In order to perform the process of MULTIPLICATION, it must be recollected that, as in (52), if a quantity be divided into several parts, and each of these parts be multiplied by a number, and the products be added, the result is the same as would arise from multiplying the whole quantity by that number.

It is required to multiply £7 . 13 . $6\frac{1}{4}$ by 13. The first quantity is made up of 7 pounds, 13 shillings, 6 pence, and 1 farthing. And

1 farth.	× 13 is	13 farth.	or	£0 . 0 . $3\frac{1}{4}$ (219)
6 pence	× 13 is	78 pence,	or	0 . 6 . 6
13 shill.	× 13 is	169 shill.	or	8 . 9 . 0
7 pounds	× 13 is	91 pounds,	or	91 . 0 . 0
		The sum of all these is		£99 . 15 . $9\frac{1}{4}$

which is therefore £7 . 13 . $6\frac{1}{4}$ × 13.

This process is usually written as follows:

$$\begin{array}{r} £7\ .\ 13\ .\ 6\frac{1}{4} \\ 13 \\ \hline £99\ .\ 15\ .\ 9\frac{1}{4} \end{array}$$

228. Division is performed upon the same principle as in (74), viz. that if a quantity be divided into any number of parts, and each part be divided by any number, the different quotients added together will make up the quotient of the whole quantity divided by that number. Suppose it required to divide £99 . 15 . $9\frac{1}{4}$ by 13. Since 99 divided by 13 gives the quotient 7, and the remainder 8, the quantity is made up of £13×7, or £91, and £8 . 15 . $9\frac{1}{4}$. The quotient of the first, 13 being the divisor, is £7: it remains to find that of the second. Since £8 is 160*s.*, £8 . 15 . $9\frac{1}{4}$ is 175*s.* $9\frac{1}{4}$*d.*, and 175 divided by 13 gives the quotient 13, and the remainder 6; that is, 175*s.* $9\frac{1}{4}$*d.* is made up of 169*s.* and 6*s.* $9\frac{1}{4}$*d.*, the quotient of the first of which is 13*s.*, and it remains to find that of the second. Since 6*s.* is 72*d.*, 6*s.* $9\frac{1}{4}$*d.* is $81\frac{1}{4}$*d.*, and 81 divided by 13 gives the quotient 6 and remainder 3; that is, $81\frac{1}{4}$*d.* is 78*d.* and $3\frac{1}{4}$*d.*, of the first of which the quotient is 6*d.* Again, since 3*d.* is $\frac{12}{4}$, or 12 farthings, $3\frac{1}{4}$*d.* is 13 farthings, the quotient of which is 1 farthing, or $\frac{1}{4}$, without remainder. We have then divided £99 . 15 . $9\frac{1}{4}$ into four parts, each of which is divisible by 13, viz. £91, 169*s.*, 78*d.*, and 13 farthings; so that the thirteenth part of this quantity is £7 . 13 . $6\frac{1}{4}$. The whole process may be written down as follows; and the same sort of process may be applied to the exercises which follow:

```
       £   s.  d.   £  s.  d.
    13)99  15  9¼(7  13  6¼
       91
        8
       20
       160+15 = 175
                13
                 45
                 39
                  6
                 12
                 72+9 = 81
                        78
                         3
                         4
                        12+1 = 13
                               13
                                0
```

Here, each of the numbers 99, 175, 81, and 13, is divided by 13 in the usual way, though the divisor is only written before the first of them.

EXERCISES.

2 cwt. 1 qr. 21 lbs. 7 oz. × 53 = 129 cwt. 1 qr. 16 lbs. 3 oz.

$2^d\ 4^h\ 3^m\ 27^s \times 109 = 236^d\ 10^h\ 16^m\ 3^s$

£27 . 10 . 8 × 569 = £15666 . 9 . 4

£7 . 4 . 8 × 123 = £889 . 14

$£166 \times \frac{8}{33} = £40 . 4 . 10\frac{6}{33}$

$£187 . 6 . 7 \times \frac{3}{100} = £5 . 12 . 4\frac{3}{4}\frac{2}{25}$

$4s.\ 6\frac{1}{2}d. \times 1121 = £254 . 11 . 2\frac{1}{2}$

4s. 4d. × 4260 = 6s. 6d. × 2840

229. Suppose it required to find how many times 1s. $4\frac{1}{4}d.$ is contained in £3 . 19 . $10\frac{3}{4}$. The way to do this is to find the number of farthings in each. By (219), in the first there are 65, and in the second 3835 farthings. Now, 3835 contains 65 59 times; and therefore the second quantity is 59 times as great as the first. In the case, however, of pounds, shillings, and pence, it would be best to use decimals of a pound, which will give a sufficiently exact answer. Thus 1s. $4\frac{1}{4}d.$ is £·067, and £3 . 19 . $10\frac{3}{4}$ is £3·994, and 3·994 divided by ·067 is 3994 by 67, or $59\frac{41}{67}$. This is an extreme case, for the smaller the divisor, the greater the effect of an error in a given place of decimals.

EXERCISES.

How many times does 6 cwt. 2 qrs. contain 1 qr. 14 lbs. 1 oz.? and $1^d\ 2^h\ 0^m\ 47^s$ contain $3^m\ 46^s$? *Answer*, 17·30758 and 414·367257.

If 2 cwt. 3 qrs. 1 lb. cost £150 . 13 . 10, how much does 1 lb. cost?

Answer, 9s. 9d. $\frac{13}{309}$.

A grocer mixes 2 cwt. 15 lbs. of sugar at 11*d.* per pound with 14 cwt. 3 lbs. at 5*d.* per pound. At how much per pound must he sell the mixture so as not to lose by mixing them? *Answer*, 5d. $\frac{3}{4}\frac{153}{905}$.

230. There is a convenient method of multiplication called PRACTICE. Suppose I ask, How much do 153 tons cost if each ton cost £2 . 15 . $7\frac{1}{2}$? It is plain that if this sum be multiplied by 153, the

product is the price of the whole. But this is also evident, that, if I buy 153 tons at £2 . 15 . $7\frac{1}{2}$ each ton, payment may be made by first putting down £2 for each ton, then 10*s*. for each, then 5*s*., then 6*d*., and then $1\frac{1}{2}d$. These sums together make up £2 . 15 . $7\frac{1}{2}$, and the reason for this separation of £2 . 15 . $7\frac{1}{2}$ into different parts will be soon apparent. The process may be carried on as follows:

1. 153 tons, at £2 each ton, will cost	£306	0	0
2. Since 10*s*. is £$\frac{1}{2}$, 153 tons, at 10*s*. each, will cost £$\frac{153}{2}$, which is	76	10	0
3. Since 5*s*. is $\frac{1}{2}$ of 10*s*., 153 tons, at 5*s*., will cost half as much as the same number at 10*s*. each, that is, $\frac{1}{2}$ of £76 . 10, which is	38	5	0
4. Since 6*d*. is $\frac{1}{10}$ of 5*s*., 153 tons, at 6*d*. each, will cost $\frac{1}{10}$ of what the same number costs at 5*s*. each, that is, $\frac{1}{10}$ of £38 . 5, which is	3	16	6
5. Since $1\frac{1}{2}$ or 3 halfpence is $\frac{1}{4}$ of 6*d*. or 12 halfpence, 153 tons, at $1\frac{1}{2}d$. each, will cost $\frac{1}{4}$ of what the same number costs at 6*d*. each, that is, $\frac{1}{4}$ of £3 . 16 . 6, which is	0	19	$1\frac{1}{2}$
The sum of all these quantities is	425	10	$7\frac{1}{2}$

which is, therefore, £2 . 15 . $7\frac{1}{2} \times 153$.

The whole process may be written down as follows:

	£153 0 0		£1 per ton.
£2 is 2 × £1	306 0 0	or what 153 tons would cost at	2 0 0
10*s*. is $\frac{1}{2}$ of £1	76 10 0		0 10 0
5*s*. is $\frac{1}{2}$ of 10*s*.	38 5 0		0 5 0
6*d*. is $\frac{1}{10}$ of 5*s*.	3 16 6		0 0 6
$1\frac{1}{2}d$. is $\frac{1}{4}$ of 6*d*.	0 19 $1\frac{1}{2}$		0 0 $1\frac{1}{2}$
Sum . . .	£425 10 $7\frac{1}{2}$		£2 15 $7\frac{1}{2}$

ANOTHER EXAMPLE.

What do 1735 lbs. cost at 9*s.* $10\frac{3}{4}$*d.* per lb.? The price 9*s.* $10\frac{3}{4}$*d.* is made up of 5*s.*, 4*s.*, 10*d.*, $\frac{1}{2}$*d.*, and $\frac{1}{4}$*d.*; of which 5*s.* is $\frac{1}{4}$ of £1, 4*s.* is $\frac{1}{5}$ of £1, 10*d.* is $\frac{1}{6}$ of 5*s.*, $\frac{1}{2}$*d.* is $\frac{1}{20}$ of 10*d.*, and $\frac{1}{4}$*d.* is $\frac{1}{2}$ of $\frac{1}{2}$*d.* Follow the same method as in the last example, which gives the following:

	£1735 0 0	or what 1735 lbs. would cost at	£1 per lb.
5*s.* is $\frac{1}{4}$ of £1	433 15 0		0 5 0
4*s.* is $\frac{1}{5}$ of £1	347 0 0		0 4 0
10*d.* is $\frac{1}{6}$ of 5*s.*	72 5 10		0 0 10
$\frac{1}{2}$*d.* is $\frac{1}{20}$ of 10*d.*	3 12 $3\frac{1}{2}$		0 0 $0\frac{1}{2}$
$\frac{1}{4}$*d.* is $\frac{1}{2}$ of $\frac{1}{2}$*d.*	1 16 $1\frac{3}{4}$		0 0 $0\frac{1}{4}$
by addition ...	£858 9 $3\frac{1}{4}$		£0 9 $10\frac{3}{4}$

In all cases, the price must first be divided into a number of parts, each of which is a simple fraction* of some one which goes before. No rule can be given for doing this, but practice will enable the student immediately to find out the best method for each case. When that is done, he must find how much the whole quantity would cost if each of these parts were the price, and then add the results together.

EXERCISES.

What is the cost of

243 cwt. at £14 . 18 . $8\frac{1}{4}$ per cwt.?—*Answer*, £3629 . 1 . $0\frac{3}{4}$.

169 bushels at £2 . 1 . $3\frac{1}{4}$ per bushel?—*Answer*, £348 . 14 . $9\frac{1}{4}$.

273 qrs. at 19*s.* 2*d.* per quarter?—*Answer*, £261 . 12 . 6.

2627 sacks at 7*s.* $8\frac{1}{2}$*d.* per sack?—*Answer*, £1012 . 9 . $9\frac{1}{2}$.

* Any fraction of a unit, whose numerator is unity, is generally called an *aliquot part* of that unit. Thus, 2*s.* and 10*s.* are both aliquot parts of a pound, being £$\frac{1}{10}$ and £$\frac{1}{2}$.

231. Throughout this section it must be observed, that the rules can be applied to cases where the quantities given are expressed in common or decimal fractions, instead of the measures in the tables. The following are examples:

What is the price of 272·3479 cwt. at £2 . 1 . $3\frac{1}{2}$ per cwt.?

Answer, £562·2849, or £562 . 5 . $8\frac{1}{4}$.

$66\frac{1}{2}$lbs. at 1*s*. $4\frac{1}{2}$*d*. per lb. cost £4 . 11 . $5\frac{1}{4}$.

How many pounds, shillings, and pence, will 279·301 acres let for if each acre lets for £3·1076?—*Answer*, £867·9558, or £867 . 19 . $1\frac{1}{4}$.

What does $\frac{1}{4}$ of $\frac{3}{13}$ of 17 bush. cost at $\frac{1}{5}$ of $\frac{2}{3}$ of £17 . 14 per bushel?

Answer, £2·3146, or £2 . 6 . $3\frac{1}{2}$.

What is the cost of 19lbs. 8oz. 12dwt. 8gr. at £4 . 4 . 6 per ounce?—*Answer*, £999 . 14 . $1\frac{1}{4}\frac{1}{5}$.

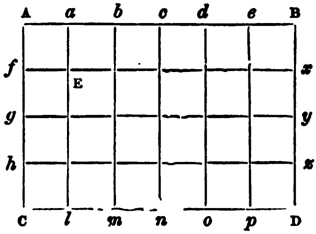
232. It is often required to find to how much a certain sum per day will amount in a year. This may be shortly done, since it happens that the number of days in a year is 240+120+5; so that a penny per day is a pound, half a pound, and 5 pence per year. Hence the following rule: To find how much any sum per day amounts to in a year, turn it into pence and fractions of a penny; to this add the half of itself, and let the pence be pounds, and each farthing five shillings; then add five times the daily sum, and the total is the yearly amount. For example, what does 12*s*. $3\frac{3}{4}$*d*. amount to in a year? This is $147\frac{3}{4}$*d*., and its half is $73\frac{7}{8}$*d*., which added to $147\frac{3}{4}$*d*. gives $221\frac{5}{8}$*d*., which turned into pounds is £221 . 12 . 6. Also, 12*s*. $3\frac{3}{4}d.\times 5$ is £3 . 1 . $6\frac{3}{4}$, which added to the former sum gives £224 . 14 . $0\frac{3}{4}$ for the yearly amount. In the same way the yearly amount of 2*s*. $3\frac{1}{2}$*d*. is £41 . 16 . $5\frac{1}{2}$; that of $6\frac{3}{4}$*d*. is £10 . 5 . $3\frac{3}{4}$; and that of 11*d*. is £16 . 14 . 7.

233. An inverse rule may be formed, sufficiently correct for every purpose, in the following way: If the year consisted of 360 days, or $\frac{3}{2}$ of 240, the subtraction of one-third from any sum per year would give the proportion which belongs to 240 days; and every pound so obtained would be one penny per day. But as the year is not 360, but 365 days, if we divide each day's share into 365 parts, and take 5 away, the whole of the subtracted sum, or 360×5 such parts, will give 360 parts for each

of the 5 days which we neglected at first. But 360 such parts are left behind for each of the 360 first days; therefore, this additional process divides the whole annual amount equally among the 365 days. Now, 5 parts out of 365 is one out of 73, or the 73d part of the first result must be subtracted from it to produce the true result. Unless the daily sum be very large, the 72d part will do equally well, which, as 72 farthings are 18 pence, is equivalent to subtracting at the rate of one farthing for 18*d.*, or $\frac{1}{2}$*d.* for 3*s.*, or 10*d.* for £3. The rule, then, is as follows: To find how much per day will produce a given sum per year, turn the shillings, &c. in the given sum into decimals of a pound (221); subtract one-third; consider the result as pence; and diminish it by one farthing for every eighteen pence, or ten pence for every £3. For example, how much per day will give £224 . 14 . $0\frac{3}{4}$ per year? This is 224·703, and its third is 74·901, which subtracted from 224·703, gives 149·802, which, if they be pence, amounts to 12*s.* 5·802*d.*, in which 1*s.* 6*d.* is contained 8 times. Subtract 8 farthings, or 2*d.*, and we have 12*s.* 3·802*d.*, which differs from the truth only about $\frac{1}{20}$ of a farthing. In the same way, £100 per year is 5*s.* $5\frac{3}{4}$*d.* per day.

234. The following connexion between the measures of length and the measures of surface is the foundation of the application of arithmetic to geometry.

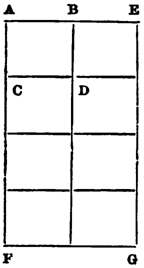
Suppose an oblong figure, A, B, C, D, as here drawn (which is called a *rectangle* in geometry), with the side A B 6 inches, and the side A C 4



inches. Divide A B and C D (which are equal) each into 6 inches by the points *a*, *b*, *c*, *l*, *m*, &c.; and A C and B D (which are also equal) into 4 inches by the points *f*, *g*, *h*, *x*, *y*, and *z*. Join *a* and *l*, *b* and *m*,

&c., and f and x, &c. Then, the figure A B C D is divided into a number of squares; for a square is a rectangle whose sides are equal, and therefore A a f E is square, since A a is of the same length as A f, both being 1 inch. There are also four rows of these squares, with six squares in each row; that is, there are 6×4, or 24 squares altogether. Each of these squares has its sides 1 inch in length, and is what was called in (215) *a square inch*. By the same reasoning, if one side had contained 6 *yards*, and the other 4 *yards*, the surface would have contained 6×4 *square yards;* and so on.

235. Let us now suppose that the sides of A B C D, instead of being a whole number of inches, contain some inches and a fraction. For example, let A B be $3\frac{1}{2}$ inches, or (114) $\frac{7}{2}$ of an inch, and let A C contain $2\frac{1}{4}$ inches, or $\frac{9}{4}$ of an inch. Draw A E twice as long as A B, and A F four times as long as A C, and complete the rectangle A E F G. The rest of the figure needs no description. Then, since A E is twice A B, or twice $\frac{7}{2}$ inches, it is 7 inches. And since A F is four times A C, or four times $\frac{9}{4}$ inches, it is 9 inches. Therefore, the whole rectangle A E F G contains, by (234), 7×9 or 63 square inches. But the rectangle A E F G contains 8 rectangles, all of the same figure as A B C D; and therefore A B C D is one-eighth part of A E F G, and contains $\frac{63}{8}$ square inches. But $\frac{63}{8}$ is made by multiplying $\frac{9}{4}$ and $\frac{7}{2}$ together (118). From this and the last article it appears, that, whether the sides of a rectangle be a whole or a fractional number of inches, the number of square inches in its surface is the product of the numbers of inches in its sides. The square itself is a rectangle whose sides are all equal, and therefore the number of square inches which a square contains is found by multiplying the number of inches in its side by itself. For example, a square whose side is 13 inches in length contains 13×13 or 169 square inches.



236. EXERCISES.

What is the content, in square feet and inches, of a room whose sides are 42 ft. 5 inch. and 31 ft. 9 inch.? and supposing the piece from which its carpet is taken to be three quarters of a yard in breadth, what length of it must be cut off?—*Answer*, The content is 1346 square feet 105 square inches, and the length of carpet required is 598 feet $6\frac{5}{9}$ inches.

The sides of a rectangular field are 253 yards and a quarter of a mile; how many acres does it contain?—*Answer*, 23.

What is the difference between 18 *square miles*, and a square of 18 miles long, or 18 *miles square*?—*Answer*, 306 square miles.

237. It is by this rule that the measure in (215) is deduced from that in (214); for it is evident that twelve inches being a foot, the square foot is 12×12 or 144 square inches, and so on. In a similar way it may be shewn that the content in cubic inches of a cube, or parallelepiped,* may be found by multiplying together the number of inches in those three sides which meet in a point. Thus, a cube of 6 inches contains 6×6×6, or 216 cubic inches; a chest whose sides are 6, 8, and 5 feet, contains 6×8×5, or 240 cubic feet. By this rule the measure in (216) was deduced from that in (214).

SECTION II.

RULE OF THREE.

238. Suppose it required to find what 156 yards will cost, if 22 yards cost 17*s*. 4*d*. This quantity, reduced to pence, is 208*d*.; and if 22 yards cost 208*d*., each yard costs $\frac{208}{22}d$. But 156 yards cost 156 times the price of one yard, and therefore cost $\frac{208}{22}\times 156$ pence, or $\frac{208\times 156}{22}$ pence (117). Again, if $25\frac{1}{2}$ French francs be 20 shillings sterling, how many francs are in £20 . 15? Since $25\frac{1}{2}$ francs are 20 shillings, twice the number of francs must be twice the number of shillings; that is, 51 francs are 40 shillings, and one shilling is the

* A parallelepiped, or more properly, a *rectangular* parallelepiped, is a figure of the form of a brick; its sides, however, may be of any length; thus, the figure of a plank has the same name. A cube is a parallelepiped with equal sides, such as is a die.

fortieth part of 51 francs, or $\frac{51}{40}$ francs. But £20 15*s*. contain 415 shillings (219); and since 1 shilling is $\frac{51}{40}$ francs, 415 shillings is $\frac{51}{40}$ ×415 francs, or (117) $\frac{51\times415}{40}$ francs.

