A FIRST BOOK IN LOGIC

HENRY BRADFORD SMITH

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SMITH

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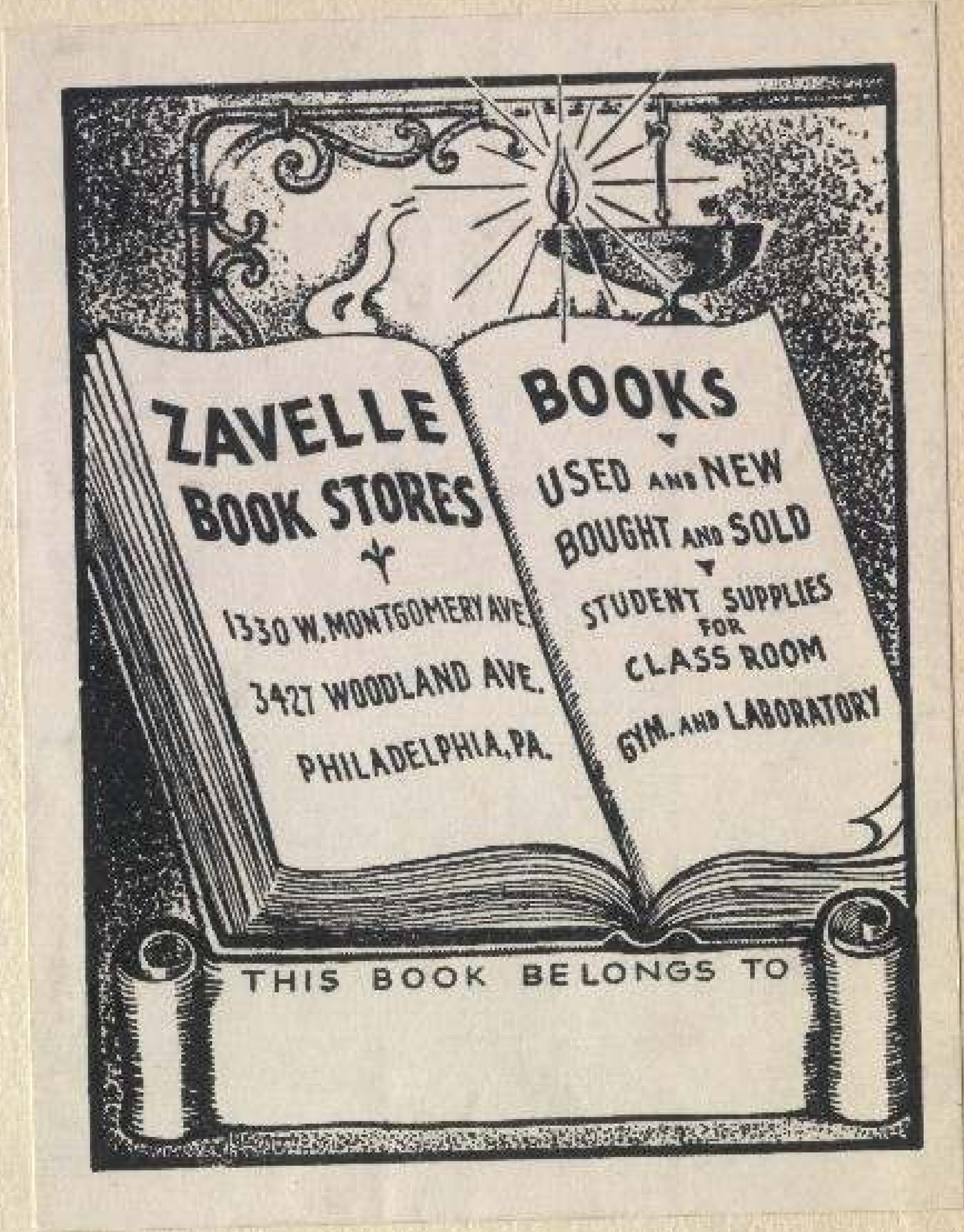
Samuel Skulsky



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BY
HENRY BRADFORD SMITH

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TABLE OF CONTENTS

CONTENTS

CONTENTS

PREFACE

The recent developments in logical theory following upon the contributions of Boole, Pierce, and Schröder have seemed to place the subject beyond the reach of the average student and even in most instances beyond the reach of the technically equipped philosopher. The result has been to reduce the attention that was formerly given to elementary instruction in logic and to displace the traditional course from its originally dominant position in the university curriculum. The conviction is abroad that the ancient *organon* is so far inferior to the modern instrument perfected by the critical labors of Peano, Frege, Russell, and others, that it no longer deserves the attention once bestowed upon it. These objections the writer has endeavored to meet, in the first instance by introducing no symbols whatever, save the ones employed by traditional logic itself, so that the treatment may be followed by any intelligent reader, and secondly by keeping modern developments always in mind while following the traditional order of treatment. Finally, he has met the recent contention that the classical system does not hold true in all of its parts, by showing in the last chapter of this work that this view is based upon a misapprehension. In completing the chapters dealing with the theoretical portion of the subject the writer has availed himself of parts of his *Letters on Logic,* which in turn is based upon a brief syllabus which Professor Singer once placed at his disposal, and also of parts of his recently published *Foundations of Formal Logic.*

H. B. S.

In the second printing of the text it has seemed well to include a more extended treatment of fallacies, a treatment which had already been published under the title of *How The Mind Falls Into Error* and which now appears bound at the end of the book. The fifth chapter on fallacies is retained in order not to disturb the page numbering even though the material is contained in the part appended. The reading of it may be dispensed with by those who read the last part. Finally, the last chapter is omitted altogether and its place is supplied by Appendix I. Appendix II gives a brief account of multiple implication on which the solution of the general problem of logic depends. *Logic as the Art of Symbols,* appended at the very end, will introduce the reader to these more abstract studies.

In the second revised edition, a new part, *Abstract Logic* has been added as an introduction to modern symbolic logic.

H. B. S.

TERMS, RELATIONS, PROPOSITIONS

CHAPTER I

INTRODUCTION TO INFERENCE

Every deductive science is concerned with a set of **relationships** which are peculiar to its domain or field of application, and a set of **objects** of which it is meaningful to say that they stand in these relationships to one another. Suppose a set of objects, *a, b, c,* etc., which stand for classes or groups of things, and the relation of **inclusion,** and consider the statement,

a is included in *b*.

If *a* stands for *Athenians* and *b* stands for *Greeks,* it is clear that the members of the *a* class are contained among the members of the *b* class and the statement that *a* is included in *b* is true. But if the case should be reversed and we should say,

b is included in *a*,

then it would be untrue, for not all Greeks are Athenians. Any statement that is either true or false is *meaningful; meaningless* if it be neither true nor false.

Suppose a set of objects, x, y, z, etc., and the relation of **implication**, and consider the statement,

x implies y.

Substitute here for x the expression, Socrates is a man, and for y the expression, Socrates is a mortal, and we have,

Socrates is a man implies
Socrates is a mortal,

the result, whether true or false, being at least meaningful. If we were to say, Socrates implies Greeks, we would say something that is neither true nor false. The expression would be meaningless because only of propositions (not of classes) can we say that the one *implies* the other.