239. Such questions as the last two belong to the most extensive rule in Commercial Arithmetic, which is called the RULE OF THREE, because in it three quantities are given, and a fourth is required to be found. From both the preceding examples the following rule may be deduced, which the same reasoning will shew to apply to all similar cases.

It must be observed, that in these questions there are two quantities which are of the same sort, and a third of another sort, of which last the answer must be. Thus, in the first question there are 22 and 156 yards and 208 pence, and the thing required to be found is a number of pence. In the second question there are 20 and 415 shillings and $25\frac{1}{2}$ francs, and what is to be found is a number of francs. Write the three quantities in a line, putting that one last which is the only one of its kind, and that one first which is connected with the last in the question.* Put the third quantity in the middle. In the first question the quantities will be placed thus:

22 yds. 156 yds. 17*s*. 4*d*.

In the second question they will be placed thus:

20*s*. £20 15*s*. $25\frac{1}{2}$ francs.

Reduce the first and second quantities, if necessary, to quantities of the same denomination. Thus, in the second question, £20 15*s*. must be reduced to shillings (219). The third quantity may also be reduced to any other denomination, if convenient; or the first and third may be multiplied by any quantity we please, as was done in the second

* This generally comes in the same member of the sentence. In some cases the ingenuity of the student must be employed in detecting it. The reasoning of (238) is the best guide. The following may be very often applied. If it be evident that the answer must be less than the given quantity of its kind, multiply that given quantity by the less of the other two; if greater, by the greater. Thus, in the first question, 156 yards must cost more than 22; multiply, therefore, by 156.

question; and, on looking at the answer in (238), and at (108), it will be seen that no change is made by that multiplication. Multiply the second and third quantities together, and divide by the first. The result is a quantity of the same sort as the third in the line, and is the answer required. Thus, to the first question the answer is (238) $\frac{208 \times 156}{22}$ pence, or, which is the same thing, $\frac{17s.\ 4d. \times 156}{22}$.

240. The whole process in the first question is as follows:*

```
yds.     yds.      s.  d.
 22  :   156  ::  17 . 4
                  12
                 ----
                 208 pence.
                 156
                ----
                1248
               1040
               208
               -----
          22)32448(1474 3/4 d. and 14/22, or 7/11 of a farthing,
             22                or (219) £6 . 2 . 10 3/4 7/11.
             ---
             104
              88
             ---
              164
              154
              ---
               108
                88
               ---
                20
  (228)          4
                ---
                80
                66
                --
                14
```

The question might have been solved without reducing 17s. 4d. to pence, thus:

* It is usual to place points, in the manner here shewn, between the quantities. Those who have read Section VIII. will see that the Rule of Three is no more than the process for finding the fourth term of a proportion from the other three.

yds.		yds.		s. d.	
22	:	156	::	17 . 4	
				156	(227)

22)£135 . 4 . 0(£6 . 2 . $10\frac{3}{4}\frac{7}{11}$ (228)
132

$3\times 20+4 = 64$
44

$20\times 12 = 240$
220

$20\times 4 = 80$
66

14

The student must learn by practice which is the most convenient method for any particular case, as no rule can be given.

241. It may happen that the three given quantities are all of one denomination; nevertheless it will be found that two of them are of one, and the third of another sort. For example: What must an income of £400 pay towards an income-tax of 4*s.* 6*d.* in the pound? Here the three given quantities are, £400, 4*s.* 6*d.*, and £1, which are all of the same species, viz. money. Nevertheless, the first and third are income; the second is a tax, and the answer is also a tax; and therefore, by (152), the quantities must be placed thus:

£1 : £400 :: 4*s.* 6*d.*

242. The following exercises either depend directly upon this rule, or can be shewn to do so by a little consideration. There are many questions of the sort, which will require some exercise of ingenuity before the method of applying the rule can be found.

EXERCISES.

If 15 cwt. 2 qrs. cost £198 . 15 . 4, what does 1 qr. 22 lbs. cost?

Answer, £5 . 14 . $5\frac{3}{4}\frac{185}{217}$.

If a horse go 14 m. 3 fur. 27 yds. in $3^h\ 26^m\ 12^s$, how long will he be in going 23 miles? *Answer*, $5^h\ 29^m\ 34^s\ \frac{2462}{25327}$.

Two persons, A and B, are bankrupts, and owe exactly the same sum; A can pay 15*s.* $4\frac{1}{2}$*d.* in the pound, and B only 7*s.* $6\frac{3}{4}$*d.* At the same time A has in his possession £1304 . 17 more than B; what do the debts of each amount to? *Answer*, £3340 . 8 . $3\frac{3}{4}\frac{9}{25}$.

For every $12\frac{1}{2}$ acres which one country contains, a second contains $56\frac{1}{4}$. The second country contains 17,300 square miles. How much does the first contain? Again, for every 3 people in the first, there are 5 in the second; and there are in the first 27 people on every 20 acres. How many are there in each country?—*Answer*, The number of square miles in the first is $3844\frac{4}{9}$, and its population 3,321,600; and the population of the second is 5,536,000.

If $42\frac{1}{2}$ yds. of cloth, 18 in. wide, cost £59 . 14 . 2, how much will $118\frac{1}{4}$ yds. cost, if the width be 1 yd.? *Answer*, £332 . 5 . $2\frac{4}{17}$.

If £9 . 3 . 6 last six weeks, how long will £100 last?

Answer, $65\frac{145}{367}$ weeks.

How much sugar, worth $9\frac{3}{4}$*d.* a pound, must be given for 2 cwt. of tea, worth 10*d.* an ounce? *Answer*, 32 cwt. 3 qrs. 7 lbs. $\frac{35}{39}$.

243. Suppose the following question asked: How long will it take 15 men to do that which 45 men can finish in 10 days? It is evident that one man would take 45×10, or 450 days, to do the same thing, and that 15 men would do it in one-fifteenth part of the time which it employs one man, that is, in $\frac{450}{15}$, or 30 days. By this and similar reasoning the following questions can be solved.

EXERCISES.

If 15 oxen eat an acre of grass in 12 days, how long will it take 26 oxen to eat 14 acres? *Answer*, $96\frac{12}{13}$ days.

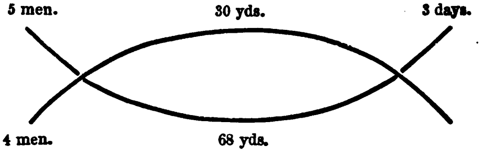
If 22 masons build a wall 5 feet high in 6 days, how long will it take 43 masons to build 10 feet? *Answer*, $6\frac{6}{43}$ days.

244. The questions in the preceding article form part of a more general class of questions, whose solution is called the Double Rule of Three, but which might, with more correctness, be called the Rule of *Five*, since five quantities are given, and a sixth is to be found. The following is an example: If 5 men can make 30 yards of cloth in 3 days, how long will it take 4 men to make 68 yards? The first

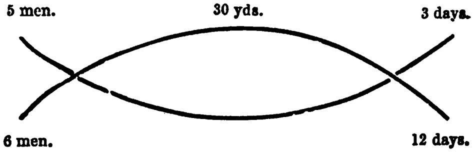
thing to be done is to find out, from the first part of the question, the time it will take one man to make one yard. Now, since one man, in 3 days, will do the fifth part of what 5 men can do, he will in 3 days make $\frac{30}{5}$, or 6 yards. He will, therefore, make one yard in $\frac{3}{6}$ or in $\frac{3\times5}{30}$ of a day. From this we are to find how long it will take 4 men to make 68 yards. Since one man makes a yard in $\frac{3\times5}{30}$ of a day, he will make 68 yards in $\frac{3\times5}{30}\times68$ days, or (116) in $\frac{3\times5\times68}{30}$ days; and 4 men will do this in one-fourth of the time, that is (123), in $\frac{3\times5\times68}{30\times4}$ days, or in $8\frac{1}{2}$ days.

Again, suppose the question to be: If 5 men can make 30 yards in 3 days, how much can 6 men do in 12 days? Here we must first find the quantity one man can do in one day, which appears, on reasoning similar to that in the last example, to be $\frac{30}{3\times5}$ yards. Hence, 6 men, in one day, will make $\frac{6\times30}{5\times3}$ yards, and in 12 days will make $\frac{12\times6\times30}{5\times3}$ or 144 yards.

From these examples the following rule may be drawn. Write the given quantities in two lines, keeping quantities of the same sort under one another, and those which are connected with each other, in the same line. In the two examples above given, the quantities must be written thus:



SECOND EXAMPLE.



Draw a curve through the middle of each line, and the extremities of the other. There will be three quantities on one curve and two on the other. Divide the product of the three by the product of the two, and the quotient is the answer to the question.

If necessary, the quantities in each line must be reduced to more simple denominations (219), as was done in the common Rule of Three (238).

EXERCISES.

If 6 horses can, in 2 days, plough 17 acres, how many acres will 93 horses plough in $4\frac{1}{2}$ days? *Answer*, $592\frac{7}{8}$.

If 20 men, in $3\frac{1}{4}$ days, can dig 7 rectangular fields, the sides of each of which are 40 and 50 yards, how long will 37 men be in digging 53 fields, the sides of each of which are 90 and $125\frac{1}{2}$ yards?

Answer, $75\frac{2451}{20720}$ days.

If the carriage of 60 cwt. through 20 miles cost £14 10*s*., what weight ought to be carried 30 miles for £5 . 8 . 9? *Answer*, 15 cwt.

If £100 gain £5 in a year, how much will £850 gain in 3 years and 8 months? *Answer*, £155 . 16 . 8.

SECTION III.

INTEREST, ETC.

245. In the questions contained in this Section, almost the only process which will be employed is the taking a fractional part of a sum of money, which has been done before in several cases. Suppose it required to take 7 parts out of 40 from £16, that is, to divide £16 into 40 equal parts, and take 7 of them. Each of these parts is £$\frac{16}{40}$, and 7 of them make $\frac{16}{40}\times 7$, or $\frac{16\times 7}{40}$ pounds (116). The process may be written as below:

```
      £16
        7
  40)112(£2 . 16s.
      80
      --
      32
       20
      ---
      640
      40
      ---
      240
      240
      ---
        0
```

Suppose it required to take 13 parts out of a hundred from £56 . 13 . $7\frac{1}{2}$.

$$56 \,.\, 13 \,.\, 7\frac{1}{2}$$
$$13$$
$$100)736 \,.\, 17 \,.\, 1\frac{1}{2}(£7 \,.\, 7 \,.\, 4\frac{1}{4}\frac{41}{50}$$
$$700$$
$$36 \times 20 + 17 = 737$$
$$700$$
$$37 \times 12 + 1 = 445$$
$$400$$
$$45 \times 4 + 2 = 182$$
$$100$$
$$82$$

Let it be required to take $2\frac{1}{2}$ parts out of a hundred from £3 12*s.* The result, by the same rule is $\frac{£3\ 12s. \times 2\frac{1}{2}}{100}$, or (123) $\frac{£3\ 12s. \times 5}{200}$; so that taking $2\frac{1}{2}$ out of a hundred is the same as taking 5 parts out of 200.

EXERCISES.

Take $7\frac{1}{3}$ parts out of 53 from £1 10*s.* *Answer*, 4*s.* $1\frac{129}{159}$*d.*

Take 5 parts out of 100 from £107 13*s.* $4\frac{3}{4}$*d.*

Answer, £5 . 7 . 8 and $\frac{3}{20}$ of a farthing.

£56 3*s.* 2*d.* is equally divided among 32 persons. How much does the share of 23 of them exceed that of the rest?

Answer, £24 . 11 . $4\frac{1}{2}\frac{1}{2}$.

246. It is usual, in mercantile business, to mention the fraction which one sum is of another, by saying how many parts out of a hundred must be taken from the second in order to make the first. Thus, instead of saying that £16 12*s.* is the half of £33 4*s.*, it is said that the first is 50 per cent of the second. Thus, £5 is $2\frac{1}{2}$ per cent of £200; because, if £200 be divided into 100 parts, $2\frac{1}{2}$ of those parts are £5. Also, £13 is 150 per cent of £8 . 13 . 4, since the first is the second and half the second. Suppose it asked, How much per cent is 23

parts out of 56 of any sum? The question amounts to this: If he who has £56 gets £100 for them, how much will he who has 23 receive? This, by (238), is $\frac{23\times100}{56}$, or $\frac{2300}{56}$, or $41\frac{1}{14}$. Hence, 23 out of 56 is $41\frac{1}{14}$ per cent.

Similarly 16 parts out of 18 is $\frac{16\times100}{18}$, or $88\frac{8}{9}$ per cent, and 2 parts out of 5 is $\frac{2\times100}{5}$, or 40 per cent.

From which the method of reducing other fractions to the rate per cent is evident.

Suppose it asked, How much per cent is £6 . 12 . 2 of £12 . 3? Since the first contains 1586*d.*, and the second 2916*d.*, the first is 1586 out of 2916 parts of the second; that is, by the last rule, it is $\frac{158600}{2916}$, or $54\frac{1136}{2916}$, or £54 . 7 . $9\frac{1}{2}$ per cent, very nearly. The more expeditious way of doing this is to reduce the shillings, &c. to decimals of a pound. Three decimal places will give the rate per cent to the nearest shilling, which is near enough for all practical purposes. For instance, in the last example, which is to find how much £6·608 is of £12·15, 6·608×100 is 660·8, which divided by 12·15 gives £54·38, or £54 . 7. Greater correctness may be had, if necessary, as in the Appendix.

EXERCISES.

How much per cent is $198\frac{1}{4}$ out of 233 parts?—*Ans.* £85 . 1 . $8\frac{3}{4}$.

Goods which are bought for £193 . 12, are sold for £216 . 13 . 4; how much per cent has been gained by them?

Answer, A little less than £11 . 18 . 6.

A sells goods for B to the amount of £230 . 12, and is allowed a commission* of 3 per cent; what does that amount to?

Answer, £6 . 18 $4\frac{1}{4}\frac{7}{25}$.

A stockbroker buys £1700 stock, brokerage being at £$\frac{1}{8}$ per cent; what does he receive?—*Answer*, £2 . 2 . 6.

* Commission is what is allowed by one merchant to another for buying or selling goods for him, and is usually a per-centage on the whole sum employed. Brokerage is an allowance similar to commission, under a different name, principally used in the buying and selling of stock in the funds.

Insurance is a per-centage paid to those who engage to make good to the payers

A ship whose value is £15,423 is insured at $19\frac{2}{3}$ per cent, what does the insurance amount to?—*Answer*, £3033 . 3 . $9\frac{1}{2}\frac{2}{5}$.

247. In reckoning how much a bankrupt is able to pay his creditors, as also to how much a tax or rate amounts, it is usual to find how many shillings in the pound is paid. Thus, if a person who owes £100 can only pay £50, he is said to pay 10*s.* in the pound. The rule is easily derived from the same reasoning as in (246). For example, £50 out of £82 is £$\frac{50}{82}$ out of £1, or $\frac{50\times20}{82}$ shillings, or 12*s.* $2\frac{1}{4}\frac{15}{41}$ in the pound.

248. Interest is money paid for the use of other money, and is always a per-centage upon the sum lent. It may be paid either yearly, half-yearly, or quarterly; but when it is said that £100 is lent at 4 per cent, it must be understood to mean 4 per cent per annum; that is, that 4 pounds are paid every year for the use of £100

The sum lent is called the *principal*, and the interest upon it is of two kinds. If the borrower pay the interest as soon as, from the agreement, it becomes due, it is evident that he has to pay the same sum every year; and that the whole of the interest which he has to pay in any number of years is one year's interest multiplied by the number of years. But if he do not pay the interest at once, but keeps it in his hands until he returns the principal, he will then have more of his creditor's money in his hands every year, and (if it were so agreed) will have to pay interest upon each year's interest for the time during which he keeps it after it becomes due. In the first case, the interest is called *simple*, and in the second *compound*. The interest and principal together are called the *amount*.

249. What is the simple interest of £1049 . 16 . 6 for 6 years and one-third, at $4\frac{1}{2}$ per cent? This interest must be $6\frac{1}{3}$ times the interest

any loss they may sustain by accidents from fire, or storms, according to the agreement, up to a certain amount which is named, and is a per-centage upon this amount. Tare, tret, and cloff, are allowances made in selling goods by wholesale, for the weight of the boxes or barrels which contain them, waste, &c.; and are usually either the price of a certain number of pounds of the goods for each box or barrel, or a certain allowance on each cwt.

of the same sum for one year, which (245) is found by multiplying the sum by $4\frac{1}{2}$, and dividing by 100. The process is as follows:

$$
\begin{array}{lr|l}
(230) & (a) & £1049 . 16 . 6 \\
\hline
 & a\times 4 & 4199 . \ 6 . 0 \\
 & a\times\frac{1}{2} & 524 . 18 . 3 \\
\hline
\end{array}
$$

$$
\begin{array}{ll}
(82) & 100)47,24 . 4 . 3(£47 . 4 . 10\frac{11}{100} \\
 & \quad\ \ 20 \\
(228) & \quad\ \ 4,84^{*} \\
 & \quad\ \ 12 \\
 & \quad\ \ 10,11^{\dagger}
\end{array}
$$

$$
\begin{array}{lr}
(b) & £47 . \ 4 . 10\frac{11}{100} \text{ Int. for one yr.} \\
\hline
b\times 6 & 283 . \ 9 . \ 0\frac{66}{100} \\
b\times\frac{1}{3} & 15 . 14 . 11\frac{37}{100} \\
\hline
 & £299 . \ 4 . \ 0\frac{3}{100} \text{ Int. for } 6\frac{1}{3} \text{ yrs.}
\end{array}
$$

EXERCISES.

What is the interest of £105 . 6 . 2 for 19 years and 7 weeks at 3 per cent? *Answer*, £60 . 9, very nearly.

What is the difference between the interest of £50 . 19 for 7 years at 3 per cent, and for 8 years at $2\frac{1}{2}$ per cent? *Answer*, 10*s*. $2\frac{1}{2}$*d.*

What is the interest of £157 . 17 . 6 for one year at 5 per cent? *Answer*, £7 . 17 . $10\frac{1}{2}$.

Shew that the interest of any sum for 9 years at 4 per cent is the same as that of the same sum for 4 years at 9 per cent?

250. In order to find the interest of any sum at compound interest, it is necessary to find the amount of the principal and interest at the end of every year; because in this case (248) it is the amount of both

* Here the 4*s*. from the dividend is taken in.

† Here the 3*d*. from the dividend is taken in.

principal and interest at the end of the first year, upon which interest accumulates during the second year. Suppose, for example, it is required to find the interest, for 3 years, on £100, at 5 per cent, compound interest. The following is the process:

	£100	First principal.
	5	First year's interest.
	105	Amount at the end of the first year.
(249)	5 . 5	Interest for the second year on £105.
	110 . 5	Amount at the end of two years.
	5 . 10 . 3	Interest due for the third year.
	115 . 15 . 3	Amount at the end of three years.
	100 . 0 . 0	First principal.
	15 . 15 . 3	Interest gained in the three years.

When the number of years is great, and the sum considerable, this process is very troublesome; on which account tables* are constructed to shew the amount of one pound, for different numbers of years, at different rates of interest. To make use of these tables in the present example, look into the column headed "5 per cent;" and opposite to the number 3, in the column headed "Number of years," is found 1·157625; meaning that £1 will become £1·157625 in 3 years. Now, £100 must become 100 times as great; and 1·157625×100 is 115·7625 (141); but (221) £·7625 is 15*s.* 3*d.*; therefore the whole amount of £100 is £115 . 15 . 3, as before.

251. Suppose that a sum of money has lain at simple interest 4 years, at 5 per cent, and has, with its interest, amounted to £350; it is required to find what the sum was at first. Whatever the sum was, if we suppose it divided into 100 parts, 5 of those parts were added every year for 4 years, as interest; that is, 20 of those parts have been added to the first sum to make £350. If, therefore, £350 be divided into 120 parts, 100 of those parts are the principal which we want to find,

* Sufficient tables for all common purposes are contained in the article on Interest in the Penny Cyclopædia; and ample ones in the Treatise on Annuities and Reversions, in the Library of Useful Knowledge.

and 20 parts are interest upon it; that is, the principal is £$\frac{350 \times 100}{150}$, or £291 . 13 . 4.