Since the meaning of the objects and relationships with which a science deals will only appear when the science has been developed, it will not be possible to give an exact definition of the science with which we shall be concerned; but we may say that *logic* deals with propositions and with *inferences,* with classes and their *inclusion,* total or partial, with respect to one another.

PRELIMINARY DEFINITION OF THE SCIENCE

The most important matter with which logic is concerned is the matter of **inference.** Every man reasons, and oftentimes, correctly enough, before he possesses any science of inference. It is the purpose of these chapters to enable the student to become aware of the *processes* which he employs habitually in his thinking; to enable him to distinguish, in as many cases as possible, a good argument from one that is fallacious; to enable him to argue with some consciousness of the misapprehensions that lead men to adopt erroneous opinions. *Inference* being, then, the most important matter with which the logician has to deal, we shall content ourselves with the following *preliminary* definition: *Logic is the science of inference necessary and probable.* Necessary inference corresponds broadly to what is called **deductive** logic, whereas probable inference is treated in another department of the subject, which is known as **inductive** logic. It is only with the science of necessary inference that the present work undertakes to deal.

LOGIC AND THE SCIENCES

Upon reflection the student will at once become aware that in all the sciences the drawing of inferences is customary and habitual, and we may adduce this fact as an additional reason for making a study of inference at first hand. Jevons remarks: "One name which has been given to Logic, namely the Science of Sciences, very aptly describes the all-extensive power of logical principles. The cultivators of special branches of knowledge appear to have been fully aware of the allegiance they owe to the highest of the sciences, for they have usually given names implying this allegiance. The very name of logic occurs as part of nearly all the names recently adopted for the sciences, which are often vulgarly called the 'ologies,' but are really the 'logics,' the 'o' being only a connecting vowel or part of the previous word. Thus geology is logic applied to explain the formation of the earth's crust; biology is logic applied to the phenomena of life; psychology is logic applied to the nature of the mind. . . . Each science is thus distinctly confessed to be a special logic. The name of logic itself is derived from the common Greek word λόγος, which usually means *word,* or the sign and outward manifestation of any inward thought. But the same word was also used to denote the inward thought or reasoning of which words are the expression, and it is thus, probably, that later Greek writers on reasoning were led to call their science ἐπιστήμη λογική, or logical science; also τέχνη λογική, or logical art. The adjective λογική, being used alone, soon came to be the name of the science, just as Mathematic, Rhetoric, and other names ending in 'ic' were originally adjectives, but have been converted into substantives."

INTRODUCTION TO INFERENCE

INFERENCE AND ITS SIGNS

Whenever an inference is intended, we are usually made aware of the fact by the occurrence of some typical word: *hence; therefore; accordingly; if, then; implies; whence; it follows.* Thus:

> "If a strict definition of logic were stated at the outset, the student would not comprehend its intention,"

is an inference, by which it is intended to assert that the second part of the statement *follows* from the first. Or, again:

> "The definition of logic just given is exact. Hence the student must not expect to comprehend its full meaning at once."

But the presence of such words as these does not infallibly suggest that the first proposition *implies* the second. Suppose the case:

> "*If* there be any virtue, think on these things."—*Phil.* iv: 8.

This is in the form of an admonition or command, and, since it cannot be said to be either true or false, it is not properly a proposition at all.

It is important, in order to gain precise notions of the meaning of inference, to become aware that a proposition quite false in itself, may yet be truly implied by another. Thus it is untrue that Cæsar is an Athenian and untrue that Cæsar is a Greek, and yet the first proposition *implies* the second:

> If Cæsar is an Athenian,
> then Cæsar is a Greek.

The omission of the statement, "All the Athenians are Greeks," which is understood, if not expressed, a procedure common in ordinary speech, is called **enthymeme.** By this term we intend to signify that our speech is silent on a matter that is tacitly understood.

THE CASE OF FALSE PREMISES

Moreover, it is to be remarked that one or more assertions, untrue in themselves, may yet imply a proposition that is true. If we were to argue:

> Alexander is one of the heroes of the Iliad and all of these heroes died young; therefore, Alexander died in his youth,

our premises, or initial statements, would be false, although our conclusion would be true and the argument would be formally correct. The fact to be observed in this connection is this: that the conclusion of an argument can only be asserted when the premises are true and the implication formally valid, *except* (as in the case above) when the conclusion is true in independence of the argument. We should then say (that is, in the case last contemplated) that the truth of the conclusion is based on *extralogical* information, that it is true in point of fact and not *because* of the premises.

We may, in this connection, call attention to an error into which uninstructed common sense sometimes falls, through a failure to recognize this distinction. A speaker, we will say, is trying to convince us of the wisdom of some political policy, or, it may be, the truth of some political theory. He begins with premises that we are altogether disinclined to allow; but, as the result of a series of cogent inferences, we find him, toward the end of his discourse, asserting matters that all of us agree to be so. Unconsciously, as we retrace his argument and discover each one of his steps to be validly taken and the outcome of his original statements to be true—unconsciously, I say, we are apt to conclude that there must be some truth in the premises with which he began. But such a conclusion would be in no way justified. The error involved in such a step is known as the fallacy of **asserting the consequent.**

NOTATION FOR CLASSES

We shall, in the chapters which follow, employ the small letters at the beginning of the alphabet, *a, b, c,* etc., to designate groups—Athenians, Greeks, Triangles, etc.—groups the members of which are ordinarily conceived by the aid of some quality or characteristic which they have in common. We shall thus learn to habituate ourselves to such expressions as,

All of the *a's* are *b's*

or, as we shall phrase it more commonly,

All *a* is *b,*

and by this we shall understand it to be asserted that all the members that belong to the *a* class are contained among the members of the *b* class. If *a* and *b* be particularized, so that the assertion becomes,

All metals are elements,

it will be true; whereas, if it were to read,

All metals are compounds,

it will manifestly be untrue.

In the light of this illustration it will be clear that the truth or untruth of assertions similar to this, like,

All *a* is *b*,
some *a* is *b*,
no *a* is *b*,
some *a* is not *b*,

will depend upon what specific meaning is assigned to the symbols *a* and *b*.

TRUTH-VALUES INDEPENDENT OF THE TERMS

But the student will have to habituate himself as well to expressions whose truth or untruth is *independent* of the meaning of *a* or *b*. Suppose that in the first and third expressions listed above, *b* should take on the specific meaning non-*a*, we should then have,

All *a* is non-*a*,
no *a* is non-*a*,

which would become, if we were further to particularize the meaning of *a*,

All Athenians are non-Athenians,
no Athenian is a non-Athenian.

It would be evident that the last of these expressions will be true for all possible values of *a* and that the first will be untrue in the same way, without any restriction being placed on the meaning of *a*.

Similarly, the first member of our original set implies the second, that is,

> If all *a* is *b*,
> then some *a* is *b*,

quite in independence of what substantives may be substituted for these symbols. Thus,

> If all squares are circles,
> then some squares are circles,

and the implication holds even when *a* and *b* stand for **empty classes**—that is, for classes which contain no objects at all, for,

> If all square-circles are squares,
> then some square-circles are squares.

Another implication of a somewhat more general character than the cases that have just been cited, and whose truth is independent of the meaning of *a*, *b*, and *c*, would be,

> If some *b* is not *a*
> and all *b* is *c*,
> then some *c* is not *a*.

A case of this sort may occasion the student some difficulty when he meets it for the first time, but by continued attention to its sense he will soon be able to assure himself of its general truth. Particularized, it might read:

> If some scholars are not Englishmen
> and all scholars are cultivated,
> then some who are cultivated are not Englishmen.

By these examples we have sought to provide the student with a *preliminary* conception of the abstract nature of inference. To broaden and deepen this conception is a part of the task of the chapters which follow.

THE PROPERTIES OF A RELATIONSHIP

Relationships as well as their objects possess properties, and, since it is often important for the student of pure science to have these in mind, we shall enumerate those that are most characteristic. Incidentally, we shall illustrate further some of the distinctions already set down.

Corresponding to any relationship there will be a set of objects, of which it is meaningful to say that they stand in this relationship to one another. Such a set will be termed a **system**. A relation is said to be **reflexive** when it holds of any one of its objects and that object itself. Thus, *implication* is reflexive, for,

If x (is true) then x (is true);

and so is *inclusion,* for,

(What is) a is included in (what is) a

and the same holds of numerical equality.

(The number) p equals (the number) p.

But *less than* is not reflexive:

(The number) p is less than p;

nor is *perpendicularity* reflexive, at least in the ordinary geometry:

(The line) m is perpendicular to m.

A relation is said to be **reciprocal** or **symmetrical** when, if it holds of any two of its objects, x and y, it holds also of y and x. Thus, *parallelism* and *perpendicularity* are symmetrical:

If x is parallel to y,
then y is parallel to x;
if x is perpendicular to y,
then y is perpendicular to x.

But, of events in time, *subsequent to* is not symmetrical:

> If p is subsequent to q,
> then q is subsequent to p;

nor is *to the right of* symmetrical:

> If a is to the right of b,
> then b is to the right of a.

A relation is said to be **transitive** if, when it holds of x and y and of y and z (three of its objects), it holds also of x and z. Accordingly, *implication* and *inclusion* are transitive, but *is the father of* is not transitive. Consider the assertion:

If p follows q and q follows r, then p follows r, p, q, and r being regarded as three events in time. Obviously, the statement will be verified if the events occur in the order,

$$r\ q\ p.$$

But suppose that they *actually* occur in the order,

$$p\ q\ r.$$

Then all of the *parts* of the original assertion, viz., p follows q, q follows r, and p follows r, are false. It is very important for the student to realize that the assertion *as a whole*, in spite of this fact, holds true. The first two parts taken together form what is called the **antecedent** of the implication, and the last part is called its **consequent.** It may be remarked in general, that whenever one of the parts of the antecedent becomes false, or whenever the consequent becomes true, then the consequent *follows* in the particular case, whether it follows generally or not. There are six permutations of the three letters *p, q,* and *r.* The student will find it a valuable exercise to verify the *transitivity* of the relation *follow* (in point of time) by taking its objects in each one of their six possible orders.

MEANING OF DEFINITION

We have now made clear what it means for a relationship to possess or not to possess a given property. We remark, further, that it is by its properties that it is defined. Suppose three relationships are given and it is said of the first that it is symmetrical, but neither reflexive nor transitive; of the second that it is transitive, but neither reflexive or symmetrical; and of the third that it is reflexive and transitive, but not symmetrical.

Now, while you cannot say of the first relation precisely what it is—it might be the relation of *spouse* or the relation of *perpendicularity*—you may yet name a good many relations which it is not. For example, it is not *father of,* or *subsequent to,* or *to the right of.* But (and this is the important thing) its possible meaning is delimited by the conditions imposed upon it. The second might be the relation *greater than* or it might be *subsequent to.* The third might be *implication* or it might be *inclusion.* In each case there will be a great many things which it could not be.

Our meaning will now be clear when we say that a relationship is defined when enough of its properties have been enumerated to distinguish it from whatever other relationships are in question, and that these properties are to be found by constructing all the true and all the false propositions into which this relationship may enter in a meaningful way. The task of any deductive science, then, is to completely develop its own system, for it is precisely within its own system that the propositions in question may be found. A deductive science, therefore, is defined by the task or problem which it sets for itself, and its *full* meaning, accordingly, will only appear when this task has reached completion.

Exercises

1. Are the following consequences justly drawn from the stated conditions:

   Upon experiment it is found that a musical note p and a musical note q cannot be distinguished by ear, and that the same holds of q and r. It is inferred that p and r are indistinguishable by ear.

   Two straight lines have no common perpendicular. It is inferred that they approach each other.

A candle is burned in air under a jar, until the oxygen is exhausted. It is inferred that the inert residuum ("nitrogen") is one chemical substance.

A chair and a lead ball of equal weight are held, one in each hand. It is inferred that they will be judged of equal weight.

The pitch of an engine whistle is rising. We infer that the train is approaching.

An ice vender has two weightless scales. He hangs one scale on the other and ten pounds of ice on the lower scale. We infer that each scale will register five pounds.

The number of prime numbers is less than the number of numbers, for not every number is prime.

A thick glass and a thin glass are filled with boiling water by different observers in different rooms. The observer with the thin glass infers that the glass in the other room did not break because his own did not.

2. Examine the following statements in order to determine their truth or untruth:

War is war and Germany is Germany.

"No man can lose what he never had."

A work of art is either moral or immoral.

"Hang sorrow! Care will kill a cat,
And therefore let's be merry."

"O wad some power the giftie gie us
To see oursels as ithers see us!"

It is true that not all prime numbers are odd.

If no proposition is true, then one proposition is true.

If the moon is green cheese, Cæsar and Socrates are the same person.

3. In the following propositions:

All metals are elements,
no elements are compounds,
some metals are red,
not all metals are white,

the relations are: "all, are," "no, are," "some, are," "not all, are." Classify these under the heads, reflexive, symmetrical, and transitive.

CHAPTER II

CLASSIFICATION OF TERMS

The word **term** is from the Latin *terminus* and is so designated because it forms one end of a proposition. For our purposes a term is synonymous with a **class,** a group of objects which have some characteristic in common, every substantive in the language being the symbol for such a group. If a term denotes a perfectly definite group of objects, it is called a **constant** term; if it stands indifferently for any class whatever, it is called **variable.**

THE PREDICABLES

Many are the ways in which terms may be classified, but only those divisions will be noticed which have some bearing on our subsequent theory and its applications. In the first instance, because of the importance of the terminology, we must describe the Aristotelian **predicables.** These are said to be the kinds of terms or attributes which may be predicated of any subject. They are called genus, species, difference, property, and accident. When a class is conceived as divided into a number of smaller classes it is called the **genus,** and each smaller class that goes to make it up is called a **species** of the genus. Thus, if the genus be the class of numbers, odd numbers and even numbers are two species of the genus; if the genus be odd numbers, all prime numbers, except the number two, will be one of its species. It is important to observe that, while the genus contains more objects or individuals than the species, it is defined by fewer attributes. If the genus be number and the species be prime number, the genus is defined by whatever is characteristic of number, and the species by these same attributes with the addition of whatever is characteristic of a prime.