252. Suppose that A was engaged to pay B £350 at the end of four years from this time, and that it is agreed between them that the debt shall be paid immediately; suppose, also, that money can be employed at 5 per cent, simple interest; it is plain that A ought not to pay the whole sum, £350, because, if he did, he would lose 4 years' interest of the money, and B would gain it. It is fair, therefore, that he should only pay to B as much as will, *with interest*, amount in four years to £350, that is (251), £291 . 13 . 4. Therefore, £58 . 6 . 8 must be struck off the debt in consideration of its being paid before the time. This is called DISCOUNT;* and £291 . 13 . 4 is called the *present value* of £350 due four years hence, discount being at 5 per cent. The rule for finding the present value of a sum of money (251) is: Multiply the sum by 100, and divide the product by 100 increased by the product of the rate per cent and number of years. If the time that the debt has yet to run be expressed in years and months, or months only, the months must be reduced to the equivalent fraction of a year.

EXERCISES.

What is the discount on a bill of £138 . 14 . 4, due 2 years hence, discount being at $4\frac{1}{2}$ per cent? *Answer*, £11 . 9 . 1.

What is the present value of £1031 . 17, due 6 months hence, interest being at 3 per cent? *Answer*, £1016 . 12.

253. If we multiply by $a+b$, or by $a-b$, when we should multiply by a, the result is wrong by the fraction $\frac{b}{a+b}$, or $\frac{b}{a-b}$, of itself: being too great in the first case, and too small in the second. Again, if we divide by $a+b$, where we should have divided by a, the result is too small by the fraction $\frac{b}{a}$ of itself; while, if we divide by $a-b$ instead of a, the result is too great by the same fraction of itself. Thus, if we divide by 20 instead of 17, the result is $\frac{3}{17}$ of itself too small; and if

* This rule is obsolete in business. When a bill, for instance, of £100 having a year to run, is *discounted* (as people now say) at 5 per cent, this means that 5 per cent of £100, or £5, is struck off.

we divide by 360 instead of 365, the result is too great by $\frac{5}{365}$, or $\frac{1}{73}$ of itself.

If, then, we wish to find the interest of a sum of money for a portion of a year, and have not the assistance of tables, it will be found convenient to suppose the year to contain only 360 days, in which case its 73d part (the 72d part will generally do) must be subtracted from the result, to make the alteration of 360 into 365. The number 360 has so large a number of divisors, that the rule of Practice (230) may always be readily applied. Thus, it is required to find the portion which belongs to 274 days, the yearly interest being £18 . 9 . 10, or 18·491.

274	18·491
180 is $\frac{1}{2}$ of 360	9·246
94	
90 is $\frac{1}{2}$ of 180	4·623
4 is $\frac{1}{90}$ of 360	·205
	9)14·074
	8)1·564
	·196
	13·878 = £13 . 17 . 7 *Answer.*

But if the nearest farthing be wanted, the best way is to take 2-tenths of the number of days as a multiplier, and 73 as a divisor; since $m \div 365$ is $2m \div 730$, or $\frac{2}{10}m \div 73$. Thus, in the preceding instance, we multiply by 54·8 and divide by 73; and 54·8×18·491 = 1013·3068, which divided by 73 gives 13·881, very nearly agreeing with the former, and giving £13 . 17 . $7\frac{1}{2}$, which is certainly within a farthing of the truth.

254. Suppose it required to divide £100 among three persons in such a way that their shares may be as 6, 5, and 9; that is, so that for every £6 which the first has, the second may have £5, and the third £9. It is plain that if we divide the £100 into 6+5+9, or 20 parts, the first must have 6 of those parts, the second 5, and the third 9. Therefore (245) their shares are respectively, £$\frac{100\times6}{20}$, £$\frac{100\times5}{20}$, and £$\frac{100\times9}{20}$, or £30, £25, and £45.

EXERCISES.

Divide £394 . 12 among four persons, so that their shares may be as 1, 6, 7, and 18.—*Answer*, £12 . 6 . $7\frac{1}{2}$; £73 . 19 . 9; £86 . 6 . $4\frac{1}{2}$; £221 . 19 . 3.

Divide £20 among 6 persons, so that the share of each may be as much as those of all who come before put together.—*Answer*, The first two have 12*s*. 6*d*.; the third £1 . 5; the fourth £2 . 10; the fifth £5; and the sixth £10.

255. When two or more persons employ their money together, and gain or lose a certain sum, it is evidently not fair that the gain or loss should be equally divided among them all, unless each contributed the same sum. Suppose, for example, A contributes twice as much as B, and they gain £15, A ought to gain twice as much as B; that is, if the whole gain be divided into 3 parts, A ought to have two of them and B one, or A should gain £10 and B £5. Suppose that A, B, and C engage in an adventure, in which A embarks £250, B £130, and C £45. They gain £1000. How much of it ought each to have? Each one ought to gain as much for £1 as the others. Now, since there are 250+130+45, or 425 pounds embarked, which gain £1000, for each pound there is a gain of £$\frac{1000}{425}$. Therefore A should gain $\frac{1000\times250}{425}$ pounds, B should gain $\frac{1000\times130}{425}$ pounds, and C $\frac{1000\times45}{425}$ pounds. On these principles, by the process in (245), the following questions may be answered.

A ship is to be insured, in which A has ventured £1928, and B £4963. The expense of insurance is £474 . 10 . 2. How much ought each to pay of it? *Answer*, A must pay £132 . 15 . $2\frac{1}{2}$.

A loss of £149 is to be made good by three persons, A, B, and C. Had there been a gain, A would have gained 4 times as much as B, and C as much as A and B together. How much of the loss must each bear? *Answer*, A pays £59 . 12, B £14 . 18, and C £74 . 10.

256. It may happen that several individuals employ several sums of money together for different times. In such a case, unless there be a special agreement to the contrary, it is right that the more time a sum

is employed, the more profit should be made upon it. If, for example, A and B employ the same sum for the same purpose, but A's money is employed twice as long as B's, A ought to gain twice as much as B. The principle is, that one pound employed for one month, or one year, ought to give the same return to each. Suppose, for example, that A employs £3 for 6 months, B £4 for 7 months, and C £12 for 2 months, and the gain is £100; how much ought each to have of it? Now, since A employs £3 for six months, he must gain 6 times as much as if he employed it one month only; that is, as much as if he employed £6×3, or £18, for one month; also, B gains as much as if he had employed £4×7 for one month; and C as if he had employed £12×2 for one month. If, then, we divide £100 into 6×3+4×7+12×2, or 70 parts, A must have 6×3, or 18, B must have 4×7, or 28, and C 12×2, or 24 of those parts. The shares of the three are, therefore,

$$£\frac{6\times3\times100}{6\times3+4\times7+12\times2},\ £\frac{4\times7\times100}{6\times3+4\times7+12\times2},\ \text{and}\ £\frac{12\times2\times100}{6\times3+4\times7+12\times2}.$$

EXERCISES.

A, B, and C embark in an undertaking; A placing £3 . 6 for 2 years, B £100 for 1 year, and C £12 for $1\frac{1}{2}$ years. They gain £4276 . 7 How much must each receive of the gain?

Answer, A £226 . 10 . 4; B £3432 . 1 . 3; C £617 . 15 . 5.

A, B, and C rent a house together for 2 years, at £150 per annum. A remains in it the whole time, B 16 months, and C $4\frac{1}{2}$ months, during the occupancy of B. How much must each pay of the rent?*

Answer, A should pay £190 . 12 . 6; B £90 . 12 . 6; C £18 . 15.

257. These are the principal rules employed in the application of arithmetic to commerce. There are others, which, as no one who understands the principles here laid down can fail to see, are virtually contained in those which have been given. Such is what is commonly called the Rule of Exchange, for such questions as the following: If

* This question does not at first appear to fall under the rule. A little thought will serve to shew that what probably will be the first idea of the proper method of solution is erroneous.

20 shillings be worth $25\frac{1}{2}$ francs, in France, what is £160 worth? This may evidently be done by the Rule of Three. The rules here given are those which are most useful in common life; and the student who understands them need not fear that any ordinary question will be above his reach. But no student must imagine that from this or any other book of arithmetic he will learn precisely the modes of operation which are best adapted to the wants of the particular kind of business in which his future life may be passed. There is no such thing as a set of rules which are at once most convenient for a butcher and a banker's clerk, a grocer and an actuary, a farmer and a bill-broker; but a person with a good knowledge of the *principles* laid down in this work, will be able to examine and meet his own future wants, or, at worst, to catch with readiness the manner in which those who have gone before him have done so for themselves.

APPENDIX

TO THE

FIFTH EDITION

OF

DE MORGAN'S ELEMENTS OF ARITHMETIC.

I. ON THE MODE OF COMPUTING.

THE rules in the preceding work are given in the usual form, and the examples are worked in the usual manner. But if the student really wish to become a ready computer, he should strictly follow the methods laid down in this Appendix; and he may depend upon it that he will thereby save himself trouble in the end, as well as acquire habits of quick and accurate calculation.

I. In numeration learn to connect each primary decimal number, 10, 100, 1000, &c. not with the place in which the unit falls, but with the number of ciphers following. Call ten a *one-cipher* number, a hundred a *two-cipher* number, a million a *six-cipher* number, and so on. If *five* figures be cut off from a number, those that are left are hundred-thousands; for 100,000 is a *five*-cipher number. Learn to connect tens, hundreds, thousands, tens of thousands, hundreds of thousands, millions, &c. with 1, 2, 3, 4, 5, 6, &c. in the mind. What is a *seventeen-cipher* number? For every 6 in seventeen say *million*, for the remaining 5 say *hundred-thousand:* the answer is a hundred thousand millions of millions. If twelve places be cut off from the right of a number, what does the remaining number stand for?—*Answer*, As many millions of millions as there are units in it when standing by itself.

II. After learning to count forwards and backwards with rapidity, as in 1, 2, 3, 4, &c. or 30, 29, 28, 27, &c., learn to count forwards or backwards by twos, threes, &c. up to nines at least, beginning from any number. Thus, beginning from four and proceeding by sevens, we have 4, 11, 18, 25, 32, &c., along which series you must learn to go as easily as along the series 1, 2, 3, 4, &c.; that is, as quick as you can pronounce the words. The act of addition must be made in the mind without assistance: you must not permit yourself to say, 4 and 7 are 11, 11 and 7 are 18, &c.; but only 4, 11, 18, &c. And it would be desirable, though not so necessary, that you should go back as readily as forward; by sevens for instance, from sixty, as in 60, 53, 46, 39, &c.

III. Seeing a number and another both of one figure, learn to catch instantly the number you must add to the smaller to get the greater. Seeing 3 and 8, learn by practice to think of 5 without the necessity of saying 3 *from* 8 *and there remains* 5. And if the second number be the less, as 8 and 3, learn also by practice how to pass *up* from 8 to the next number which ends with 3 (or 13), and to catch the necessary augmentation, *five*, without the necessity of formally undertaking in words to subtract 8 from 13. Take rows of numbers, such as

4 2 6 0 5 0 1 8 6 4

and practise this rule upon every figure and the next, not permitting yourself in this simple case ever to name the higher one. Thus, say 4 and 8 (4 first, 2 second, 4 from the next number that ends with 2, or 12, leaves 8), 2 and 4, 6 and 4, 0 and 5, 5 and 5, 0 and 1, 1 and 7, 8 and 8, 6 and 8.

IV. Study the same exercise as the last one with two figures and one. Thus, seeing 27 and 6, pass from 27 up to the next number that ends with 6 (or 36), catch the 9 through which you have to pass, and allow yourself to repeat as much as "27 and 9 are 36." Thus, the row of figures 17729638109 will give the following practice: 17 and 0 are 17; 77 and 5 are 82; 72 and 7 are 79; 29 and 7 are 36; 96 and 7 are 103; 63 and 5 are 68; 38 and 3 are 41; 81 and 9 are 90; 10 and 9 are 19.

V. In a number of two figures, practise writing down the units at the moment that you are keeping the attention fixed upon the tens. In the preceding exercise, for instance, write down the results, repeating the tens with emphasis at the instant of writing down the units.

VI. Learn the multiplication-table so well as to name the product the instant the factors are seen; that is, until 8 and 7, or 7 and 8, suggest 56 at once, without the necessity of saying "7 times 8 are 56." Thus looking along a row of numbers, as 39706548, learn to name the products of every successive pair of digits as fast as you can repeat them, namely, 27, 63, 0, 0, 30, 20, 32.

VII. Having thoroughly mastered the last exercise, learn further, on seeing three numbers, to augment the product of the first and second by the third without any repetition of words. Practise until 3, 8, 4, for instance, suggest 3 times 8 and 4, or 28, without the necessity of saying "3 times 8 are 24, and 4 is 28." Thus, 179236408 will suggest the following practice, 16, 65, 21, 12, 22, 24, 8.

VIII. Now, carry the last still further, as follows: Seeing four figures, as 2, 7, 6, 9, catch up the product of the first and second, increased by the third, as in the last, without a helping word; name the result, and add the next figure, name the whole result, laying emphasis upon the tens. Thus, 2, 7, 6, 9, must immediately suggest "20 and 9 are 29." The row of figures 773698974 will give the instances 52 and 6 are 58; 27 and 9 are 36; 27 and 8 are 35; 62 and 9 are 71; 81 and 7 are 88; 79 and 4 are 83.

IX. Having four numbers, as 2, 4, 7, 9, vary the last exercise as follows: Catch the product of the first and second, increased by the third; but instead of adding the fourth, go up to the next number that ends with the fourth, as in exercise IV. Thus, 2, 4, 7, 9, are to suggest "15 and 4 are 19." And the row of figures 1723968929 will afford the instances 9 and 4 are 13; 17 and 2 are 19; 15 and 1 are 16; 33 and 5 are 38; 62 and 7 are 69; 57 and 5 are 62; 74 and 5 are 79.

X. Learn to find rapidly the number of times a digit is contained in given units and tens, with the remainder. Thus, seeing 8 and 53, arrive at and repeat "6 and 5 over." Common short division is the best practice. Thus, in dividing 236410792 by 7,

$$7)\underline{236410792}$$
$$33772970\text{, remainder } 2.$$

All that is repeated should be 3 and 2; 3 and 5; 7 and 5; 7 and 2; 2 and 6; 9 and 4; 7 and 0; 0 and 2.

In performing the several rules, proceed as follows:

Addition. Not one word more than repeating the numbers written in the following process: the accented figure is the one to be written down; the doubly accented figure is carried (and don't *say* "carry 3," but do it).

```
 47963
  1598
 26316
 54792
   819
  6686
------
138174
```

6, 15, 17, 23, 31, 3″4′; 11, 12, 21, 22, 31, 3″7′; 9, 17, 24, 27, 32, 4″1′; 10, 14, 20, 21, 2″8′; 7, 9, 1′3′.

In verifying additions, instead of the usual way of omitting one line, adding without it, and then adding the line omitted, verify each column by adding it both upwards and downwards.

Subtraction. The following process is enough. The carriages, being always of *one*, need not be mentioned.

```
From 79436258190
Take 58645962738
     -----------
     20790295452
```

8 and 2′, 4 and 5′, 7 and 4′, 3 and 5′, 6 and 9′, 10 and 2′, 6 and 0′, 4 and 9′, 7 and 7′, 9 and 0′, 5 and 2′. It is useless to stop and say, 8 and 2 make 10; for as soon as the 2 is obtained, there is no occasion to remember what it came from.

Multiplication. The following, put into words, is all that need be repeated in the multiplying part; the addition is then done as usual The unaccented figures are carried.

```
    670383
      9876
   -------
   4022298      18′, 49′, 22′, 2′, 42′, 4′0′,
  4692681       21′, 58′, 26′, 2′, 49′, 4′6′,
 5363064        24′, 66′, 30′, 3′, 56′, 5′3′,
6033447         27′, 74′, 34′, 3′, 63′, 6′0′.
----------
6620702508
```

Verify each line of the multiplication and the final result by casting out the nines. (*Appendix* II. p. 166.)

It would be almost as easy, for a person who has well practised the 8th exercise, to add each line to the one before in the process, thus:

670383	
9876	
4022298	8; 21 and 9 are 30′; 59 and 2
50949108	are 61′; 27 and 2 are 29; 2
587255508	and 2 are 4′; 49 and 0 are 49′;
6620702508	46 and 4 are 5′0′.

On the right is all the process of forming the second line, which completes the multiplication by 76, as the third line completes that by 876, and the fourth line that by 9876.

Division. Make each multiplication and the following subtraction in one step, by help of the process in the 9th exercise, as follows:

```
27693)441972809662(15959730
          165042
           265778
            165410
             269459
              202226
                83756
                  6772
```

The number of words by which 26577 is obtained from 165402 (the multiplier being 5) is as follows: 15 and 7′ are 2″2; 47 and 7′ are 5″4; 35 and 5′ are 4″0; 39 and 6′ are 4″5; 14 and 2′ are 16.

The processes for extracting the square root, and for the solution of equations (*Appendix* XI.), should be abbreviated in the same manner as the division.*

* The teacher will find further remarks on this subject in the *Companion to the Almanac* for 1844, and in the *Supplement to the Penny Cyclopædia*, article *Computation*.

APPENDIX II.

ON VERIFICATION BY CASTING OUT NINES AND ELEVENS.

THE process of *casting out the nines*, as it is called, is one which the young computer should learn and practise, as a check upon his computations. It is not a complete check, since if one figure were made too small, and another as much too great, it would not detect this double error; but as it is very unlikely that such a double error should take place, the check furnishes a strong presumption of accuracy.

The proposition upon which this method depends is the following: If a, b, c, d be four numbers, such that

$$a = bc+d,$$

and if m be any other number whatsoever, and if a, b, c, d, severally divided by m, give the remainders p, q, r, s, then

$$p \text{ and } qr+s$$

give the same remainder when divided by m (and perhaps are themselves equal).

For instance, $334 = 17\times19+11$;
divide these four numbers by 7, the remainders are 5, 3, 5, and 4. And 5 and $5\times3+4$, or 5 and 19, both leave the remainder 5 when divided by 7.

Any number, therefore, being used as a divisor, may be made a check upon the correctness of an operation. To provide a check which may be most fit for use, we must take a divisor the remainder to which is most easily found. The most convenient divisors are 3, 9, and 11, of which 9 is far the most useful.

As to the numbers 3 and 9, the remainder is always the same as that of the sum of the digits. For instance, required the remainder of 246120377 divided by 9. The sum of the digits is $2+4+6+1+2+0+3+7+7$, or 32, which gives the remainder 5. But the easiest way of proceeding is by throwing out nines as fast as they arise in the sum. Thus, repeat 2, 6 $(2+4)$, 12 $(6+6)$, say 3 (throwing out 9), 4, 6, 9 (throw this away), 7, 14, (or throwing out the 9) 5. This is the remainder required, as would appear by dividing 246120377 by 9. A proof may be given thus: It is obvious that each of the numbers, 1, 10, 100, 1000, &c. divided by 9, leaves a remainder 1, since they are 1, 9+1, 99+1, &c. Consequently, 2, 20, 200, &c. leave the remainder 2; 3, 30, 300, the remainder 3; and so on. If, then, we divide, say 1764 by 9 in parcels, 1000 will be one more than an exact number of nines, 700 will be seven more, and 60 will be six more. So, then, from 1, 7, 6, 4, put together, and the nines taken out, comes the only remainder which can come from 1764.

To apply this process to a multiplication: It is asserted, in page 32, that

$$10004569 \times 3163 = 31644451747.$$

In casting out the nines from the first, all that is necessary to repeat is, one, five, ten, one, *seven;* in the second, three, four, ten, one, *four;* in the third, three, four, ten, one, five, nine, four, nine, eight, twelve, three, ten, *one*. The remainders then are, 7, 4, 1. Now, 7×4 is 28, which, casting out the nines, gives 1, the same as the product.

Again, in page 43, it is asserted that

$$23796484 = 130000 \times 183 + 6484.$$

Cast out the nines from 13000, 183, 6484, and we have 4, 3, and 4. Now, 4×3+4, with the nines cast out, gives 7; and so does 23796484.

To avoid having to remember the result of one side of the equation, or to write it down, in order to confront it with the result of the other side, proceed as follows: Having got the remainder of the more complicated side, into which two or more numbers enter, subtract it from 9, and carry the remainder into the simple side, in which there is only one number. Then the remainder of that side ought to be 0. Thus, having got 7 from the left hand of the preceding, take 2, the rest of 9, forget 7, and carry in 2 as a beginning to the left-hand side, giving 2, 4, 7, 14, 5, 11, 2, 6, 14, 5, 9, 0.

Practice will enable the student to cast out nines with great rapidity.

This process of casting out the nines does not detect any errors in which the remainder to 9 happens to be correct. If a process be tedious, and some additional check be desirable, the method of casting out *elevens* may be followed after that of casting out the nines. Observe that 10+1, 100−1, 1000+1, 10000−1, &c. are all divisible by eleven. From this the following rule for the remainder of division by

11 may be deduced, and readily used by those who know the algebraical process of subtraction. For those who have not got so far, it may be doubted whether the rule can be made easier than the actual division by 11.

Subtract the first figure from the second, the result from the third, the result from the fourth, and so on. The final result, or the rest of 11 if the figure be negative, is the remainder required. Thus, to divide 1642915 by 11, and find the remainder, we have 1 from 6, 5; 5 from 4, −1; −1 from 2, 3; 3 from 9, 6; 6 from 1, −5; −5 from 5, 10; and 10 is the remainder. But 164 gives −1, and 10 is the remainder; 164291 gives −5, and 6 is the remainder. With very little practice these remainders may be read as rapidly as the number itself. Thus, for 127619833424 need only be repeated, 1, 6, 0, 1, 8, 0, 3, 0, 4, −2, 6, and 6 is the remainder.

When a question has been tried both by nines and elevens, there can be no error unless it be one which makes the result wrong by a number of times 99 exactly.

APPENDIX III.

ON SCALES OF NOTATION.

We are so well accustomed to 10, 100, &c., as standing for ten, ten tens, &c., that we are not apt to remember that there is no reason why 10 might not stand for five, 100 for five fives, &c., or for twelve, twelve twelves, &c. Because we invent different columns of numbers, and let units in the different columns stand for collections of the units in the preceding columns, we are not therefore bound to allow of no collections except in tens.

If 10 stood for 2, that is, if every column had its unit double of the unit in the column on the right, what we now represent by 1, 2, 3, 4, 5, 6, &c., would be represented by 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, &c. This is the *binary* scale. If we take the *ternary* scale, in which 10 stands for 3, we have 1, 2, 10, 11, 12, 20, 21, 22, 100, 101, 102, 110, &c. In the *quinary* scale, in which 10 is five, 234 stands for 2 twenty-fives, 3 fives, and 4, or sixty-nine. If we take the *duodenary* scale, in which 10 is twelve, we must invent new symbols for ten and eleven, because 10 and 11 now stand for twelve and thirteen; use the letters *t* and *e*. Then 176 means 1 twelve-twelves, 7 twelves, and 6, or two hundred and thirty-four; and 1*te* means two hundred and seventy-five.

The number which 10 stands for is called the *radix* of the *scale of notation*. To change a number from one scale into another, divide the number, written as in the first scale, by the number which is to be the radix of the new scale; repeat this division again and again, and the remainders are the digits required. For example, what, in the quinary scale, is that number which, in the decimal scale, is 17036?

5)17036	
5)3407	Rem^r. 1
5)681	 2
5)136	 1
5)27	 1
5)5	 2
5)1	 0
0	 1

Answer . . 1021121

	Quinary.		Decimal.
Verification,	1000000	means	15625
	20000	......	1250
	1000	......	125
	100	......	25
	20	......	10
	1	......	1
	1021121	......	17036

The reason of this rule is easy. Our process of division is nothing but telling off 17036 into 3407 fives and 1 over; we then find 3407 fives to be 681 fives of fives and 2 *fives* over. Next we form 681 fives of fives into 136 fives of fives of fives and 1 five of fives over; and so on.

It is a useful exercise to multiply and divide numbers represented in other scales of notation than the common or decimal one. The rules are in all respects the same for all systems, *the number carried being always the radix of the system.* Thus, in the quinary system we carry fives instead of tens. I now give an example of multiplication and division:

```
     Quinary.                    Decimal.
       42143        means          2798
        1234        .......         194
     -------                     ------
      324232                      11192
     232034                      25182
    134341                       2798
   42143                         ------
   ---------                     542812
   114332222
```

```
     Duodecimal.                  Decimal.
 4t9)76t4e08(16687         705)22610744(32071
   4t9                          1460
   ----                          5074
   2814                           1394
   2546                            689
   ----
    28te
    2546
    ----
     3650
     3320
     ----
      3308
      2t33
      ----
       495
```

Another way of turning a number from one scale into another is as follows: Multiply the first digit by the *old* radix *in the new scale*, and add the next digit; multiply the result again by the old radix in the new scale, and take in the next digit, and so on to the end, always using the radix of the scale you want to leave, and the notation of the scale you want to end in.

Thus, suppose it required to turn 16687 (duodecimal) into the decimal scale, and 16432 (septenary) into the quaternary scale:

```
          16687                              16432
  Duodecimals into Decimals.      Septenaries into Quaternaries.
       1×12+6 = 18                        1×7+6 = 31
              ×12+6                              ×7+4
              -----                           ------
               222                             1133
              ×12+8                              ×7+3
              -----                           ------
              2672                            22130
              ×12+7                              ×7+2
              -----                           -------
Answer . . . . 32071                          1021012
```

Owing to our division of a foot into 12 equal parts, the duodecimal scale often becomes very convenient. Let the square foot be also divided into 12 parts, each part is 12 square inches, and the 12th of the 12th is one square inch. Suppose, now, that the two sides of an oblong piece of ground are 176 feet 9 inches 7-12ths of an inch, and 65 feet 11 inches 5-12ths of an inch. Using the duodecimal scale, and *duodecimal fractions*, these numbers are 128·97 and 55·*e*5. Their product, the number of square feet required, is thus found:

```
    128·97
     55·e5
    ------
    617ee
  116095
  617ee
 617ee
 ---------
 68e8144e
```

Answer, 68*e*8·144*e* (duod.) square feet, or 11660 square feet 16 square inches $\frac{4}{12}$ and $\frac{11}{144}$ of a square inch.

It would, however, be exact enough to allow 2-hundredths of a foot for every quarter of an inch, an additional hundredth for every 3 inches,* and 1-hundredth more if there be a 12th or 2-12ths above the quarter of an inch. Thus, $9\frac{7}{12}$ inches should be ·76+·03+·01, or ·80, and $11\frac{5}{12}$ would be ·95; and the preceding might then be found decimally as 176·8×65·95 as 11659·96 square feet, near enough for every practical purpose.

APPENDIX IV.

ON THE DEFINITION OF FRACTIONS.

The definition of a fraction given in the text shews that $\frac{7}{9}$, for instance, is the *ninth* part of *seven*, which is shewn to be the same thing as *seven-ninths* of a unit. But there are various modes of speech under which a fraction may be signified, all of which are more or less in use.

1. In $\frac{7}{9}$ we have the 9th part of 7.
2. 7-9ths of a unit.
3. The fraction which 7 is of 9.

* And at discretion one hundredth more for a large fraction of three inches.

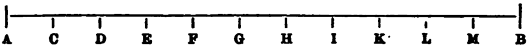
4. The times and parts of a time (in this case part of a time only) which 7 contains 9.
5. The multiplier which turns *nines* into *sevens*.
6. The *ratio* of 7 to 9, or the *proportion* of 7 to 9.
7. The multiplier which alters a number in the ratio of 9 to 7.
8. The 4th proportional to 9, 1, and 7.

The first two views are in the text. The third is deduced thus: If we divide 9 into 9 equal parts, each is 1, and 7 of the parts are 7; consequently the fraction which 7 is of 9 is $\frac{7}{9}$. The fourth view follows immediately: For *a time* is only a word used to express one of the repetitions which take place in multiplication, and we allow ourselves, by an easy extension of language, to speak of a portion of a number as being that number taken a *part of a time*. The fifth view is nothing more than a change of words: A number reduced to $\frac{7}{9}$ of its amount has every 9 converted into a 7, and any fraction of a 9 which may remain over into the corresponding fraction of 7. This is completely proved when we prove the equation $\frac{7}{9}$ of $a = 7$ times $\frac{a}{9}$. The sixth, seventh, and eighth views are illustrated in the chapter on proportion.

When the student comes to algebra, he will find that, in all the applications of that science, fractions such as $\frac{a}{b}$ most frequently require that a and b should be themselves supposed to be fractions. It is, therefore, of importance that he should learn to accommodate his views of a fraction to this more complicated case.

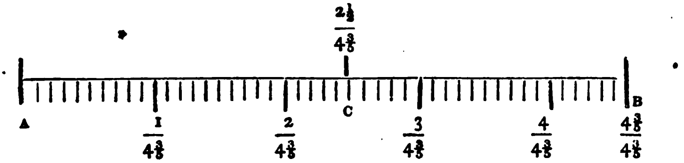
Suppose we take $\frac{2\frac{1}{2}}{4\frac{3}{5}}$. We shall find that we have, in this case, a better idea of the views from and after the third inclusive, than of the first and second, which are certainly the most simple ways of conceiving $\frac{7}{9}$. We have no notion of the $\left(4\frac{3}{5}\right)$th part of $2\frac{1}{2}$, nor of $2\frac{1}{2}$ $\left(4\frac{3}{5}\right)$ths of a unit; indeed, we coin a new species of adjective when we talk of the $\left(4\frac{3}{5}\right)$th part of anything. But we can readily imagine that $2\frac{1}{2}$ is some fraction of $4\frac{3}{5}$; that the first is *some* part of a time the second; that there must be *some* multiplier which turns every $4\frac{3}{5}$ in a number into $2\frac{1}{2}$; and so on. Let us now see whether we can invent a distinct mode of applying the first and second views to such a compound fraction as the above.

We can easily imagine a fourth part of a length, and a fifth part, meaning the lines of which 4 and 5 make up the length in question; and there is also in existence a length of which four lengths and two-fifths of a length make up the original length in question. For instance, we might say that 6, 6, 2 is a division of 14 into $2\frac{1}{3}$ equal parts—2 equal parts, 6, 6, and a third of a part, 2. So we might agree to say, that the $\left(2\frac{1}{3}\right)$th, or $\left(2\frac{1}{3}\right)$rd, or $\left(2\frac{1}{3}\right)$st (the reader may coin the adjective as he pleases) part of 14 is 6. If we divide the line A B into eleven equal parts in C, D, E, &c., we must then say that A C is the 11th part,



A D the $\left(5\frac{1}{2}\right)$th, A E the $\left(3\frac{2}{3}\right)$th, A F the $\left(2\frac{3}{4}\right)$th, A G the $\left(2\frac{1}{5}\right)$th, A H the $\left(1\frac{5}{6}\right)$th, A I the $\left(1\frac{4}{7}\right)$th, A K the $\left(1\frac{3}{8}\right)$th, A L the $\left(1\frac{2}{9}\right)$th, A M the $\left(1\frac{1}{10}\right)$th, and A B itself the 1st part of A B. The reader may refuse the language if he likes (though it is not so much in defiance of etymology as talking of *multiplying* by $\frac{1}{2}$); but when A B is called 1, he must either call A F $\frac{1}{2\frac{3}{4}}$, or make one definition of one class of fractions and another of another. Whatever abbreviations they may choose, all persons will agree that $\frac{a}{b}$ is a direction to find such a fraction as, repeated b times, will give 1, and then to take that fraction a times.

So, to get $\frac{2\frac{1}{2}}{4\frac{3}{5}}$, the simplest way is to divide the whole unit into 46 parts; 10 of these parts, repeated $4\frac{3}{5}$ times, give the whole. The



$\left(4\frac{3}{5}\right)$th is then $\frac{10}{46}$, and $2\frac{1}{2}$ such parts is $\frac{25}{46}$, or A C. The student should try several examples of this mode of interpreting complex fractions.

But what are we to say when the denominator itself is less than unity, as in $\frac{3\frac{1}{2}}{\frac{2}{5}}$? Are we to have a $\left(\frac{2}{5}\right)$th part of a unit? and what is it? Had there been a 5 in the denominator, we should have taken the part of which 5 will make a unit. As there is $\frac{2}{5}$ in the denominator, we must take the part of which $\frac{2}{5}$ will be a unit. That part is larger than a unit; it is $2\frac{1}{2}$ units; $2\frac{1}{2}$ is that of which $\frac{2}{5}$ is 1. The above fraction then directs us to repeat $2\frac{1}{2}$ units $3\frac{1}{4}$ times. By extending our word 'multiplication' to the taking of a part of a time, all multiplications are also divisions, and all divisions multiplications, and all the terms connected with either are subject to be applied to the results of the other.

If $2\frac{1}{3}$ yards cost $3\frac{1}{2}$ shillings, how much does one yard cost? In such a case as this, the student looks at a more simple question. If 5 yards cost 10 shillings, he sees that each yard costs $\frac{10}{5}$, or 2 shillings, and, concluding that the same process will give the true result when the data are fractional, he forms $\frac{3\frac{1}{2}}{2\frac{1}{3}}$, reduces it by rules to $\frac{3}{2}$ or $1\frac{1}{2}$, and concludes that 1 yard costs 18 pence. The answer happens to be correct; but he is not to suppose that this rule of copying for fractions whatever is seen to be true of integers is one which requires no demonstration. In the above question we want money which, repeated $2\frac{1}{3}$ times, shall give $3\frac{1}{2}$ shillings. If we divide the shilling into 14 equal parts, 6 of these parts repeated $2\frac{1}{3}$ times give the shilling. To get $3\frac{1}{2}$ times as much by the same repetition, we must take $3\frac{1}{2}$ of these 6 parts at each step, or 21 parts. Hence, $\frac{21}{14}$, or $1\frac{1}{2}$, is the number of shillings in the price.

APPENDIX V.

ON CHARACTERISTICS.

WHEN the student comes to use logarithms, he will find what follows very useful. In the mean while, I give it merely as furnishing a rapid rule for finding the place of a decimal point in the quotient before the division is commenced.

When a bar is written over a number, thus, $\bar{7}$, let the number be called negative, and let it be thus used: Let it be augmented by additions of its own species, and diminished by subtractions; thus, $\bar{7}$ and $\bar{2}$ give $\bar{9}$, and let $\bar{7}$ with $\bar{2}$ subtracted give $\bar{5}$. But let the *addition* of a number without the bar *diminish* the negative number, and the *subtraction increase* it. Thus, $\bar{7}$ and 4 are $\bar{3}$, $\bar{7}$ and 12 make 5, $\bar{7}$ with 8 subtracted is $\overline{15}$. In fact, consider 1, 2, 3, &c., as if they were gains, and $\bar{1}$, $\bar{2}$, $\bar{3}$, as if they were losses: let the addition of a gain or the removal of a loss be equivalent things, and also the removal of a gain and the addition of a loss. Thus, when we say that $\bar{4}$ diminished by $\overline{11}$ gives 7, we say that a loss of 4 incurred at the moment when a loss of 11 is removed, is, on the whole, equivalent to a gain of 7; and saying that $\bar{4}$ diminished by 2 is $\bar{6}$, we say that a loss of 4, accompanied by the removal of a gain of 2, is altogether a loss of 6.

By the *characteristic* of a number understand as follows: When there are places before the decimal point, it is one less than the number of such places. Thus, 3·214, 1·0083, 8 (which is 8·00...) 9·999, all have 0 for their characteristics. But 17·32, 48, 93·116, all have 1; 126·03 and 126 have 2; 11937264·666 has 7. But when there are no places before the decimal point, look at the first decimal place which is significant, and make the characteristic negative accordingly. Thus, ·612, ·121, ·9004, in all of which significance begins in the first decimal place, have the characteristic $\bar{1}$; but ·018 and ·099 have $\bar{2}$; ·00017 has $\bar{4}$; ·000000001 has $\bar{9}$.

To find the characteristic of a quotient, subtract the characteristic of the divisor from that of the dividend, carrying one before subtraction if the first significant figures of the divisor are greater than those of the dividend. For instance, in dividing 146·08 by ·00279. The characteristics are 2 and $\bar{3}$; and 2 with $\bar{3}$ removed would be 5. But on looking, we see that the first significant figures of the divisor, 27, taken by themselves, and without reference to their local value, mean a larger number than 14, the first two figures of the dividend. Consequently, to $\bar{3}$ we carry 1 before subtracting, and it then becomes $\bar{2}$, which, taken from 2, gives 4. And this 4 is the characteristic of the quotient, so that the quotient has 5 places before the decimal point. Or, if *abcdef* be the first figures of the quotient, the decimal point must be thus placed, *abcde·f*. But if it had been to divide ·00279 by 146·08, no carriage would have been required; and $\bar{3}$ diminished by 2 is $\bar{5}$; that is, the first significant figure of the quotient is in the 5th place. The quotient, then, has ·0000 before any significant figure. A few applications of this rule will make it easy to do it in the head, and thus to assign the meaning of the first figure of the quotient even before it is found.

APPENDIX VI.

ON DECIMAL MONEY.

Of all the simplifications of commercial arithmetic, none is comparable to that of expressing shillings, pence, and farthings as decimals of a pound. The rules are thereby put almost upon as good a footing as if the country possessed the advantage of a real decimal coinage.

Any fraction of a pound sterling may be decimalised by rules which can be made to give the result at once.

Two shillings is	£·100	
One shilling is	£·050	
Sixpence is	£·025	
One farthing is	£·001	$04\frac{1}{6}$

Thus, every pair of shillings is a unit in the first decimal place; an odd shilling is a 50 in the second and third places; a farthing is so nearly the thousandth part of a pound, that to say one farthing is ·001, two farthings is ·002, &c., is so near the truth that it makes no error in the first three decimals till we arrive at sixpence, and then 24 farthings is exactly ·025 or 25 thousandths. But 25 farthings is ·026, 26 farthings is ·027, &c. Hence the rule for the *first three places* is

One in the first for every pair of shillings: 50 in the second and third for the odd shilling, if any; and 1 for every farthing additional, with 1 extra for sixpence.

Thus,	0s. $3\frac{1}{2}d.$ = £·014	2s. 6d. = £·125
	0s. $7\frac{3}{4}d.$ = £·032	2s. $9\frac{1}{2}d.$ = £·139
	1s. $2\frac{1}{2}d.$ = £·060	3s. $2\frac{3}{4}d.$ = £·161
	1s. $11\frac{1}{4}d.$ = £·096	13s. $10\frac{3}{4}d.$ = £·694

In the fourth and fifth places, and those which follow, it is obvious that we have no produce from any farthings except those above sixpence. For at every sixpence, $\cdot00004\frac{1}{6}$ is converted into ·001, and this has been already accounted for. Consequently, to fill up the *fourth and fifth* places,

Take 4 *for every farthing* above the last sixpence, and an additional* 1 *for every six farthings, or three halfpence.*

The remaining places arise altogether from $\cdot00000\frac{1}{6}$ for every farthing above the last three halfpence; for at every three halfpence complete, $\cdot00000\frac{1}{6}$ is converted into ·00001, and has been already accounted for. Consequently, to fill up *all the places after the fifth,*

Let the number of farthings above the last three halfpence be a numerator, 6 *a denominator, and annex the figures of the corresponding decimal fraction.*

It may be easily remembered that

The figures of $\frac{1}{6}$ are 166666...	The figures of $\frac{4}{6}$ are 666666...
. . . . $\frac{2}{6}$ ... 333333...	. . . . $\frac{5}{6}$ ... 833333...
. . . . $\frac{3}{6}$ ... 5	

0s. $3\frac{1}{2}d.$ = ·014 | 58 | 3333...	2s. 6d. = ·125 | 00 | 0000...
0s. $7\frac{3}{4}d.$ = ·032 | 29 | 1666...	2s. $9\frac{1}{2}d.$ = ·139 | 58 | 3333...