This double meaning of genus and species is fixed by different names. Thus, the individual things to which the term applies, comprise its meaning in **extension** (extent, breadth, or denotation). The qualities that serve to define the term comprise its meaning in **intension** (intent, depth, or connotation). Ordinarily, if we add qualities to a term—that is, if we increase its meaning in connotation or intension—we thereby diminish its meaning in denotation or extension. Two species of the genus substances would be elements and compounds, the first being gotten by adding the quality "chemically simple," or "not further reducible," to the genus. Again, two species of the genus element would be metals and non-metals. The first would be gotten by adding, it may be, the characteristics, possessing luster and being an easy conductor of electricity. It is, therefore, usually said: As the connotation of a term is increased the denotation is decreased. The student must be warned, however, that this law only holds when the extension of the genus class is finite. Thus, if there be an infinite number of prime numbers, the denotation of "number" is not diminished by adding that the number is a "prime number."

The additional qualities required to distinguish the species from the genus are called the **difference,** and a class is supposed to be defined by the proximate genus (next higher genus) and the difference. The old manuals of logic, following tradition, speak often of a lowest species (*infima species*), a class not further divisible, and a highest genus (*summum genus, genus generalissimum*), which cannot in turn be taken as the species of a higher class. The modern counterparts of these are the **null class,** or class which contains no objects, such as square-circles or alien Americans, and the so-called **universe** of discourse, which contains all the objects that happen to be in question.

The remaining predicables, property and accident, are thus defined: A **property** is any quality that may be predicated of a class, and which, implicitly or explicitly, is essential to the meaning of the class. An **accident** is any quality which may be predicated of a class, but which is not essential to the meaning of the class. It would be accidental for an isopod to be red, but essentially it must have equal legs. An odd number has the property of being not divisible by two; it is accidental to its meaning for it to be a prime. Water is essentially composed of hydrogen and oxygen and only by way of accident is it a solid, a liquid, or a gas.

REDUCTION OF TERMS TO THE CONCRETE GENERAL

Terms are, according to another important distinction, either singular or general. A **singular** term denotes an individual. Such names as London, Socrates, the Vatican, or the son of Napoleon I, Kant, or either one of the Kilkenny cats, designate a single object. A **general** term denotes any one of a number of objects. Cabbages and kings, monkeys and prime ministers would be examples of the case in question. In the logic of Peano, whose work has inspired much of the recent research in this subject, the relation of an individual to a class is conceived as different from the relation of a class to a class. What distinguishes the case of the inclusion of one class in another and the case of the inclusion of an individual in a class, is the property of transivity, which is taken to hold of the first relation, but not of the second. If I say, "Athenians are men and men are a class, therefore Athenians are a class," I say something that is not only meaningful, but true. But if I assert, "Socrates is a man and men are a class, therefore Socrates is a class," I am supposed to say something meaningless, or at least something untrue, the conclusion being false and the premises being true.

We shall, however, in our subsequent theory reject this distinction of Peano's making, as one whose nature is extralogical, as one which turns on a question of fact or is a matter of application. That is, we shall say, "Socrates is a man" is exactly expressed by the phrase, "Every Socrates is a man," it being a matter of extralogical information—that is, a matter of fact—that there is only one Socrates. Or, to take a case in which a singular term occurs both as subject and as predicate, the proposition, "St. Paul's is the largest cathedral," is exactly, if awkwardly, expressed by, "Every St. Paul's is every largest cathedral," for it is a geographical fact that there is only one St. Paul's, and it is an arithmetical fact that there is only one largest member of a class. Accordingly, in translating grammatical expressions into the forms which are recognized by the logician, a singular term may always be reduced to a term which is general.

A further division of terms, which it is necessary to recognize, is the distinction between those which are abstract and those which are concrete. All general terms are either concrete or abstract. A **concrete** term is the name of a group of things conceived as individuals. An **abstract** term is the name of a group of things conceived by the aid of a common property. It will not be necessary to multiply illustrations of this distinction. Truth and beauty are abstract terms; true things and beautiful things are concrete. If I say "Truth is beauty," the same meaning is advanced in the expression, "True things are beautiful things." Observe, then, that just as the singular term may always be reduced to the concrete general, so terms that are abstract may always be made concrete.

In concluding a classification of terms we must notice their division into those that are positive and those that are negative. All general terms are either positive or negative. A **positive** term connotes the possession of a quality. A **negative** term connotes the absence of a quality. The genus substances is made up of two species, elements and non-elements, and of these the first is positive and the second negative. But this distinction is relative, for elements and non-elements may be called, respectively, non-compounds and compounds. Whenever a negative term describes some class important in itself, language has generally invented a positive term to correspond. For the chemist non-elements are as important as elements, so that the word compounds has been invented to stand for that class. There are, however, notable failures to create such a word. The genus elements is divided into metals and non-metals, but there is no corresponding positive term with which to designate this latter class. In general we observe that is an accident of language, whether or not a negative term possesses a synonyme which expresses its positive sense. Not all terms which have a negative prefix, however, convey a negative intent. Jevons remarks: "The participle *unloosed* certainly appears to be the negative of *loosed;* but the two words mean exactly the same thing, the prefix *un-* not being really the negative; *invaluable,* again, means not what is devoid of value, but what cannot be measured; and a *shameless* action can equally be called by the positive term, a *shameful* action."

UNIVERSE OF DISCOURSE

It is necessary to have very clearly in mind the distinction between a positive term and its corresponding negative, or the distinction between a class and its contrary (contradictory), if we are to understand the meaning of what is called the **universe of discourse,** a conception which now plays a very important rôle in the present-day logical theory. The view that in any argument there is presupposed a limited class, which stands for all of the objects under discussion, is due to De Morgan and we can not do better than to quote this author in full: "Let us take a pair of contrary names, as man and not-man. It is plain that between them they represent everything imaginable or real, in the universe. But the contraries of common language usually embrace, not the whole universe, but some one general idea. Thus, of men, Briton and alien are contraries: every man must be one of the two; no man can be both. Not-Briton and alien are identical names, and so are not-alien and Briton. The same may be said of integer and fractions among numbers, peer and commoner among subjects of the realm, male and female among animals, and so on. In order to express this, let us say that the whole idea under consideration is the *universe* (meaning merely the whole of which we are considering parts) and let names which have nothing in common, but which between them contain the whole idea under consideration, be called contraries *in or with respect to that universe*. Thus, the universe being mankind, Briton and alien are contraries, as are soldier and civilian, male and female, etc.: the universe being animal, man and brute are contraries, etc."

Exercises

1. Give a connotative and give a denotative definition of prime numbers less than ten.
2. Name three species of the following genera, which together make up the whole: animal, plant, number, element.
3. Show how the law which connects the quantity of extension with the quantity of intension holds of the series: gold, metal, element, substance.
4. Are the following characteristics, that are predicated of the planets of our system, properties or accidents?
   There is a "regular progression of distances" of the planets from the sun (expressed as Bode's law). This breaks down at Neptune.