* The student should remember all the multiples of 4 up to 4×25, or 100.

1*s*. $2\frac{1}{2}d.$ = ·060|41|6666... 3*s*. $2\frac{3}{4}d.$ = ·161|45|83333...

1*s*. $11\frac{1}{4}d.$ = ·096|87|5 13*s*. $10\frac{3}{4}d.$ = ·694|79|1666...

The following examples will shew the use of this rule, if the student will also work them in the common way.

To turn pounds, &c., into farthings: Multiply the pounds by 960, or by 1000—40, or by $1000\left(1-\frac{4}{100}\right)$; that is, from 1000 times the pounds subtract 4 per cent of itself. Thus, required the number of farthings in £1663 . 11 . $9\frac{3}{4}$.

1663·590625×1000 =	1663590·625
4 per cent of this,	66543·625
No. of farthings required,	1597047

What is $47\frac{1}{2}$ per cent of £166 . 13 . 10 and ·6148 of £2971 . 16 . 9?

	166·691		2971·837
40 p. c.	66·6764	·6	1783·1022
5 p. c.	8·3346	·01	29·7184
$2\frac{1}{2}$ p. c.	4·1673	·004	11·8873
		·0008	2·3775
	79·1783		1827·0854
£79 . 3 . $6\frac{3}{4}$		£1827 . 1 . $8\frac{1}{2}$	

The inverse rule for turning the decimal of a pound into shillings, pence, and farthings, is obviously as follows:

A pair of shillings for every unit in the first place; an odd shilling for 50 (*if there be* 50) *in the second and third places; and a farthing for every thousandth left, after abating* 1 *if the number of thousandths so left exceed* 24.

The direct rule (with three places) gives too little, the inverse rule too much, except at the end of a sixpence, when both are accurate. Thus, £·183 is rather less than 3*s*. 8*d*., and 6*s*. $4\frac{3}{4}d.$ is rather greater than £·319; or when the two do not exactly agree, the *common money is the greatest*. But £·125 and £·35 are exactly 2*s*. 6*d*. and 7*s*.

Required the price of 17 cwt. 81 lb. $13\frac{1}{2}$ oz. at £3 . 11 $9\frac{3}{4}$ per cwt. true to the hundredth of a farthing.

		3·590625
		17
		61·040625
lb. 56	$\frac{1}{2}$	1·795313
16	$\frac{1}{7}$	·512946
7	$\frac{1}{8}$	·224414
2	$\frac{1}{8}$	·064118
oz. 8	$\frac{1}{4}$	·016029
4	$\frac{1}{2}$	·008015
1	$\frac{1}{4}$	·002004
$\frac{1}{2}$	$\frac{1}{2}$	·001002
		£63·664466

£63 . 13 . $3\frac{1}{2}$

Three men, A, B, C, severally invest £191 . 12 . $7\frac{3}{4}$, £61 . 14 . 8. and £122 . 1 . $9\frac{1}{2}$ in an adventure which yields £511 . 12 . $6\frac{1}{2}$. How ought the proceeds to be divided among them?

A, 191·63229
B, 61·73333
C, 122·08958 Produce of £1.
375·45520)511·62708(1·362686
136 17188
23 53532
1 00801
25710
3183
180

1·362686	1·362686	1·362686
92 236191	33 33716	85 980221
1 362686	8 17612	1 362686
1 226417	13627	272537
13627	9538	27254
8176	409	1090
409	41	122
27	4	7
3	8 41231	1
1		1 663697
2 611346		

261·1346	. . . A's share	£261 . 2 . $8\frac{1}{4}$
84·1231	. . . B's „ . . .	84 . 2 . $5\frac{3}{4}$
166·3697	. . . C's „ . . .	166 . 7 . $4\frac{3}{4}$
511·6274		£511 . 12 . $6\frac{3}{4}$

If ever the fraction of a farthing be wanted, remember that the *coinage*-result is larger than the decimal of a pound, when we use only three places. From 1000 times the decimal take 4 per cent, and we get the exact number of farthings, and we need only look at the decimal then left to set the preceding right. Thus, in

134·6	123·1	369·7
5·38	4·92	14·79
·22	·18	·91

we see that (if we use four decimals only) the pence of the above results are nearly 8*d.* ·22 of a farthing, $5\frac{1}{2}$*d.* ·18, and $4\frac{1}{2}$*d.* ·91.

A man can pay £2376 . 4 . $4\frac{1}{2}$, his debts being £3293 . 11 . $0\frac{3}{4}$. How much per cent can he pay, and how much in the pound?

```
3293·553)2376·2180(·7214756
              70 7309
               4 8598
               1 5662
                 2488
                  183
                   18
```

Answer, £72 . 2 . $11\frac{1}{2}$ per cent.

0 . 14 . $5\frac{1}{4}$ per pound.

APPENDIX VII.

ON THE MAIN PRINCIPLE OF BOOK-KEEPING.

A BRIEF notice of the principle on which accounts are kept (when they are *properly* kept) may perhaps be useful to students who are learning book-keeping, as the treatises on that subject frequently give too little in the way of explanation.

Any person who is engaged in business must desire to know accurately, whenever an investigation of the state of his affairs is made.

1, What he had at the commencement of the account, or immediately after the last investigation was made; 2, What he has gained and lost in the interval in all the several branches of his business; 3, What he is now worth. From the first two of these things he obviously knows the third. In the interval between two investigations, he may at any one time desire to know how any one account stands.

An *account* is a recital of all that has happened, in reference to any class of dealings, since the last investigation. It can only consist of receipts and expenditures, and so it is said to have two sides, a *debtor* and a *creditor* side.

All accounts are kept in *money*. If goods be bought, they are estimated by the money paid for them. If a debtor give a bill of exchange, being a promise to pay a certain sum at a certain time, it is put down as worth that sum of money. All the tools, furniture, horses, &c. used in the business are rated at their value in money. All the actual coin, bank-notes, &c., which are in or come in, being the only money in the books which really is money, is called *cash*.

The accounts are kept as if every different sort of account belonged to a separate person, and had an interest of its own, which every transaction either promotes or injures. If the student find that it helps him, he may imagine a clerk to every account: one to take charge of, and regulate, the actual cash; another for the bills which the house is to receive when due; another for those which it is to pay when due; another for the cloth (if the concern deal in cloth); another for the sugar (if it deal in sugar); one for every person who has an account with the house; one for the profits and losses; and so on.

All these clerks (or accounts) belonging to one merchant, must account to him in the end—must either produce all they have taken in charge, or relieve themselves by shewing to whom it went. For all that they have received, for every responsibility they have undertaken *to the concern itself*, they are bound, or are *debtors;* for everything which has passed out of charge, or about which they are relieved from answering *to the concern*, they are unbound, or are *creditors*. These words must be taken in a very wide sense by any one to whom book-keeping is not to be a mystery. Thus, whenever any account assumes responsibility to any parties *out of the concern*, it must be creditor in the books, and debtor whenever it discharges any other parties of their responsibility. But whenever an account removes responsibility from any other account in the same books it is debtor, and creditor whenever it imposes the same.

To whom are all these parties, or accounts, bound, and from whom are they released? Undoubtedly the merchant himself, or, more properly, the *balance-clerk*, presently mentioned. But it is customary to say that the accounts are debtors *to* each other, and creditors *by* each other. Thus, cash *debtor to* bills receivable, means that the cash account (or the clerk who keeps it) is bound to answer for a sum which was paid on a bill of exchange due to the house. At full length it would be: "Mr. C (who keeps the cash-box) has received, and is answerable for, this sum which has been paid in by Mr. A, when he paid his bill of exchange." On the other hand, the corresponding entry in the account of bills receivable runs—bills receivable, *creditor* by cash. At full length: "Mr. B (who keeps the bills receivable) is freed from all responsibility for Mr. A's bill, which he once held, by handing over to Mr. C, the cash-clerk, the money with which Mr. A took it up." Bills receivable creditor *by* cash is intelligible, but cash debtor *to* bills receivable is a misnomer. The cash account is debtor *to the merchant by* the sum received for the bill, and it should be cash debtor *by* bill receivable. The fiction of debts, not one of which is ever paid to the party *to* whom it is said to be owing, though of no consequence in practice, is a stumbling-block to the learner; but he must keep the phrase, and remember its true meaning.

The account which is made *debtor*, or bound, is said to be *debited;* that which is made *creditor*, or released, is said to be *credited.* All who receive must be *debited;* all who give must be *credited.*

No cancel is ever made. If cash received be afterwards repaid, the sum paid is not struck off the receipts (or debtor-side of the cash account), but a discharge, or credit, is written on the expenditure (or credit) side.

The book in which the accounts are kept is called a *ledger*. It has double columns, or else the debtor side is on one page, and the creditor side on the opposite, of each account. The debtor-side is always the left. Other books are used, but they are only to help in keeping the ledger correct. Thus there may be a *waste-book*, in which all transactions are entered as they occur, in common language; a *journal*, in which the transactions described in the waste-book are entered at stated periods, in the language of the ledger. The items entered in the journal have references to the pages of the ledger to which they are carried, and the items in the ledger have also references to the pages of the journal from which they come; and by this mode of reference it is easy to make a great deal of abbreviation in the ledger. Thus, when it happens, in making up the journal to a certain date, that several different sums were paid or received at or near the same time, the totals may be entered in the ledger, and the cash account may be made debtor to, or creditor by, sundry accounts, or sundries; the sundry accounts being severally credited or debited for their shares of the whole. The only book that need be explained is the ledger. All the other books, and the manner in which they are kept, important as they may be, have nothing to do with the main principle of the method. Let us, then, suppose that all the items are entered at once in the ledger as they arise. It has appeared that every item is entered twice. If A pay on account of B, there is an entry, "A, creditor by B;" and another, "B, debtor to A." This is what is called *double-entry;* and the consequence of it is, that the sum of all the debtor items in the whole book is equal to the sum of all the creditor items. For what is the first set but the second with the items in a different order? If it were convenient, one entry of each sum might be made a double-entry. The multiplication table is called a table of *double-entry*, because 42, for instance, though it occurs only once, appears in two different aspects, namely, as 6 times 7 and as 7 times 6. Suppose, for example, that there are five accounts, A, B, C, D, E, and that each account has one transaction of its own with every other account; and let the debits be in the *columns*, the credits in the *rows*, as follows:

	A Debtor.	B Debtor.	C Debtor.	D Debtor.	E Debtor.
A, Creditor		23	19	32	4
B, Creditor	17		6	11	25
C, Creditor	9	41		10	2
D, Creditor	14	28	16		3
E, Creditor	15	4	60	1	

Here the 16 is supposed to appear in D's account as D creditor by C, and in C's account as C debtor to D. And to say that the sum of debtor items is the same as that of creditor items, is merely to say that the preceding numbers give the same sum, whether the rows or the columns be first added up.

If it be desired to close the ledger when it stands as above, the following is the way the accounts will stand: the lines in italics will presently be explained.

A, Debtor.		A, Creditor.		B, Debtor.		B, Creditor.	
To B . .	17	By B . .	23	To A . .	23	By A . .	17
To C . .	9	By C . .	19	To C . .	41	By C . .	6
To D . .	14	By D . .	32	To D . .	28	By D .	11
To E . .	15	By E . .	4	To E . .	4	By E . .	25
To Balance	23					*By Balance*	37
	78		78		96		96

C, Debtor.		C, Creditor.		D, Debtor.		D, Creditor.	
To A . .	19	By A . .	9	To A . .	32	By A . .	14
To B . .	6	By B . .	41	To B . .	11	By B . .	28
To D . .	16	By D . .	10	To C . .	10	By C . .	16
To E . .	60	By E . .	2	To E . .	1	By E . .	3
		By Balance	39	*To Balance*	7		
	101		101		61		61

E, Debtor.		E, Creditor.		*Balance, Debtor.*		*Balance, Cred.*	
To A . .	4	By A . .	15	*To B* . .	37	*By A* . .	23
To B . .	25	By B . .	4	*To C* . .	39	*By D* . .	7
To C . .	2	By C . .	60			*By E* . .	46
To D . .	3	By D . .	1		76		76
To Balance	46						
	80		80				

In all the part of the above which is printed in Roman letters we see nothing but the preceding table repeated. But when all the accounts have been completed, and no more entries are left to be made, there remains the last process, which is termed *balancing the ledger*. To get an idea of this, suppose a new clerk, who goes round all the accounts, collecting debts and credits, and taking them all upon himself, that he alone may be entitled to claim the debts and to be responsible for the assets of the concern. To this new clerk, whom I will call the *balance-clerk*, every account gives up what it has, whether the same be debt or credit. The cash-clerk gives up all the cash; the clerks of the two kinds of bills give up all their documents, whether bills receivable or entries of bills payable (remember that any entry against which there is money set down in the books counts as money when given up, that is, as money due or money owing); the clerks of the several accounts of goods give up all their unsold remainders at cost prices; the clerks of the several personal accounts give up vouchers for the sums owing to or from the several parties; and so on. But where more has been paid out than received, the balance-clerk adjusts these accounts by giving instead of receiving; in fact, he so acts as to make the debtor and creditor sides of the accounts he visits equal in amount. For instance, the A account is indebted to the concern 55, while payments or discharges to the amount of 78 have been made by it. The balance-clerk accordingly hands over 23 to that account, for which it becomes debtor, while the balance enters itself as creditor to the same amount. But in the B account there is 96 of receipt, and only 59 of payment or discharge. The balance-clerk then receives 37 from this account, which is therefore credited by balance, while the balance acknowledges as much of debt. The balance account must, of course, exactly balance itself, if the accounts be all right; for of all the equal and opposite entries of which the ledger consists, so far as they do not balance one another, one goes into one side of the balance account, and the other into the other. Thus the balance account becomes a test of the accuracy of one part of the work: if its two sides do not give the same sums, either there have been entries which have not had their corresponding balancing entries correctly made, or else there has been error in the additions.

But since the balance account must thus always give the *same sum* on both sides, and since *balance debtor* implies what is favourable to the concern, and *balance creditor* what is unfavourable, does it not appear as if this system could only be applied to cases in which there is neither loss nor gain? This brings us to the two accounts in which are entered all that the concern *began with*, and all that it *gains or loses*—the *stock account*, and the *profit-and-loss account*. In order to make all that there was to begin with a matter of double entry, the opening of the ledger supposes the merchant himself to put his several clerks in charge of their several departments. In the stock account, *stock*, which here stands for the owner of the books, is made creditor by all the property, and debtor by all the liabilities; while the several accounts are made debtors for all they take from the stock, and creditors by all the responsibilities they undertake. Suppose, for instance, there are £500 in cash at the commencement of the ledger. There will then appear that the merchant has handed over to the cash-box £500, and in the stock account will appear, "Stock creditor by cash, £500;" while in the cash account will appear, "Cash debtor to stock, £500." Suppose that at the beginning there is a debt outstanding of £50 to Smith and Co., then there will appear in the stock account, "Stock debtor to Smith and Co. £50," and in Smith and Co.'s account will appear "Smith and Co. creditors by stock, £50." Thus there is double entry for all that the concern begins with by this contrivance of the stock account.

The account to which everything is placed for which an actual equivalent is not seen in the books is the *profit-and-loss* account. This profit-and-loss account, or the clerk who keeps it, is made answerable for every loss, and the supposed cause of every gain. This account, then, becomes debtor for every loss, and creditor by every gain. If goods be damaged to the amount of £20 by accident, and a loss to that amount occur in their sale, say they cost £80 and sell for £60 cash, it is clear that there is an entry "Cash debtor to goods £60," and "Goods creditor by cash £60." Now, there is an entry of £80 somewhere to the debit of the goods for cash laid out, or bills given, for the whole of the goods. It would affect the accuracy of the accounts to take no notice of this; for when the balance-clerk comes to adjust this account, he would find he receives £20 less than he might have reckoned upon, without any explanation of the reason; and there would be a failure of the principle of double-entry. Since it is convenient that the balance-account of the goods should merely represent the stock in hand at the close, the account of goods therefore lays the responsibility of £20 upon the profit-and-loss account, or there is the entry "Goods creditor by profit-and-loss, £20," and also "Profit-and-loss debtor to goods, £20." Again, in all payments which are not to bring in a specific return, such as house and trade expenses, wages, &c. these several accounts are supposed to adjust matters with the profit-and-loss account before the balance begins. Thus, suppose the outgoings from the mere premises occupied exceed anything those premises yield by £200, or the debits of the house account exceed its credits by £200, the account should be balanced by transferring the responsibility to the profit-and-loss account, under the entries "House expenses creditor by profit-and-loss, £200," "Profit-and-loss debtor to house expenses, £200." In this way the profit-and-loss account steps in from time to time before the balance account commences its operations, in order that that same balance account may consist of *nothing but the necessary matters of account for the next year's ledger.*

This *transference of accounts*, or transfusion of one account into another, requires attentive consideration. The receiving account becomes creditor for the credits, and debtor for the debits, of the transmitting account. The rule, therefore, is: Make the transmitting account balance itself, and, on whichever side it is necessary to enter a balancing sum, make the account debtor or creditor, as the case may be, to the receiving account, and the latter creditor or debtor to the former. Thus, suppose account A is to be transferred to account B, and the latter is to arrange with the balance-accouut. If the two stand as in Roman letters, the processes in Italic letters will occur before the final close.

A, Debtor.	A, Creditor.	B, Debtor.	B, Creditor.
To sundries £100	By sundries £500	To sundries £600	By sundries £400
To B 400		*To Balance* 200	*By A* 400
£500	£500	£800	£800

And the entry in the balance account will be, "Creditor by B, £200," shewing that, on these two accounts, the credits exceed the debits by £200.

Still, before the balance account is made up, it is desirable that the profit-and-loss account should be transferred to the stock account; for the profit and loss of this year is of no moment as a part of next year's ledger, except in so far as it affects the stock at the commencement of the latter. Let this be done, and the balance account may then be made in the form required.

The stock account and the profit-and-loss account, the latter being the only direct channel of alteration for the former, differ in a peculiar manner* from the other preliminary accounts, and the balance account

* The treatises on book-keeping have described this difference in as peculiar a

is a species of umpire. They represent the merchant: their interests are his interests; he is solvent upon the excess of their credits over their debits, insolvent upon the excess of their debits over their credits. It is exactly the reverse in all the other accounts. If a malicious person were to get at the ledger, and put on a cipher to the pounds in various items, with a view of making the concern appear worse than it really is, he would make his alterations on the *debtor* sides of the stock and profit-and-loss accounts, and on the *creditor* sides of all the others. Accordingly, in the balance account, the net stock, after the incorporation of the profit-and-loss account, appears on the *creditor* side (if not, it should be called *amount of insolvency*, not *stock*), and the debts of the concern appear on the same side. But on the debit side of the balance account appear all the assets of the concern (for which the balance-clerk is debtor to the clerks from whom he has taken them).

The young student must endeavour to get the enlarged view of the words debtor and creditor which is requisite, and must then learn by practice (for nothing else will give it) facility in allotting the actual entries in the waste-book to the proper sides of the proper accounts. I do not here pretend to give more than such a view of the subject as may assist him in studying a treatise on book-keeping, which he will probably find to contain little more than examples.

manner. They call these accounts the *fictitious accounts*. Now they represent the merchant himself; their credits are gain to the business, their debits losses or liabilities. If the terms real and fictitious are to be used at all, they are the *real* accounts, and all the others are as *fictitious* as the clerks whom we have supposed to keep them.

APPENDIX VIII.

ON THE REDUCTION OF FRACTIONS TO OTHERS OF NEARLY EQUAL VALUE.

There is a useful method of finding fractions which shall be nearly equal to a given fraction, and with which the computer ought to be acquainted. Proceed as in the rule for finding the greatest common measure of the numerator and denominator, and bring all the quotients into a line. Then write down,

$$\frac{1}{\text{1st Quot.}}\,,\qquad \frac{\text{2d Quot.}}{\text{1st Quot.}\times\text{2d Quot.}+1}$$

Then take the third quotient, multiply the numerator and denominator of the second by it, and add to the products the preceding numerator and denominator. Form a third fraction with the results for a numerator and denominator. Then take the fourth quotient, and proceed with the third and second fractions in the same way; and so on till the quotients are exhausted. For example, let the fraction be $\frac{9131}{13128}$.

9131)13128(1, 2
1137 3997(3, 1
551 586(1, 15
201 35(1, 2
26 9(1, 8
8 1

This is the process for finding the greatest common measure of 9131 and 13128 in its most compact form, and the quotients and fractions are:

1	2	3	1	1	15	1	2	1	8
$\frac{1}{1}$	$\frac{2}{3}$	$\frac{7}{10}$	$\frac{9}{13}$	$\frac{16}{23}$	$\frac{249}{358}$	$\frac{265}{381}$	$\frac{779}{1120}$	$\frac{1044}{1501}$	$\frac{9131}{13128}$

It will be seen that we have thus a set of fractions ending with the original fraction itself, and formed by the above rule, as follows:

$$\text{1st Fraction} = \frac{1}{\text{1st Quot.}} = \frac{1}{1}$$

$$\text{2d Fraction} = \frac{\text{2d Quot.}}{\text{1st Quot.}\times\text{2d Quot.}+1} = \frac{2}{3}$$

$$\text{3d Fraction} = \frac{\text{2d Num}^{\text{r}}.\times\text{3d Quot.}+\text{1st Num}^{\text{r}}.}{\text{2d Den}^{\text{r}}.\times\text{3d Quot.}+\text{1st Den}^{\text{r}}.} = \frac{2\times3+1}{3\times3+1} = \frac{7}{10}$$

$$\text{4th Fraction} = \frac{\text{3d Num}^{\text{r}}.\times\text{4th Quot.}+\text{2d Num}^{\text{r}}.}{\text{3d Den}^{\text{r}}.\times\text{4th Quot.}+\text{2d Den}^{\text{r}}.} = \frac{7\times1+2}{10\times1+3} = \frac{9}{13};$$

and so on. But we have done something more than merely reascend to the original fraction by means of the quotients. The set of fractions, $\frac{1}{1}, \frac{2}{3}, \frac{7}{10}, \frac{9}{13}$, &c. are continually approaching in value to the original fraction, the first being too great, the second too small, the third too great, and so on alternately, but each one being nearer to the given fraction than any of those before it. Thus, $\frac{1}{1}$ is too great, and $\frac{2}{3}$ is too small; but $\frac{2}{3}$ is not so much too small as $\frac{1}{1}$ is too great. And again, $\frac{7}{10}$, though too great, is not so much too great as $\frac{2}{3}$ is too small.

Moreover, the difference of any of the fractions from the original fraction is never greater than a fraction having unity for its numerator and the product of the denominator and the next denominator for its denominator. Thus, $\frac{1}{1}$ does not err by so much as $\frac{1}{3}$, nor $\frac{2}{3}$ by so much as $\frac{1}{30}$, nor $\frac{7}{10}$ by so much as $\frac{1}{130}$, nor $\frac{9}{13}$ by so much as $\frac{1}{299}$, &c.

Lastly, no fraction of a less numerator and denominator can come so near to the given fraction as any one of the fractions in the list. Thus, no fraction with a less numerator than 249, and a less denominator than 358, can come so near to $\frac{9131}{13128}$ as $\frac{249}{358}$.

The reader may take any example for himself, and the test of the accuracy of the process is the ultimate return to the fraction begun with. Another test is as follows: The numerator of the difference of any two consecutive approximating fractions ought to be unity. Thus, in our instance, we have $\frac{16}{23}$ and $\frac{249}{358}$, which, with a common denominator, 23×358, have 5728 and 5727 for their numerators.

As another example, let us examine this question: The length of the year is 365·24224 days, which is called in common life $365\frac{1}{4}$ days. Take the fraction $\frac{24224}{100000}$, and proceed as in the rule.

```
24224)100000(4, 7, 1, 4, 9, 2
  2496    3104
    64     608
     0      32
```

$$\frac{1}{4} \quad \frac{7}{29} \quad \frac{8}{33} \quad \frac{39}{161} \quad \frac{359}{1482} \quad \frac{757}{3125}$$

and $\frac{757}{3125}$ is ·24224 in its lowest terms. Hence, it appears that the excess of the year over 365 days amounts to about 1 day in 4 years, which is not wrong by so much as 1 day in 116 years; more accurately, to 7 days in 29 years, which is not wrong by so much as 1 day in 957 years; more accurately still, to 8 days in 33 years, which is not wrong by so much as 1 day in 5313 years; and so on.

This method may be applied to finding fractions nearly equal to the square roots of integers, in the following manner:

$\sqrt{43} = 6 + \ldots.$

6	1 5 4 5 5 4 5 1 6 6	1 5 4, &c.
1	7 6 3 9 2 9 3 6 7 1	7 6 3, &c.
6	1 1 3 1 5 1 3 1 1 1 2	1 1 3, &c.

Set down the number whose square root is wanted, say 43. This square root is 6 and a fraction. Set down the integer 6 in the first and third row, and 1 in the second row always. Form the successive rows each from the one before, in the following manner:

One row being	The next row has b', a', c', formed in this order, thus,
a	a' = excess of $b'c'$, already formed, over a.
b	b' = quotient of $43-a^2$ divided by b.
c	c' = integer in the quotient of $6+a$ divided by b'.

Thus the second row is formed from the first, as under:

6	1	= excess of 7×1 (both just found) over 6.
1	7	= 43—6×6 divided by 1.
6	1	= integer of 6+6 divided by 7 (just found).

The third row is formed from the second, thus:

1	5	= excess of 1×6 over 1.
7	6	= 43—1×1 divided by 7.
1	1	= integer of 6+1 divided by 6;

and so on. In process of time the second column, 1, 7, 1, occurs again, after which the several columns are repeated in the same order. As a final process, take the set in the lowest line (excluding the first, 6), namely, 1, 1, 3, 1, 5, 1, 3, &c. and use them by the rule given at the beginning of this article, as follows:

1	1	3	1	5	1	3	1	1, &c.
$\frac{1}{1}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{5}{9}$	$\frac{29}{52}$	$\frac{34}{61}$	$\frac{131}{235}$	$\frac{165}{296}$	$\frac{296}{531}$

Hence, $6\frac{165}{296}$ is very near the square root of 43, not erring by so much as $\frac{1}{296 \times 531}$.

If we try it, we shall find $6\frac{165}{296}$ to be $\frac{1941}{296}$, the square of which is $\frac{3767481}{87616}$, or $43-\frac{7}{87616}$.

This rule is of use when it is frequently wanted to use one square root, and therefore desirable to ascertain whether any easy approximation exists by means of a common fraction. For example, $\sqrt{2}$ is often used.

$$\begin{array}{l} \sqrt{2} = 1 + \dots \\ \begin{array}{c|cccccc} 1 & 1 & 1 & & & & \\ 1 & 1 & 1 & & & & \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 \end{array} \\ \quad \frac{1}{2}\ \frac{2}{5}\ \frac{5}{12}\ \frac{12}{29}\ \frac{29}{70}\ \frac{70}{169}, \text{ \&c.} \end{array}$$

Here it appears that $1\frac{29}{70}$ does not err by $\frac{1}{70 \times 169}$; consequently, $\frac{99}{70}$ or $\frac{100-1}{70}$ is, considering the ease of the operation, a fair approximation. In fact, $\frac{99}{70}$ is 1·4142857... the truth being 1·4142135...

The following is an additional example:

$$\begin{array}{l} \sqrt{19} = 4 + \dots \\ \begin{array}{c|ccccccccccc} 4 & 2 & 3 & 3 & 2 & 4 & 4 & 2 & & & & \\ 1 & 3 & 5 & 2 & 5 & 3 & 1 & 3 & & & & \\ 4 & 2 & 1 & 3 & 1 & 2 & 8 & 2 & 1 & 3 & 1 & 2, \text{ \&c.} \end{array} \\ \quad \frac{1}{2}\ \frac{1}{3}\ \frac{4}{11}\ \frac{5}{14}\ \frac{14}{39}, \text{ \&c.} \end{array}$$

APPENDIX IX.

ON SOME GENERAL PROPERTIES OF NUMBERS.

Prop. 1. If a fraction be reduced to its lowest terms, *so called*,* that is, if neither numerator nor denominator be divisible by any integer greater than unity, then no fraction of a smaller numerator and denominator can have the same value.

Let $\frac{a}{b}$ be a fraction in which a and b have no common measure greater than unity: and, if possible, let $\frac{c}{d}$ be a fraction of the same value, c being less than a, and d less than b. Now, since $\frac{a}{b} = \frac{c}{d}$, we have $\frac{a}{c} = \frac{b}{d}$;

* This theorem shews that what is *called* reducing a fraction to its lowest terms (namely, dividing numerator and denominator by their greatest common measure), is correctly so called.

let m be the integer quotient of these last fractions (which must exist, since $a>c$, $b>d$), and let e and f be the remainders. Then

$$\frac{a}{b} \text{ or } \frac{mc+e}{md+f} = \frac{c}{d} = \frac{mc}{md}$$

Hence, $\frac{e}{f}$ and $\frac{mc}{md}$ must be equal, for if not, $\frac{mc+e}{md+f}$ would lie between $\frac{mc}{md}$ and $\frac{e}{f}$, instead of being equal to the former. Hence, $\frac{a}{b} = \frac{e}{f}$; so that if a fraction whose numerator and denominator have no common measure greater than unity, be equal to a fraction of lower numerator and denominator, it is equal to another in which the numerator and denominator are still lower. If we proceed with $\frac{a}{b} = \frac{e}{f}$ in a similar manner, we find $\frac{a}{b} = \frac{g}{h}$ where $g<e$, $h<f$, and so on. Now, if there be any process which perpetually diminishes the terms of a fraction by one or more units at every step, it must at last bring either the numerator or denominator, or both, to o. Let $\frac{a}{b} = \frac{v}{w}$ be one of the steps, and let $a = kv+x$, $b = kw+y$; so that $\frac{kv+x}{kw+y} = \frac{v}{w}$. Now, if $x = 0$ but not y, this is absurd, for it gives $\frac{kv}{kw+y} = \frac{kv}{kw}$. A similar absurdity follows if y be o, but not x; and if both x and y be $= 0$, then $a = kv$, $b = kw$, or a and b have a common measure, k. Now k must be greater than 1, for v and w are less than c and d, which by hypothesis are less than a and b. Consequently a and b have a common measure k greater than 1, which by hypothesis they have not. If, then, a and b be integers not divisible by any integer greater than 1, the fraction $\frac{a}{b}$ is really *in its lowest terms*. Also a and b are said to be *prime to one another*.

Prop. 2. If the product ab be divisible by c, and if c be prime to b, it must divide a. Let $\frac{ab}{c} = d$, then $\frac{b}{c} = \frac{d}{a}$. Now $\frac{b}{c}$ is in its lowest terms; therefore, by the last proposition, d and a must have a common measure. Let the greatest common measure be k, and let $a = kl$, $d = km$. Then $\frac{b}{c} = \frac{km}{kl} = \frac{m}{l}$, and $\frac{m}{l}$ is also in its lowest terms; but so is $\frac{b}{c}$; therefore we must have $m = b$, $l = c$, for otherwise a fraction in its lowest terms would be equal to another of lower terms. Therefore $a = kc$, or a is divisible by c. And from this it follows, that if a number be prime to two others, it is prime to their product. Let a be prime to b and c, then no measure of a can measure either b or c, and no such measure can measure the product bc; for any measure of bc which is prime to one must measure the other.

PROP. 3. If a be prime to b, it is prime to all the powers of b. Every measure* of a is prime to b, and therefore does not divide b. Hence, by the last, no measure of a divides b^2; hence, a is prime to b^2, and so is every measure of it; therefore, no measure of a divides bb^2, consequently a is prime to b^3, and so on.

Hence, if a be prime to b, a cannot divide without remainder any power of b. This is the reason why no fraction can be made into a decimal unless its denominator be measured by no prime† numbers except 2 and 5. For if $\frac{a}{b} = \frac{c}{10^n}$, which last is the general form of a decimal fraction, let $\frac{a}{b}$ be in its lowest terms; then $\frac{10^n a}{b}$ is an integer, whence (Prop. 2) b must divide 10^n, and so must all the divisors of b. If, then, among the divisors of b there be any prime numbers except 2 and 5, we have a prime number (which is of course a number prime to 10) not dividing 10, but dividing one of its powers, which is absurd.

PROP. 4. If b be prime to a, all the multiples of b, as b, $2b$, ... up to $(a-1)b$ must leave different remainders when divided by a. For if, m being greater than n, and both less than a, we have mb and nb giving the same remainder, it follows that $mb-nb$, or $(m-n)b$, is divisible by a; whence (Prop. 2), a divides $m-n$, a number less than itself, which is absurd.

If a number be divided into its prime factors, or reduced to a product of prime numbers only (as in $360 = 2\times2\times2\times3\times3\times5$), and if a, b, c, &c. be the prime factors, and α, β, γ, &c. the number of times they severally enter, so that the number is $a^\alpha \times b^\beta \times c^\gamma \times$ &c., then this can be done in only one way: For any prime number v, not included in the above

* For that which measures a measure is itself a measure; so that if a measure of a could have a measure in common with b, a itself would have a common measure with b.

† A prime number is one which is prime to all numbers except its own multiples, or has no divisors except 1 and itself.

list, is prime to a, and therefore to a^α, to b and therefore to b^β, and therefore to $a^\alpha \times b^\beta$. Proceeding in this way, we prove that v is prime to the complete product above, or to the given number itself.

The number of divisors which the preceding number $a^\alpha b^\beta c^\gamma$.... can have, 0 and itself included, is $(\alpha+1)(\beta+1)(\gamma+1)$.... For a^α has the divisors 1, a, a^2 ... a^α and no others, $\alpha+1$ in all. Similarly, b^β has $\beta+1$ divisors, and so on. Now as all the divisors are made by multiplying together one out of each set, their number (page 202) is $(\alpha+1)(\beta+1)(\gamma+1)$....

If a number, n, be divisible by certain prime numbers, say 3, 5, 7, 11, then the third part of all the numbers up to n is divisible by 3, the fifth part by 5, and so on. But more than this: when the multiples of 3 are omitted, exactly the fifth part of *those which remain* are divisible by 5; for the fifth part of the whole are divisible by 5, and the fifth part of those which are removed are divisible by 5, therefore the fifth part of those which are left are divisible by 5. Again, because the seventh part of the whole are divisible by 7, and the seventh part of those which are divisible by 3, or by 5, or by 15, it follows that when all those which are multiples of 3 or 5, or both, are removed, the seventh part of those which remain are divisible by 7; and so on. Hence, the number of numbers not exceeding n, which are not divisible by 3, 5, 7, or 11, is $\frac{10}{11}$ of $\frac{6}{7}$ of $\frac{4}{5}$ of $\frac{2}{3}$ of n. Proceeding in this way, we find that the number of numbers which are prime to n, that is, which are not divisible by any one of its prime factors, a, b, c,... is

$$n\frac{a-1}{a}\;\frac{b-1}{b}\;\frac{c-1}{c}\;\ldots\ldots \text{ or } a^{\alpha-1}b^{\beta-1}c^{\gamma-1}\ldots(a-1)(b-1)(c-1)\ldots.$$

Thus, 360 being $2^3 3^2 5$, its number of divisors is $4\times3\times2$, or 24, and there are $2^2 3.1.2.4$ or 96 numbers less than 360 which are prime to it.

Prop. 5. If a be prime to b, then the terms of the series, a, a^2, a^3,... severally divided by b, must all leave different remainders, until 1 occurs as a remainder, after which the cycle of remainders will be again repeated.

Let $a \div b$ give the remainder r (not unity); then $a^2 \div b$ gives the same remainder as $ra \div b$, which (Prop. 4) cannot be r: let it be s. Then

a^s+b gives the same remainder as $sa+b$, which (Prop. 4) cannot be either r or s, unless s be 1: let it be t. Then a^4+b gives the same remainder as $ta+b$; if t be not 1, this cannot be either r, s, or t: let it be u. So we go on getting different remainders, until 1 occurs as a remainder; after which, at the next step, the remainder of $a+b$ is repeated. Now, 1 must come at last; for division by b cannot give any remainders but 0, 1, 2, ... $b-1$; and 0 never arrives (Prop. 3), so that as soon as $b-2$ *different* remainders have occurred, no one of which is unity, the next, which must be different from all that precede, must be 1. If not before, then at a^{b-1} we must have a remainder 1; after which the cycle will obviously be repeated.

Thus, 7, 7^2, 7^3, 7^4, &c. will, when divided by 5, be found to give the remainders 2, 4, 3, 1, &c.

Prop. 6. The difference of two mth powers is always divisible without remainder by the difference of the roots; or a^m-b^m is divisible by $a-b$; for

$$a^m-b^m = a^m-a^{m-1}b+a^{m-1}b-b^m = a^{m-1}(a-b)+b(a^{m-1}-b^{m-1})$$

From which, if $a^{m-1}-b^{m-1}$ is divisible by $a-b$, so is a^m-b^m. But $a-b$ is divisible by $a-b$; so therefore is a^2-b^2; so therefore is a^3-b^3; and so on.

Therefore, if a and b, divided by c, leave the same remainder, a^2 and b^2, a^3 and b^3, &c. severally divided by c, leave the same remainders; for this means that $a-b$ is divisible by c. But a^m-b^m is divisible by $a-b$, and therefore by every measure of $a-b$, or by c; but a^m-b^m cannot be divisible by c, unless a^m and b^m, severally divided by c, give the same remainder.

Prop. 7. If b be a prime number, and a be not divisible by b, then a^b and $(a-1)^b+1$ leave the same remainder when divided by b. This proposition cannot be proved here, as it requires a little more of algebra than the reader of this work possesses.*

Prop. 8. In the last case, a^{b-1} divided by b leaves a remainder 1.

* Expand $(a-1)^b$ by the binomial theorem; shew that *when b is a prime number* every coefficient which is not unity is divisible by b; and the proposition follows.

From the last, a^b-a leaves the same remainder as $(a-1)^b+1-a$ or $(a-1)^b-(a-1)$; that is, the remainder of a^b-a is not altered if a be reduced by a unit. By the same rule, it may be reduced another unit, and so on, still without any alteration of the remainder. At last it becomes 1^b-1, or 0, the remainder of which is 0. Accordingly, a^b-a, which is $a(a^{b-1}-1)$, is divisible by b; and since b is prime to a, it must (Prop. 2) divide $a^{b-1}-1$; that is, a^{b-1}, divided by b, leaves a remainder 1, if b be a prime number and a be not divisible by b.

From the above it appears (Prop. 5 and 7), that if a be prime to b, the set 1, a, a^2, a^3, &c. successively divided by b, give a set of remainders beginning with 1, and in which 1 occurs again at a^{b-1}, if not before, and at a^{b-1} certainly (whether before or not), if b be a prime number. From the point at which 1 occurs, the cycle of remainders recommences, and 1 is always the beginning of a cycle. If, then, a^m be the first power which gives 1 for remainder, m must either be $b-1$, or a measure of it, *when b is a prime number.*

But if we divide the terms of the series m, ma, ma^2, ma^3, &c. by b, m being less than b, we have cycles of remainders beginning with m. If 1, r, s, t, &c. be the first set of remainders, then the second set is the set of remainders arising from m, mr, ms, mt, &c. If 1 never occur in the first set before a^{b-1} (except at the beginning), then all the numbers under $b-1$ inclusive are found among the set 1, r, s, t, &c.; and if m be prime to b (Prop. 4), all the same numbers are found, in a different order, among the remainders of m, mr, &c. But should it happen that the set 1, r, s, t, &c. is not complete, then m, mr, ms, &c. may give a different set of remainders.

All these last theorems are constantly verified in the process for reducing a fraction to a decimal fraction. If m be prime to b, or the fraction $\frac{m}{b}$ in its lowest terms, the process involves the successive division of m, $m\times10$, $m\times10^2$, &c. by b. This process can never come to an end unless some power of 10, say 10^n, is divisible by b; which cannot be, if b contain any prime factors except 2 and 5. In every other case the quotient repeats itself, the repeating part sometimes commencing from the first figure, sometimes from a later figure. Thus.