The plane of a planet's rotation practically coincides with that of the orbit of each—probably excepting Uranus.
The direction of rotation is the same as that of the orbital revolution—probably excepting Uranus and Neptune.

5. Does the term property, as used in the statement which follows, conform to the definition of this chapter? No
The properties of the elements are periodic functions of the atomic weights;—elements arranged in a series of increasing atomic weights show steady increase or steady decrease in a property within any one period. Exceptions are iodine, tellurium, iron, cobalt, and nickel.
6. What universe of discourse is most naturally suggested by the following terms and their negatives? Odd numbers, straight lines, carbon compounds, protective tariffs, foreign policies.
7. Define the following words in terms of what you take to be the proximate genus and the difference: triangle, mammal, proposition, system, species.
8. Can the following words be termed *indefinables* in the above sense? Class, point, number, time, length.
9. Classify the following words under the heads, singular and general, abstract and concrete, positive and negative: unwieldy, Jupiter, protoplasm, Senate, incognito, respiration.

CHAPTER III

LOGICAL AND GRAMMATICAL FORMS

At the conclusion of our introductory chapter we were at pains to distinguish between statements whose truth or falsity depends upon the meaning of the terms, and those whose truth or untruth is independent of any meaning that our general terms, *a, b, c,* etc., may happen to take on. Statements of the first sort are commonly designated **propositional functions;** those of the second sort, **propositions.** In spite of this distinction we shall employ the word *proposition* to denote any sentence that is either true or false, and the word will now be understood to be defined in this more general sense.

Not all the sentences which the grammarian recognizes are propositions. Thus, the **interrogative,** the **hortatory,** and the **imperative** modes of expression will be sentences, but not propositions in the sense defined. It is neither true nor false to say,

> "What's Hecuba to him or he to Hecuba,
> That he should weep for her?"

Or to say,

> "Eat, drink and be merry."

Or, again,

> "Stand not upon the order of your going,
> But go at once."

Of any sentence, however, which the grammarian calls **declarative, optative,** or **exclamatory,** either truth or falsehood may be predicated. It is our purpose here to indicate the manner in which any proposition, of whatever grammatical form, may be expressed by means of the few simple forms which the logician recognizes. It is manifest, then, that our analysis need not concern itself with sentences of the interrogative, the hortatory, or the imperative type.

In the first place, we observe that any expression in the optative form may always be made declarative. If one were to say,

> "Would that ignorance were bliss,"

he might, presumably, substitute for this, without changing the meaning in any way,

> "I wish that ignorance were bliss."

Or, again,

> "The devil damn thee black,
> Thou cream-faced loon,"

is clearly susceptible to the same modification, it may be at the cost of some rhetorical advantage.

A reduction of the exclamatory to the declarative form is equally possible. Thus, "A Daniel come to judgment!"

means, we may presume,

"Another Daniel's come to judgment,"

Or, again, the exclamation,

"How sharper than a serpent's tooth it is
To have a thankless child!"

may easily be made direct assertion. We may, therefore, confine our attention to declarative sentences alone.

In effecting a further reduction of grammatical forms, in order to show that the forms employed by the logician are sufficient for the expression of any truth, we observe that any verb other than the verb "to be" may be rendered by the copula, by absorbing its meaning into the predicate term. The expression "it rains" has always reference to some particular place as,

"It is raining in London,"

and this in turn is equivalent to

"London is a place where rain is falling."

If it be remarked that,

"A favorite has no friend,"

the same intention is expressed by,

"Every favorite is friendless."

The distinction of *number* offers, in turn, no real difficulty. The proposition,

"All the Athenians are Greeks,"

may be written in the equivalent form,

"Every Athenian is a Greek."

Moreover, the word "all" need not be replaced by "every," for we may say, at the cost of some awkwardness:

"All the members of the class Athenians is a member of the class Greeks."

Further, differences of *tense* are easily reduced to the *present* by attaching a date, or the suggestion of a date, to the predicate term. Thus, in place of,

"Achilles was celebrated as the swift of foot,"

it may be stated that,

"Achilles is celebrated in the 'Iliad' as the swift of foot."

The student will have no difficulty in effecting a reduction of other grammatical distinctions, such as those of *person, voice,* and *mood.* Assuming, then, that these reductions have been made, it will be taken for granted that any proposition, of whatever grammatical structure, may be cast into one or more of the relational functions that belong to the domain of which logic treats.

LOGICAL AND GRAMMATICAL FORMS

FORMS RECOGNIZED BY THE LOGICIAN

The logician recognizes the following propositions as necessary and sufficient for the expression of any truth:

(1) The **disjunctive** form (either, or),

either x or y,

(2) The **conjunctive** form (and),

x and y,

(3) The **hypothetical** form (if, then),

x implies y,

(4) The **categorical** forms (adjective of quantity, copula),

A(ab) All a is b,
E(ab) No a is b,
I(ab) Some a is b,
O(ab) Some a is not b.

THE DOUBLE MEANING OF DISJUNCTION

The disjunction, *either, or,* in ordinary speech is ambiguous. If we were to say of such and such a person,

"This man is either an Englishman or a Shakespearean scholar,"

we may not intend to exclude the possibility of his being both, educated Englishmen being supposed, we will say, to know the plays well, while the same assumption is not made of men of other nationalities. But if we were to say,

> "I know that I shall either like this man or dislike him very much,"

the intention is clearly to exclude the possibility of my liking him for some of his qualities and disliking him for others.

Here, however, the disjunction is not one between classes, as in our first illustration, but one between propositions, as in our second example. Commonly, when the options of every-day life are expressed in the form of a disjunction, when they are what William James calls options of the living, forced or momentous kind, we intend the sense: either the one or the other, *but not both.* Thus to transform some of James's illustrations,

> "This man is either Christian or he is agnostic," or
> "You must either accept this truth or go without it,"

the two parts of the disjunction *cannot both be true.*

If, on the other hand, I should meet a man whose knowledge of Shakespeare is extraordinary and should exclaim in surprise,

> "Either this man is an Englishman or else he is a Shakespearean specialist,"

I mean at least *one* of the parts of the disjunction to be true, and *possibly both.* The student should always bear in mind that this last case conveys the true meaning of *logical* disjunction and that disjunction will always be understood in this sense.

Whenever the letters x, y, etc., are used to designate propositions, they are commonly taken to be true; that is, the words, *is true,* are understood to follow each one. Thus, if we assert,

either x or y,

we are only shortening the expression,

either x is true or y is true, or both,

and the shorter phrase,

x and y

means in expanded form,

x is true and y is true.

Whenever the proposition x or the proposition y is taken to be false ,the words, *is untrue,* will have to be expressly written down.

LAW CONNECTING CONJUNCTION AND DISJUNCTION

It will always be possible to express the denial of a disjunction—that is, the assertion that the whole statement, either x or y, is untrue, in the form of a conjunction of the two propositions involved. Thus, the untruth of

either x is true or y is true

is exactly rendered by

x is untrue and y is untrue.

This is an important fact and is one of the reasons for taking logical disjunction in the sense above defined. If we deny of some person "that he is either foreign born or that he is foreign bred," we assert the same thing when we say:

> "That he is foreign born is untrue and it is untrue that he is foreign bred."

In the same way the denial of the conjunction of two propositions may always be expressed as a logical disjunction. Thus the untruth of

x is true and y is true

is precisely equivalent to

either x is untrue or y is untrue.

If we deny "that this student is dull and that he needs no stimulus," we express the same thing by saying,

> Either this student is not dull or else he needs a stimulus.

This law which connects the conjunctive and the disjunctive forms may be expressed generally as follows: the denial of the disjunction of two propositions is the conjunction of the two separately denied; the denial of the conjunction of two propositions is the disjunction of the two separately denied.

LAW CONNECTING DISJUNCTION AND IMPLICATION

There is another fact, which is universally assumed in modern logical investigations and to which it is right to direct the student's attention. This is a certain equivalence, which is assumed to exist between the hypothetical and the disjunctive forms. The assertions,

> "If he speaks not in jest, then
> he speaks in earnest," and
> "Either he speaks in jest or in earnest,"

are taken to convey precisely the same meaning. A general statement of this truth would be: x implies y is equivalent to the phrase, either x is untrue or y is true. In the subsequent chapters we shall express the *denial* of "x implies y" by means of the expression, "x does not imply y."

THE CATEGORICAL FORMS AND THEIR DIAGRAMMATICAL REPRESENTATION

It only remains to explain the categorical forms and the notation which has been introduced to represent them. In the propositions, A(ab), E(ab), I(*ab*), and O(*ab*), the terms are the **subject** *a,* which is written first in the bracket, and the **predicate** *b,* which is written second, and the **term-order** is the order subject-predicate. When we wish to indicate that the term-order is not settled, we shall place a comma in the bracket between the terms. Thus, A(*a, b*) may mean either "all *a* is *b*" or "all *b* is *a*."

"All *a* is *b*" asserts that all the members of the *a* class are contained among the members of the *b* class, leaving it undetermined whether the members of the subject class are related to the members of the predicate class through identity, or exhaust only a part of the members of that class. Accordingly, the word *some,* in the sense of *some at least, possibly all,* is understood, if not expressed, before the predicate term, and it is this sense which the word will always convey in our subsequent theory. The meaning of the assertion, "all *a* is *b*," may be illustrated by the following diagram (Fig. 1), that is, if "all *a* is *b*" is a true proposition, then the class *a* is related to the class *b* in one of these two ways,

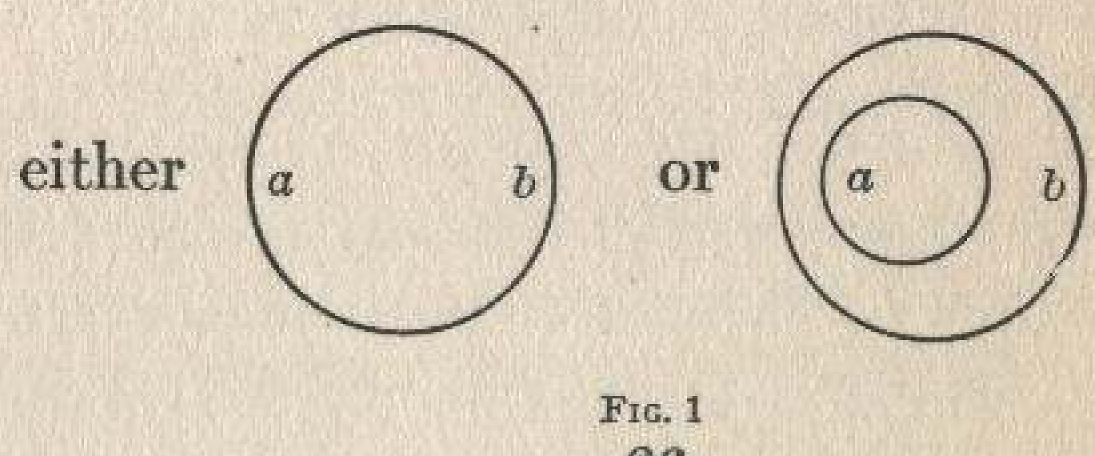


Fig. 1

The diagrammatic representation of the other categorical forms is given (in Figs. 2, 3, 4) below:

No *a* is *b*

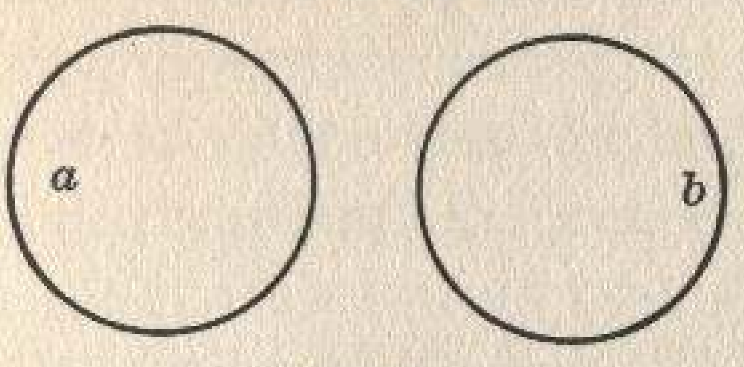


Fig. 2

Some *a* is *b*

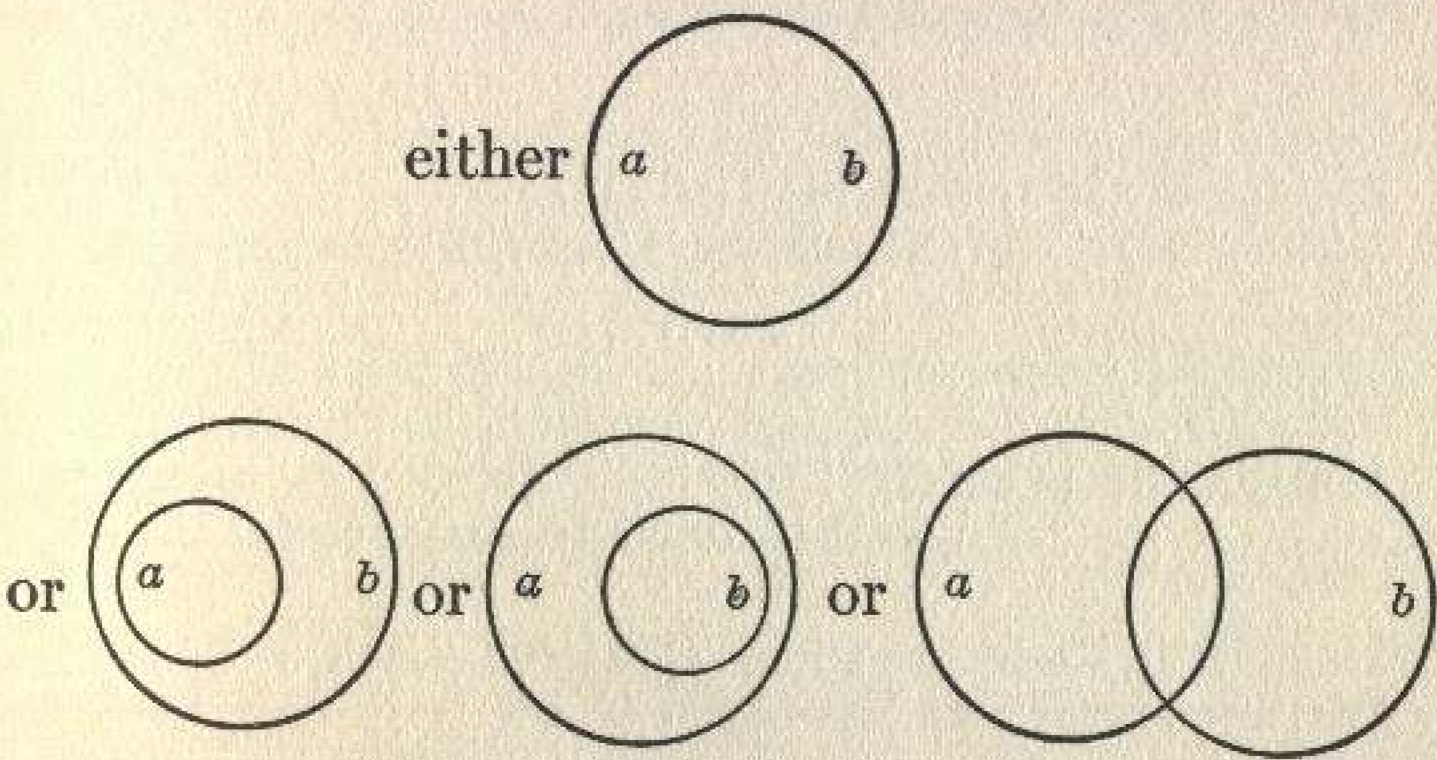


Fig. 3

Some *a* is not *b*

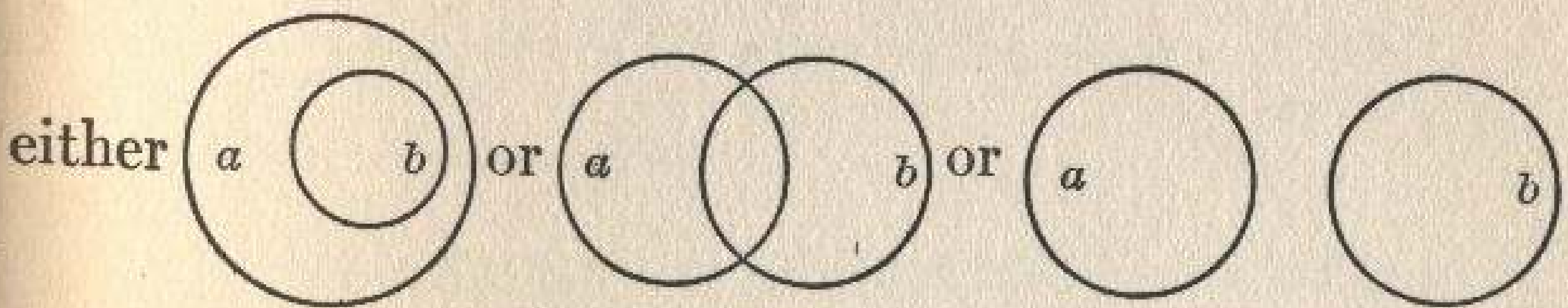


Fig. 4

These propositions are the ones employed in the classical logic, the science which has descended to us from Aristotle. The peculiar simplicity which is introduced into logic by the selection of this particular set of categorical forms depends upon the fact that the denial of any one involves the assertion of one of the others. Thus the denial of A is the assertion of O, and conversely; the denial of E is the assertion of I, and conversely.

Exercises

1. Are the following sentences propositions, and why?

> "Can it be possible that this old hermit has heard nothing of the report that God is dead!"
>
> —Nietzsche, *Thus spake Zarathustra.*

> "Necessity knows no law. . . . Behold the eleventh commandment, the message you bring to the world to-day, sons of Kant!"
>
> —Rolland, *Au-dessus de la mêlée.*

> "Render to Cæsar the things that are Cæsar's, and to God the things that are God's."
>
> —*Mark,* xii: 17.

2. Render the sense of the following sentence in the form of a conjunction:

> "The race is not to the swift, nor the battle to the strong."
>
> —*Ecclesiastes,* ix: 11.

and the sense of the following sentence in the form of a disjunction:

"Of making many books there is no end; and much study is a weariness of the flesh."

—*Ecclesiastes,* xii: 12.

3. Express the following disjunction in the hypothetical form:

"Either Bacon and Shakespeare are not the same person or else the moon is made of green cheese,"

and the following implication in the form of a disjunction:

"If you do not accept this truth, you must go without it."

4. Express the following sentences in categorical form:

"The heart hath its own reasons, which are unknown to reason."

—PASCAL.

"A nation free from prejudice soon becomes a free nation."

—CONDORCET.

"The brain in some sort digests impressions; it produces an organic secretion of thought."

—CABANIS.

5. The word *true* means true in all instances. Distinguish between the meaning of the phrase, *not necessarily true,* and the phrase, *necessarily not true,* and state general propositions which illustrate each meaning.

FALLACIES

CHAPTER IV

HISTORICAL CHANGES IN MEANING

Since ambiguity in the meaning of a word or phrase is perhaps the most fertile source of error, when we come to apply the rules of correct thinking, it is important for the student to be aware of some of the ways in which these ambiguities arise. **Semasiology,** whose problem it is to set forth the causes of these changes in meaning, is a field as yet but imperfectly explored. We shall be content to enumerate a few of the commonest cases in which old signs tend to disappear, and new ones arise to replace them.[1]

We note in the first instance that two *synonymes which are exact tend to differentiate in meaning.* De Quincey observes: "all languages tend to clear themselves of synonymes, as intellectual culture advances; the superfluous words being taken up and appropriated by new shades and combinations of thought evolved in the progress of society. And long before this appropriation is fixed and petrified, as it were, into the acknowledged vocabu-

1 Some of the examples in this chapter are borrowed from Jevons, but by far the larger number are taken from Bréal, *Essai de Sémantique,* Paris, 1897.

lary of the language, an insensible *clinamen* (to borrow a Lucretian word) prepares the way for it. Thus, for instance, before Mr. Wordsworth had unveiled the great philosophic distinction between the powers of *fancy* and *imagination,* the two words had begun to diverge from each other, the first being used to express a faculty somewhat capricious and exempted from law, the other to express a faculty more self-determined.

When, therefore, it was at length perceived that under an apparent unity of meaning there lurked a real dualism, and for philosophic purposes it was necessary that this distinction should have its appropriate expression, this necessity was met half way by the *clinamen* which had already affected the popular usage of the words."