$\frac{1}{7}$ yields $\cdot142857142857$, &c., but $\frac{1}{14}$ gives $\cdot07(142857)(142857)$, &c., and $\frac{1}{28}$ gives $\cdot03(571428)(571428)$, &c.

In $\frac{m}{b}$, the quotient always repeats from the very beginning whenever b is a prime number and m is less than b; and the number of figures in the repeating part is then always $b-1$, or a measure of it. That it must be so, appears from the above propositions.

Before proceeding farther, we write down the repeating part of a quotient, with the remainders which are left after the several figures are formed. Let the fraction be $\frac{1}{17}$, we have

$$0_{10}5_{15}8_{14}8_{4}2_{6}3_{9}5_{5}2_{16}9_{7}4_{2}1_{3}1_{13}7_{11}6_{8}4_{12}7_{1}$$

This may be read thus: 10 by 17, quotient 0, remainder 10; 10^2 by 17, quotient 05, remainder 15; 10^3 by 17, quotient 058, remainder 14; and so on. It thus appears that 10^{16} by 17 leaves a remainder 1, which is according to the theorem.

If we multiply 0588, &c. by *any number under* 17, the same cycle is obtained with a different beginning. Thus, if we multiply by 13, we have 7647058823529411

beginning with what comes after remainder 13 in the first number. If we multiply by 7, we have 4117, &c. The reason is obvious: $\frac{1}{17}\times 13$, or $\frac{13}{17}$, when turned into a decimal fraction, starts with the divisor 130, and we proceed just as we do in forming $\frac{1}{17}$, when within four figures of the close of the cycle.

It will also be seen, that in the last half of the cycle the quotient figures are complements to 9 of those in the first half, and that the remainders are complements to 17. Thus, in $0_{10}5_{15}8_{14}8_{4}$, &c. and $9_{7}4_{2}1_{3}1_{13}$, &c. we see $0+9 = 9$, $5+4 = 9$, $8+1 = 9$, &c., and $10+7 = 17$, $15+2 = 17$, $14+3 = 17$, &c. We may shew the necessity of this as follows: If the remainder 1 never occur till we come to use a^{b-1}, then, b being prime, $b-1$ is even; let it be $2k$. Accordingly, $a^{2k}-1$ is divisible by b; but this is the product of a^k-1 and a^k+1, one of which must be divisible by b. It cannot be a^k-1, for then a power of a preceding the $(b-1)$th would leave remainder 1, which is not the case in our instance: it must then be a^k+1, so that a^k divided by b leaves a remainder $b-1$;

and the nth step concludes the first half of the process. Accordingly, in our instance, we see, b being 17 and a being 10, that remainder 16 occurs at the 8th step of the process. At the next step, the remainder is that yielded by $10(b-1)$, or $9b+b-10$, which gives the remainder $b-10$. But the first remainder of all was 10, and $10+(b-10)=b$. If ever this complemental character occur in any step, it must continue, which we shew as follows: Let r be a remainder, and $b-r$ a subsequent remainder, the sum being b. At the next step after the first remainder, we divide $10r$ by b, and, at the next step after the second remainder, we divide $10b-10r$ by b. Now, since the sum of $10r$ and $10b-10r$ is divisible by b, the two remainders from these new steps must be such as added together will give b, and so on; and the *quotients* added together must give 9, for the sum of the remainders $10r$ and $10b-10r$ yields a quotient 10, of which the two remainders give 1.

If $\frac{1}{59}$ and $\frac{1}{61}$ be taken, the repeating parts will be found to contain 58 and 60 figures. Of these we write down only the first halves, as the reader may supply the rest by the complemental property just given.

01694915254237288135593220338, &c.

016393442622950819672131147540, &c,

Here, then, are two numbers, the first of which multiplied by any number under 59, and the second by any number under 61, can have the products formed by carrying certain of the figures from one end to the other.

But, b being still prime, it may happen that remainder 1 may occur before $b-1$ figures are obtained; in which case, as shewn, the number of figures must be a measure of $b-1$. For example, take $\frac{1}{41}$. The repeating quotient, written as above, has only 5 figures, and 5 measures $41-1$.

$$0_{10}2_{18}4_{16}3_{37}9_{1}$$

Now, this period, it will be found, has its figures merely transposed, if we multiply by 10, 18, 16, or 37. But if we multiply by any other number under 41, we convert this period into the period of another fraction whose denominator is 41. The following are 8 periods which may be found.

$0_{10}2_{18}4_{16}3_{37}9_{1}$	$1_{9}2_{8}1_{39}9_{21}5_{5}$
$0_{20}4_{36}8_{32}7_{33}8_{2}$	$1_{19}4_{26}6_{14}3_{17}4_{6}$
$0_{30}7_{13}3_{7}1_{29}7_{3}$	$2_{28}6_{34}8_{12}2_{38}9_{11}$
$0_{40}9_{31}7_{23}5_{25}6_{4}$	$3_{27}6_{24}5_{35}8_{22}5_{15}$

To find $\frac{m}{41}$, look out for m among the remainders, and take the period in which it is, beginning after the remainder. Thus, $\frac{34}{41}$ is ·8292682926, &c., and $\frac{15}{41}$ is ·3658536585, &c. These periods are complemental, four and four, as 02439 and 97560, 07317 and 92682, &c. And if the first number, 02439, be multiplied by any number under 41, look for that number among the remainders, and the product is found in the period of that remainder by beginning after the remainder. Thus, 02439 multiplied by 23 gives 56097, and by 6 gives 14634.

The reader may try to decipher for himself how it is that, with no more figures than the following, we can extend the result of our division. The fraction of which the period is to be found is $\frac{1}{87}$.

```
87)100(01149425
     130
      430
       820                   01149425×25
        370                     28735625×25
         220                     718390625×25
          460                     17959765625×25
           25                      448994140625
    01149425287356 25
                   718390625
                         1795976 5625
                                448994
    011494252873563218390804597 7|011494
                                 |
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APPENDIX X.

ON COMBINATIONS.

There are some things connected with combinations which I place in an appendix, because I intend to demonstrate them more briefly than the matters in the text.

Suppose a number of boxes, say 4, in each of which there are counters, say 5, 7, 3, and 11 severally. In how many ways can one counter be taken out of each box, the order of going to the boxes not being regarded. *Answer*, in $5\times7\times3\times11$ ways. For out of the first box we may draw a counter in 5 different ways, and to each such drawing we may annex a drawing from the second in 7 different ways—giving 5×7 ways of making a drawing from the first two. To each of these we may annex a drawing from the third box in 3 ways—giving $5\times7\times3$ drawings from the first three; and so on. The following statements may now be easily demonstrated, and similar ones made as to other cases.

If the order of going to the boxes make a difference, and if a, b, c, d be the numbers of counters in the several boxes, there are $4\times2\times3\times1\times a\times b\times c\times d$ distinct ways. If we want to draw, say 2 out of the first box, 3 out of the second, 1 out of the third, and 3 out of the fourth, and if the order of the boxes be not considered, the number of ways is

$$a\frac{a-1}{2}\times b\frac{b-1}{2}\frac{b-2}{3}\times c\times d\frac{d-1}{2}\frac{d-2}{3}$$

If the order of going to the boxes be considered, we must multiply the preceding by $4\times3\times2\times1$. If the order of the drawings out of the boxes makes a difference, but not the order of the boxes, then the number of ways is

$$a(a-1)b(b-1)(b-2)cd(d-1)(d-2)$$

The nth power of a, or a^n, represents the number of ways in which a counters *differently marked* can be distributed in n boxes, order of placing them in each box not being considered. Suppose we want to distribute 4 differently-marked counters among 7 boxes. The first counter may go into either box, which gives 7 ways; the second counter may go into either; and any of the first 7 allotments may be combined with any one of the second 7, giving 7×7 distinct ways; the third counter varies each of these in 7 different ways, giving $7\times7\times7$ in all; and so on. But if the counters be undistinguishable, the problem is a very different thing.

Required the number of ways in which a number can be compounded of other numbers, different orders counting as different ways. Thus, 1+3+1 and 1+1+3 are to be considered as distinct ways of making 5. It will be obvious, on a little examination, that each number can be composed in exactly twice as many ways as the preceding number. Take 8 for instance. If every possible way of making 7 be written down, 8 may be made either by increasing the last component by a unit, or by annexing a unit at the end. Thus, 1+3+2+1 may yield 1+3+2+2, or 1+3+2+1+1: and all the ways of making 8 will thus be obtained; for any way of making 8, say $a+b+c+d$, must proceed from the following mode of making 7, $a+b+c+(d-1)$. Now, $(d-1)$ is either 0—that is, d is unity and is struck out—or $(d-1)$ remains, a number 1 less than d. Hence it follows that the number of ways of making n is 2^{n-1}. For there is obviously 1 way of making 1, 2 of making 2; then there must be, by our rule, 2^2 ways of making 3, 2^3 ways of making 4; and so on.

1	1+1	1+1+1	1+1+1+1
			1+1+2
		1+2	1+2+1
			1+3
	2	2+1	2+1+1
			2+2
		3	3+1
			4

This table exhibits the ways of making 1, 2, 3, and 4. Hence it follows (which I leave the reader to investigate) that there are twice as many ways of forming $a+b$ as there are of forming a and then annexing to it a formation of b; four times as many ways of forming $a+b+c$ as there are of annexing to a formation of a formations of b and of c; and so on. Also, in summing numbers which make up $a+b$, there are ways in which a is a rest, and ways in which it is not, and as many of one as of the other.

Required the number of ways in which a number can be compounded of odd numbers, different orders counting as different ways. If a be the number of ways in which n can be so made, and b the number of ways in which $n+1$ can be made, then $a+b$ must be the number of ways in which $n+2$ can be made; for every way of making 12 out of odd numbers is either a way of making 10 with the last number increased by 2, or a way of making 11 with a 1 annexed. Thus, 1+5+3+3 gives 12, formed from 1+5+3+1 giving 10. But 1+9+1+1 is formed from 1+9+1 giving 11. Consequently, the number of ways of forming 12 is the sum of the number of ways of forming 10 and of forming 11. Now, 1 can only be formed in 1 way, and 2 can only be formed in 1 way; hence 3 can only be formed in 1+1 or 2 ways, 4 in only 1+2 or 3 ways. If we take the series 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, &c. in which each number is the sum of the two preceding, then the nth number of this set is the number of ways (orders counting) in which n can be formed of odd numbers. Thus, 10 can be formed in 55 ways, 11 in 89 ways, &c.

Shew that the number of ways in which mk can be made of numbers divisible by m (orders counting) is 2^{k-1}.

In the two series,	1	1	1	2	3	4	6	9	13	19	28,	&c.
	0	1	0	1	1	1	2	2	3	4	5,	&c.,

the first has each new term after the third equal to the sum of the last and last but two; the second has each new term after the third equal to the sum of the last but one and last but two. Shew that the nth number in the first is the number of ways in which n can be made up of numbers which, divided by 3, leave a remainder 1; and that the nth number in the second is the number of ways in which n can be made up of numbers which, divided by 3, leave a remainder 2.

It is very easy to shew in how many ways a number can be made up of a given number of numbers, if different orders count as different ways. Suppose, for instance, we would know in how many ways 12 can be thus made of 7 numbers. If we write down 12 units, there are 11 intervals between unit and unit. There is no way of making 12 out of 7 numbers which does not answer to distributing 6 partition-marks in the intervals, 1 in each of 6, and collecting all the units which are not separated by partition-marks. Thus, 1+1+3+2+1+2+2, which is one way of making 12 out of 7 numbers, answers to

1 | 1 | 111 | 11 | 1 | 11 | 11

in which the partition-marks come in the 1st, 2d, 5th, 7th, 8th, and 10th of the 11 intervals. Consequently, to ask in how many ways 12 can be made of 7 numbers, is to ask in how many ways 6 partition-marks can be placed in 11 intervals; or, how many combinations or selections can be made of 6 out of 11. The answer is,

$$\frac{11\times10\times9\times8\times7\times6}{1\times2\times3\times4\times5\times6}, \text{ or } 462.$$

Let us denote by m_n the number of ways in which m things can be taken out of n things, so that m_n is the abbreviation for

$$n\times\frac{n-1}{2}\times\frac{n-2}{3}\ldots\text{ as far as }\frac{n-m+1}{m}$$

Then m_n also represents the number of ways in which $m+1$ numbers can be put together to make $n+1$. What we proved above is, that 6_{11} is the number of ways in which we can put together 7 numbers to make 12. There will now be no difficulty in proving the following:

$$2^n = 1+1_n+2_n+3_n\ldots\ldots+n_n$$

In the preceding question, 0 did not enter into the list of numbers used. Thus, 3+1+0+0 was not considered as one of the ways of putting together four numbers to make 5. But let us now ask, what is the number of ways of putting together 7 numbers to make 12, allowing 0 to be in the list of numbers. There can be no more (nor fewer) ways of doing this than of putting 7 numbers together, among which 0 is *not* included, to make 19. Take every way of making 12 (0 included), and put on 1 to each number, and we get a way of making 19 (0 not included). Take any way of making 19 (0 not included), and strike off 1 from each number, and we have one of the ways of making 12 (0 included). Accordingly, 6_{18} is the number of ways of putting together 7 numbers (0 being allowed) to make 12. And $(m-1)_{n+m-1}$ is the number of ways of putting together m numbers to make n, 0 being included.

This last amounts to the solution of the following: In how many ways can n counters (undistinguishable from each other) be distributed into m boxes? And the following will now be easily proved: The number of ways of distributing c undistinguishable counters into b boxes is $(b-1)_{b+c-1}$, if any box or boxes may be left empty. But if there must be 1 at least in each box, the number of ways is $(b-1)_{c-1}$; if there must be 2 at least in each box, it is $(b-1)_{c-b-1}$; if there must be 3 at least in each box, it is $(b-1)_{c-2b-1}$; and so on.

The number of ways in which m *odd* numbers can be put together to make n, is the same as the number of ways in which m even numbers (0 included) can be put together to make $n-m$; and this is the number of ways in which m numbers (odd or even, 0 included) can be put together to make $\frac{1}{2}(n-m)$. Accordingly, the number of ways in which m odd numbers can be put together to make n is the same as the number of combinations of $m-1$ things out of $\frac{1}{2}(n-m)+m-1$, or $\frac{1}{2}(n+m)-1$. Unless n and m be both even or both odd, the problem is evidently impossible.

There are curious and useful relations existing between numbers of combinations, some of which may readily be exhibited, under the simple expression of m_n to stand for the number of ways in which m things may be taken out of n. Suppose we have to take 5 out of 12: Let the 12 things be marked A, B, C, &c. and set apart one of them, A. Every collection of 5 out of the 12 either does or does not include A. The number of the latter sort must be 5_{11}; the number of the former sort must be 4_{11}, since it is the number of ways in which the *other four* can be chosen out of all but A. Consequently, 5_{12} must be $5_{11}+4_{11}$, and thus we prove in every case,

$$m_n = m_{n-1}+(m-1)_{n-1}$$

0_n and n_n both are 1; for there is but one way of taking *none*, and but one way of taking *all*. And again m_n and $(n-m)_n$ are the same things. And if m be greater than n, m_n is 0; for there are no ways of doing it. We make one of our preceding results more symmetrical if we write it thus,

$$2^n = 0_n+1_n+2_n+\ldots+n_n$$

If we now write down the table of symbols in which the $\overline{m+1}$th

	0	1	2	3,	&c.
1	0_1	1_1	2_1	3_1,	&c.
2	0_2	1_2	2_2	3_2,	&c.
3	0_3	1_3	2_3	3_3,	&c.
&c.	&c.	&c.	&c.	&c.	

number of the *n*th row represents m_n, the number of combinations of *m* out of *n*, we see it proved above that the law of formation of this table is as follows: Each number is to be the sum of the number above it and the number preceding the number above it. Now, the first row must be 1, 1, 0, 0, 0, &c. and the first column must be 1, 1, 1, 1, &c. so that we have a table of the following kind, which may be carried as far as we please:

	0	1	2	3	4	5	6	7	8	9	10
1	1	1	0	0	0	0	0	0	0	0	0
2	1	2	1	0	0	0	0	0	0	0	0
3	1	3	3	1	0	0	0	0	0	0	0
4	1	4	6	4	1	0	0	0	0	0	0
5	1	5	10	10	5	1	0	0	0	0	0
6	1	6	15	20	15	6	1	0	0	0	0
7	1	7	21	35	35	21	7	1	0	0	0
8	1	8	28	56	70	56	28	8	1	0	0
9	1	9	36	84	126	126	84	36	9	1	0
10	1	10	45	120	210	252	210	120	45	10	1

Thus, in the row 9, under the column headed 4, we see 126, which is $9\times8\times7\times6\div(1\times2\times3\times4)$, the number of ways in which 4 can be chosen out of 9, which we represent by 4_9.

If we add the several rows, we have 1+1 or 2, 1+2+1 or 2^2, next 1+3+3+1 or 2^3, &c. which verify a theorem already announced; and the law of formation shews us that the several columns are formed thus:

$$\begin{array}{ccc} 1\ 1 & 1\ 2\ 1 & 1\ 3\ 3\ 1 \\ 1\ 1 & 1\ 2\ 1 & 1\ 3\ 3\ 1 \\ \hline 1\ 2\ 1 & 1\ 3\ 3\ 1 & 1\ 4\ 6\ 4\ 1,\ \&c. \end{array}$$

so that the sum in each row must be double of the sum in the preceding. But we can carry the consequences of this mode of formation further. If we make the powers of $1+x$ by actual algebraical multiplication, we see that the process makes the same oblique addition in the formation of the numerical multipliers of the powers of x.

$$\begin{array}{rl} 1+x & \\ 1+x & \\ \hline 1+x & \\ x+x^2 & \\ \hline 1+2x+x^2 & \end{array} \qquad \begin{array}{l} 1+2x+x^2 \\ 1+x \\ \hline 1+2x+x^2 \\ \quad x+2x^2+x^3 \\ \hline 1+3x+3x^2+x^3 \end{array}$$

Here are the second and third powers of $1+x$: the fourth, we can tell beforehand from the table, must be $1+4x+6x^2+4x^3+x^4$; and so on. Hence we have

$$(1+x)^n = 0_n+1_n x+2_n x^2+3_n x^3+\ldots\ldots+n_n x^n$$

which is usually written with the symbols 0_n, 1_n, &c. at length, thus,

$$(1+x)^n = 1+nx+n\frac{n-1}{2}x^2+n\frac{n-1}{2}\frac{n-2}{3}x^3+\&c.$$

This is the simplest case of what in algebra is called the *binomial theorem*. If instead of $1+x$ we use $x+a$, we get

$$(x+a)^n = x^n+1_n a x^{n-1}+2_n a^2 x^{n-2}+3_n a^3 x^{n-3}+\ldots\ldots+n_n a^n$$

We can make the same table in another form. If we take a row of ciphers beginning with unity, and setting down the first, add the next, and then the next, and so on, and then repeat the process with one step less, and then again with one step less, we have the following:

1	0	0	0	0	0	0
1	1	1	1	1	1	1
1	2	3	4	5	6	
1	3	6	10	15		
1	4	10	20			
1	5	15				
1	6					
1						

In the oblique columns we see 1 1, 1 2 1, 1 3 3 1, &c. the same as in the original table, and formed by the same additions. If, before making the additions, we had always multiplied by a, we should have got the several components of the powers of $1+a$, thus,

1	0	0	0	0
1	a	a^2	a^3	a^4
1	$2a$	$3a^2$	$4a^3$	
1	$3a$	$6a^2$		
1	$4a$			
1				

where the oblique columns $1+a$, $1+2a+a^2$, $1+3a+3a^2+a^3$, &c., give the several powers of $1+a$. If instead of beginning with 1, 0, 0, &c. we had begun with p, 0, 0, &c. we should have got p, $p\times 4a$, $p\times 6a^2$, &c. at the bottom of the several columns; and if we had written at the top x^4, x^3, x^2, x, 1, we should have had all the materials for forming $p(x+a)^4$ by multiplying the terms at the top and bottom of each column together, and adding the results.

Suppose we follow this mode of forming $p(x+a)^3+q(x+a)^2+r(x+a)+s$.

x^3	x^2	x	1	x^2	x	1	x	1	1
p	0	0	0	q	0	0	r	0	s
p	pa	pa^2	pa^3	q	qa	qa^2	r	ra	
p	$2pa$	$3pa^2$		q	$2qa$		r		
p	$3pa$			q					
p									

$$px^3+3pax^2+3pa^2x+pa^3+qx^2+2qax+qa^2+rx+ra+s$$
$$=px^3+(3pa+q)x^2+(3pa^2+2qa+r)x+pa^3+qa^2+ra+s$$

Now, observe that all this might be done in one process, by entering q, r, and s under their proper powers of x in the first process, as follows:

x^3	x^2	x	1
p	q	r	s
p	$pa+q$	pa^2+qa+r	pa^3+qa^2+ra+s
p	$2pa+q$	$3pa^2+2qa+r$	
p	$3pa+q$		
p			

This process* is the one used in Appendix XI., with the slight alteration of varying the sign of the last letter, and making subtractions instead of additions in the last column. As it stands, it is the most convenient mode of writing $x+a$ instead of x in a large class of algebraical expressions. For instance, what does $2x^5+x^4+3x^2+7x+9$ become when $x+5$ is written instead of x? The expression, made complete, is,

$$2x^5 + 1x^4 + 0x^3 + 3x^2 + 7x + 9$$

	1	0	3	7	9
2	11	55	278	1397	6994
2	21	160	1078	6787	
2	31	315	2653		
2	41	520			
2	51				

Answer, $2x^5+51x^4+520x^3+2653x^2+6787x+6994$.

APPENDIX XI.

ON HORNER'S METHOD OF SOLVING EQUATIONS.

THE rule given in this chapter is inserted on account of its excellence as an exercise in computation. The examples chosen will require but little use of algebraical signs, that they may be understood by those who know no more of algebra than is contained in the present work.

To solve an equation such as

$$2x^4+x^2-3x = 416793,$$

or, as it is usually written,

$$2x^4+x^2-3x-416793 = 0,$$

we must first ascertain by trial not only the first figure of the root, but also the denomination of it: if it be a 2, for instance, we must know whether it be 2, or 20, or 200, &c., or ·2, or ·02, or ·002, &c. This must

* The principle of this mode of demonstration of Horner's method was stated in Young's Algebra (1823), being the earliest elementary work in which that method was given.

be found by trial; and the shortest way of making the trial is as follows: Write the expression in its complete form. In the preceding case the form is not complete, and the complete form is

$$2x^4+0x^3+1x^2-3x-416793.$$

To find what this is when x is any number, for instance, 3000, the best way is to take the first multiplier (2), multiply it by 3000, and take in the next multiplier (0), multiply the result by 3000, and take in the next multiplier (1), and so on to the end, as follows:

$$2\times3000+0 = 6000;\ 6000\times3000+1 = 18000001$$
$$18000001\times3000-3 = 54000002997$$
$$54000002997\times3000-416793 = 162000008574207$$

Now try the value of the above when $x = 30$. We have then, for the steps, 60 $(2\times30+0)$, 1801, 54027, and lastly,

$$1620810-416793,$$

or $x = 30$ makes the first terms greater than 416793. Now try $x = 20$ which gives 40, 801, 16017, and lastly,

$$320340-416793,$$

or $x = 20$ makes the first terms less than 416793. Between 20 and 30, then, must be a value of x which makes $2x^4+x^2-3x$ equal to 416793. And this is the preliminary step of the process.

Having got thus far, write down the coefficients +2, 0, +1, −3, and −416793, each with its proper algebraical sign, except the last, in which let the sign be changed. This is the most convenient way when the last sign is −. But if the last sign be +, it may be more convenient to let it stand, and change all which come before. Thus, in solving $x^3-12x+1 = 0$, we might write

$$-1 \quad 0 \quad +12 \quad 1$$

whereas in the instance before us, we write

$$+2 \quad 0 \quad +1 \quad -3 \quad 416793$$

Having done this, take the highest figure of the root, properly named, which is 2 tens, or 20. Begin with the first column, multiply by 20, and join it to the number in the next column; multiply that by 20, and join it to the number in the next column; and so on. But when you come to the last column, subtract the product which comes out of the preceding column, or join it to the last column after changing its sign. When this has been done, repeat the process with the numbers which now stand in the columns, omitting the last, that is, the subtracting step; then repeat it again, going only as far as the last column but two, and so on, until the columns present a set of rows of the following appearance:

a	b	c	d	e
	f	g	h	i
	k	l	m	
	n	o		
	p			

to the formation of which the following is the key:

$f = 20a+b,\quad g = 20f+c,\quad h = 20g+d,\quad i = e-20h,$

$k = 20a+f,\quad l = 20k+g,\quad m = 20l+h,$

$n = 20a+k,\quad o = 20n+l,$

$p = 20a+n.$

We call this *Horner's Process*, from the name of its inventor. The result is as follows:

2	0	1	−3	416793	(20
	40	801	16017	96453	
	80	2401	64037		
	120	4801			
	160				

We have now before us the row

2   160   4801   64037   96453

which furnishes our means of guessing at the next, or units' figure of the root.

Call the last column the *dividend*, the last but one the *divisor*, and all that come before *antecedents*. See how often the dividend contains the divisor; this gives the guess at the next figure. The guess is a true one,* if, on applying Horner's process, the divisor result, augmented as it is by the antecedent processes, still go as many times in the dividend. For example, in the case before us, 96453 contains 64037 once; let 1 be put on its trial. Horner's process is found to succeed, and we have for the second process,

2	160	4801	64037	96453
	162	4963	69000	27453
	164	5127	74127	
	166	5293		
	168			

As soon as we come to the fractional portion of the root, the process assumes a more† methodical form.

The equation being of the *fourth* degree, annex *four* ciphers to the dividend, *three* to the divisor, *two* to the antecedent, and *one* to the previous antecedent, leaving the first column as it is; then find the new figure by the dividend and divisor, as before,‡ and apply Horner's process. Annex ciphers to the results, as before, and proceed in the same way. The annexing of the ciphers prevents our having any thing to do with decimal points, and enables us to use the quotient-figures without paying any attention to their *local* values. The following exhibits the whole process from the beginning, carried as far as it is here intended to go before beginning the contraction, which will give more figures, as in the rule for the square root. The following, then, is the process as far as one decimal place:

* Various exceptions may arise when an equation has two nearly equal roots. But I do not here introduce algebraical difficulties; and a student might give himself a hundred examples, taken at hazard, without much chance of lighting upon one which gives any difficulty.

† This form might be also applied to the integer portions; but it is hardly needed in such instances as usually occur. See the article *Involution and Evolution* in the *Supplement* to the *Penny Cyclopædia*.

‡ After the second step, the trial will rarely fail to give the true figure.

<table>
<tr><td>2</td><td>0</td><td>1</td><td>−3</td><td>416793(213</td></tr>
<tr><td></td><td>40</td><td>801</td><td>16017</td><td>96453</td></tr>
<tr><td></td><td>80</td><td>2401</td><td>64037</td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td>274530000</td></tr>
<tr><td></td><td>120</td><td>4801</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td>69000</td><td>47339778</td></tr>
<tr><td></td><td>160</td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td>4963</td><td>74127000</td><td></td></tr>
<tr><td></td><td>162</td><td>5127</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td>75730074</td><td></td></tr>
<tr><td></td><td>164</td><td>529300</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td>77348376</td><td></td></tr>
<tr><td></td><td>166</td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td>534358</td><td></td><td></td></tr>
<tr><td></td><td>1680</td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td>539434</td><td></td><td></td></tr>
<tr><td></td><td>1686</td><td>544528</td><td></td><td></td></tr>
<tr><td></td><td>1692</td><td></td><td></td><td></td></tr>
<tr><td></td><td>1698</td><td></td><td></td><td></td></tr>
<tr><td></td><td>1704</td><td></td><td></td><td></td></tr>
</table>

If we now begin the contraction, it is good to know beforehand on what number of additional root-figures we may reckon. We may be pretty certain of having nearly as many as there are figures in the divisor when we begin to contract—one less, or at least two less. Thus, there being now eight figures in the divisor, we may conclude that the contraction will give us at least six more figures. To begin the contraction, let the dividend stand, cut off one figure from the divisor, two from the column before that, three from the one before that, and so on. Thus, our contraction begins with

<table>
<tr><td>|0002</td><td>1|704</td><td>5445|28</td><td>7734837|6</td><td>47339778</td></tr>
</table>

The first column is rendered quite useless here. Conduct the process as before, using only the figures which are not cut off. But it will be better to go as far as the first figure cut off, carrying from the second figure cut off. We shall then have as follows:

<table>
<tr><td>1|704</td><td>5445|28</td><td>7734837|6</td><td>47339778(6</td></tr>
<tr><td></td><td>5455|5</td><td>7767570|6</td><td>734354</td></tr>
<tr><td></td><td>5465|7</td><td>7800364|8</td><td></td></tr>
<tr><td></td><td>5475|9</td><td></td><td></td></tr>
</table>

At the next contraction the column 1|704 becomes |001704, and is quite useless. The next step, separately written (which is not, however, necessary in working), is

```
54|759          780036|48          734354(0
```

Here the dividend 734354 does not contain the divisor 780036, and we, therefore, write 0 as a root figure and make another contraction, or begin with

```
|54759          78003|648          734354(9
                78008|5             32277
                78013|4
```

At the next contraction the first column becomes |0054759, and is quite useless, so that the remainder of the process is the contracted division.

```
7801|34)32277(4137
         1072
          292
           58
            3
```

and the root required is 21·36094137.

I now write down the complete process for another equation, one root of which lies between 3 and 4: it is

$$x^3 - 10x + 1 = 0$$

```
1      0                 -10                   -1(3·1110390520730990796
       3                 -1                      2000
       6                 1700                     209000
       9 0               1791                      19769000
       9 1               188300                      743369000000
       9 2               189231                      1723117102730000
       9 30              19016300                        991247447681
       9 31              19025631                         39462875420
       9 32              1903496300 0 0                    1391491559
       9 33 0            1903524299 0 9                      58993123
       9 33 1            1903552298 2 7 0 0                   1886047
       9 33 2            1903560698 0 5|9|1                    172835
       9 33 30 0         1903569097 8 5|6|3                      1515
       9 33 30 3         1903569144 5 2|2|                        183
       9 33 30 6         1903569191 1|8|8                          12
       9 33 30 90        1903569193 0|6|                            1
       9 33|30|99        1903569194|9|3|
       9 33|31|08        .........|  |  |
     |09|33|31|17
```

The student need not repeat the rows of figures so far as they come under one another: thus, it is not necessary to repeat 190356. But he must use his own discretion as to how much it would be safe for him to omit. I have set down the whole process here as a guide.

The following examples will serve for exercise:

1. $2x^3-100x-7=0$ $x=7\cdot10581133$.
2. $x^4+x^3+x^2+x=6000$ $x=8\cdot531437726$.
3. $x^3+3x^2-4x-10=0$ $x=1\cdot8956949165504$.
4. $x^3+100x^2-5x-2173=0$ $x=4\cdot5822460710 58464$.
5. $\sqrt[3]{2}=1\cdot259921049894873164767210607278$.*
6. $x^3-6x=100$ $x=5\cdot071351748731$.
7. $x^3+2x^2+3x=300$ $x=5\cdot955259671 22398$.
8. $x^3+x=1000$ $x=9\cdot96666679$.
9. $27000x^3+27000x=26999999$ $x=9\cdot9666666\ldots\ldots$
10. $x^3-6x=100$ $x=5\cdot0713517487$.
11. $x^5-4x^4+7x^3-863=0$ $x=4\cdot5195507$.
12. $x^3-20x+8=0$ $x=4\cdot66003769300087278$.
13. $x^3+x^2+x-10=0$ $x=1\cdot737370233$.
14. $x^3-46x^2-36x+18=0$ $x=46\cdot7616301847$, or $x=\cdot3471623192$.
15. $x^3+46x^2-36x-18=0$ $x=1\cdot1087925037$.
16. $8991x^3-162838x^2+746271x-81000=0$ $x=\cdot111222333444555\ldots\ldots$
17. $729x^3-486x^2+99x-6=0$ $x=\cdot1111\ldots$, or $\cdot2222\ldots$, or $\cdot3333\ldots$
18. $2x^3+3x^2-4x=500$ $x=5\cdot9348179623 1515279$.
19. $x^3+2x^2+x-150=0$ $x=4\cdot66840901455419832537429912017058 99$.
20. $x^3+x=x^2+500$ $x=8\cdot2409635581448585 26963$.
21. $x^3+2x^2+3x-10000=0$ $x=20\cdot852905526009$.
22. $x^5-4x-2000=0$ $x=4\cdot581400362$.
23. $10x^3-33x^2-11x-100=0$ $x=4\cdot14679780858 4278785$.
24. $x^4+x^3+x^2+x=127694$ $x=18\cdot64482373095$.
25. $10x^3+11x^2+12x=100000$ $x=21\cdot16559955545 08805$.
26. $x^3+x=13$ $x=2\cdot209753301208849$.
27. $x^3+x^2-4x-1600=0$ $x=11\cdot482837157$.
28. $x^3-2x=5$

$x=2\cdot09455148154232659148238654057930296385730610 5628239$.

* The solution of $x^3+0x^2+0x-2=0$.

29. $x^4-80x^3+24x^2-6x-80379639=0 \qquad x=123.$*

30. $x^3-242x^2-6315x+2577096=0 \qquad x=123.$*

31. $2x^4-3x^3+6x-8=0 \qquad x=1{\cdot}41421356237309504880{3}.$*

32. $x^4-19x^3+132x^2-302x+200=0 \qquad x=1{\cdot}02804$, or 4, or $6{\cdot}57653$, or $7{\cdot}39543$†.

33. $7x^4-11x^3+6x^2+5x=215 \qquad x=2{\cdot}70648049385791.$†

34. $7x^5+6x^4+5x^3+4x^2+3x=11 \qquad x={\cdot}7707688196226585223792965{0}5.$†

35. $4x^6+7x^5+9x^4+6x^3+5x^2+3x=792$

$x=2{\cdot}052042176879605365214043401281201973460275599545541724214.$†

36. $2187x^4-2430x^3+945x^2-150x+8=0 \qquad x={\cdot}1111\ldots$, or ${\cdot}2222\ldots$, or ${\cdot}3333\ldots$, or ${\cdot}4444\ldots$

APPENDIX XII.

RULES FOR THE APPLICATION OF ARITHMETIC TO GEOMETRY.

The student should make himself familiar with the most common terms of geometry, after which the following rules will present no difficulty. In them all, it must be understood, that when we talk of multiplying one line by another, we mean the repetition of one line as often as there are units of a given kind, as feet or inches, in another. In any other sense, it is absurd to talk of multiplying a quantity by another quantity. All quantities of the same kind should be represented in numbers of the same unit; thus, all the lines should be either feet and decimals of a foot, or inches and decimals of an inch, &c. And in whatever unit a length is represented, a surface is expressed in the corresponding square units, and a solid in the corresponding cubic units. This being understood, the rules apply to all sorts of units.

To find the area of a rectangle. Multiply together the units in

* These examples are taken from a paper on the subject, by Mr. Peter Gray, in the *Mechanics' Magazine.*

† These examples are taken from the late Mr. Peter Nicholson's Essay on Involution and Evolution.

two sides which meet, or multiply together two sides which meet; the product is the number of square units in the area. Thus, if 6 feet and 5 feet be the sides, the area is 6×5, or 30 square feet. Similarly, the area of a square of 6 feet long is 6×6, or 36 square feet (234).

To find the area of a parallelogram. Multiply one side by the perpendicular distance between it and the opposite side; the product is the area required in square units.

*To find the area of a trapezium.** Multiply either of the two sides which are not parallel by the perpendicular let fall upon it from the middle point of the other.

To find the area of a triangle. Multiply any side by the perpendicular let fall upon it from the opposite vertex, and take half the product. Or, halve the sum of the three sides, subtract the three sides severally from this half sum, multiply the four results together, and find the square root of the product. The result is the number of square units in the area; and twice this, divided by either side, is the perpendicular distance of that side from its opposite vertex.

To find the radius of the internal circle which touches the three sides of a triangle. Divide the area, found in the last paragraph, by half the sum of the sides.

Given the two sides of a right-angled triangle, to find the hypothenuse. Add the squares of the sides, and extract the square root of the sum.

Given the hypothenuse and one of the sides, to find the other side. Multiply the sum of the given lines by their difference, and extract the square root of the product.

To find the circumference of a circle from its radius, very nearly. Multiply twice the radius, or the diameter, by 3·1415927, taking as many decimal places as may be thought necessary. For a rough computation, multiply by 22 and divide by 7. For a very exact computation, in which decimals shall be avoided, multiply by 355 and divide by 113. See (131), last example.

To find the arc of a circular sector, very nearly, knowing the radius

* A four-sided figure, which has two sides parallel, and two sides not parallel.

and the angle. Turn the angle into seconds,* multiply by the radius, and divide the product by 206265. The result will be the number of units in the arc.

To find the area of a circle from its radius, very nearly. Multiply the square of the radius by 3·1415927.

To find the area of a sector, very nearly, knowing the radius and the angle. Turn the angle into seconds, multiply by the square of the radius, and divide by 206265×2, or 412530.

To find the solid content of a rectangular parallelopiped. Multiply together three sides which meet: the result is the number of cubic units required. If the figure be not rectangular, multiply the area of one of its planes by the perpendicular distance between it and its opposite plane.

To find the solid content of a pyramid. Multiply the area of the base by the perpendicular let fall from the vertex upon the base, and divide by 3.

To find the solid content of a prism. Multiply the area of the base by the perpendicular distance between the opposite bases.

To find the surface of a sphere. Multiply 4 times the square of the radius by 3·1415927.

To find the solid content of a sphere. Multiply the cube of the radius by $3{\cdot}1415927 \times \frac{4}{3}$, or 4·18879.

To find the surface of a right cone. Take half the product of the circumference of the base and slanting side. *To find the solid content*, take one-third of the product of the base and the altitude.

To find the surface of a right cylinder. Multiply the circumference of the base by the altitude. *To find the solid content*, multiply the area of the base by the altitude.

The weight of a body may be found, when its solid content is known, if the weight of one cubic inch or foot of the body be known. But it

* The right angle is divided into 90 equal parts called *degrees*, each degree into 60 equal parts called *minutes*, and each minute into 60 equal parts called *seconds*. Thus, $2^{0}\ 15'\ 40''$ means 2 degrees, 15 minutes, and 40 seconds.

is usual to form tables, not of the weights of a cubic unit of different bodies, but of the proportion which these weights bear to some one amongst them. The one chosen is usually distilled water, and the proportion just mentioned is called the *specific gravity*. Thus, the specific gravity of gold is 19·362, or a cubic ot of gold is 19·362 times as heavy as a cubic foot of distilled water. Suppose now the weight of a sphere of gold is required, whose radius is 4 inches. The content of this sphere is 4×4×4×4·1888, or 268·0832 cubic inches; and since, by (217), each cubic inch of water weighs 252·458 grains, each cubic inch of gold weighs 252·458×19·362, or 4888·091 grains; so that 268·0832 cubic inches of gold weigh 268·0832×4888·091 grains, or $227\frac{1}{2}$ pounds troy nearly. Tables of specific gravities may be found in most works of chemistry and practical mechanics.

The cubic foot of water is 908·8488 troy ounces, 75·7374 troy pounds, 997·1369691 averdupois ounces, and 62·3210606 averdupois pounds. For all rough purposes it will do to consider the cubic foot of water as being 1000 common ounces, which reduces tables of specific gravities to common terms in an obvious way. Thus, when we read of a substance which has the specific gravity 4·1172, we may take it that a cubic foot of the substance weighs 4117 ounces. For greater correctness, diminish this result by 3 parts out of a thousand.

THE END.