In Brittany, says the Abbé Rousselot, gardens were formerly called *courtils.* The rustic word which is now displaced as the usual term, is still used in a contemptuous sense. Similarly, with the introduction of the word *hotel,* the German *Wirtshaus, Wirtschaft,* has come to designate an inn of the less pretentious type. Again, to take an illustration from the philosophical dictionary, the Greek ἀρχαί (principles) and στοιχεῖα (elements) were certainly synonymous in the time of Thales, at least in the sense in which they were used by him. But this philosopher was severely reproached by a later writer for not having distinguished between them.

LAW OF GENERALIZATION

Again, *words of special signification may take on a more general sense.* English *gain,* which has been influenced by the French *gagner* (to pasture), *le gagneur* (the cultivator), *le gain* (the harvest), means now, in either language, profit of any sort. Latin *pecunia,* originally wealth in live stock, came in the end to designate riches of whatever kind. French *temps* (time) meant at first heat (temperature), and afterward the weather, until finally the abstract idea of duration came to be attached to it. The Greek word χαρακτήρ denoted an engraving tool, but it soon came to be applied to the letters or signs engraved, a sense still preserved when we speak of the Greek characters. It was then extended by a sort of metaphor to whatever is regarded as the essential sign of any object whatever. The word *prince* has a sense much less special than the source from which it came, the *princeps senatus.*

Even proper names in certain instances may become generalized. Thus, in Rome *Cæsar* soon came to designate the emperor, and its modern derivative, czar, has even produced the abstract term, czarism. An adjective *Fabian* has been formed from the name of the Roman general Quintus Fabius Maximus, and any policy of procrastination or delay which postpones a decision is characterized as Fabian. The tower built on the island of Pharos near Alexandria has produced in several languages the name "Pharos" as a synonyme for lighthouse. Again, while there was in the beginning but one sun and one moon, we now speak of the fixed stars as *suns* and we refer to the satellites of Jupiter as the *moons* of Jupiter.

LAW OF SPECIALIZATION

A tendency which is precisely the opposite of the one that has just been recorded is constantly taking place in the formation of a language. *Words of a general signification may take on a special or more restricted meaning.* Thus, in the Middle Ages the word *species* employed by the purveyors of drugs to designate the kinds of ingredients which they sold (saffron, clove, cinnamon, nutmeg) became, when it again entered the colloquial language, the French *épices,* English *spice.* German *Muth* (courage) originally of a more general signification, which is preserved in the derivatives *Hochmuth* (pride), *Grossmuth* (generosity) etc., and roughly rendered by English *mood,* probably derived its special meaning from words like *Rittersmuth, Mannesmuth,* etc. English *wit,* while tending to become specialized, still preserves its archaic sense of "shrewdness" or "intelligence" in the phrase *mother wit.* Many other examples might be given. *Urbs,* the name of Rome for the country folk of Latium, became, because of the Roman legions, the name familiar to the whole of the ancient world. *Physician,* from φύσις (nature), has become so far specialized that a new word, *physicist,* had to be invented to express the original meaning. This word in turn has shown the same tendency, and there is now no commonly accepted term by which may be designated a scientist, whose interest is primarily in nature—*i.e.,* in nature as contrasted with man. The word *pope* (Latin *papa*) may have originated as a term of endearment (little father), employed, perhaps, in much the way in which the French now speak of Papa Joffre.

The following remarks of De Morgan deserve to be quoted in full: "The word *publication* has gradually changed its meaning, except in the courts of law. It stood for *communication to others,* without reference to the mode of communication or the number of recipients. Gradually, as printing became the easiest and most usual mode of publication, and consequently the one most frequently resorted to, the word acquired its modern meaning; if we say a man publishes his travels, we mean that he writes and prints a book descriptive of them. I suspect that many persons have come within the danger of the law, by not knowing that to write a letter which contains defamation, and to send it to another person to read, is *publishing a libel;* that is, by imagining that they were safe from the consequences of publishing, as long as they did not print. In the same manner, the well-established rule that the first publisher of a discovery is to be held the discoverer, unless the contrary can be proved, is misunderstood by many, who put the word printer in the place of publisher. I could almost fancy that some persons think rules ought to travel in meaning, with the words in which they are expressed."

Sometimes an abstract word may become the name of an object. The Latin *vestis* (the act of clothing oneself) became in course of time the name of a particular garment. The Latin ending *-tas* (as in *dignitas, cupiditas*) served to form a substantive expressing a quality. But *civitas,* which meant at first the quality of being a citizen, came to designate the totality of citizens and finally stood for "the city" itself. German *Kind* came in course of time to mean "infant," though originally it referred to "the race."

TRANSFER OF MEANING BY ANALOGY

Besides the two processes described above as **generalization** and **specialization,** the senses in which a word may be employed may be merely *multiplied.* This transpires most commonly by the transfer of meaning through the use of **analogy** or **metaphor.** Thus, in Latin "intelligence" is like a point which penetrates (*acumen*), while "folly" resembles a blunt knife (*hebes*). At Rome there took place, every five years, a census accompanied by a religious ceremony called "purification" (*lustrum, lustratio*). Since on this occasion the magistrates and the priests walked among the crowds, the verb *lustrare* came to mean "to pass in review." Cicero expressed astonishment that the Roman peasants should have given the name pearl (*gemma*) to the buds of trees. Actually the change by metaphor had been in the opposite sense, for pearls were so named because of their resemblance to buds about to burst. Again, *rivalis* designated neighbors on the opposite sides of a stream or who used the same water supply, but came to designate later on any sort of rivalry whatever. The word *influence* goes back to ancient astrological superstitions; it was supposed that a certain fluid escaped from the stars to predetermine the destiny of men and events.

Some of Archbishop Whately's examples of the transfer of meaning by analogy are worth recording. He says: "A blade, of grass, or of a sword, have the same name from direct resemblance between the things themselves. But instances of this kind are far less common than those in which the same name is applied to two things, not from their being *themselves* similar, but from their having *similar relations* to certain other things. And this is what is called analogy. Thus, the sweetness of a sound and of a taste can have no *resemblance:* but the word is applied to both, by *analogy,* because as a sweet taste *gratifies* the palate, so does a sweet sound, the ear. Thus, we speak in the analogical sense of the hands of a clock, the legs of a table, the foot of a mountain, the mouth of a river, . . . from the *similar relations* in which they stand to other things, respectively, in reference to use, position, action, etc.

"The words pertaining to *mind* may in general be traced up, as borrowed (which no doubt they all were, originally) by analogy, from those pertaining to *matter:* though in many cases the primary sense has become obsolete. Thus 'edify' in its primary sense of 'build up' is used, and the origin of it often forgotten; although the substantive 'edifice' remains in common use in a corresponding sense. When, however, we speak of 'weighing' the reasons on both sides—of 'seeing' or 'feeling' the force of an argument—'imprinting' anything on the memory, etc., we are aware of these words being used analogically."

EXERCISES[1]

1. How would you explain the fact that the first two words in the following lists are Saxon and the last two Norman?

   (a) home, hearth; palace, castle.
   (b) boor, churl; duke, count.
   (c) ox, deer; beef, venison.

2. How did *miscreant* (misbeliever) acquire its present sense?

[1] These examples have been taken from Trench, *On the Study of Words.*

3. How could *dunce* have been derived from Duns Scotus, the "subtle doctor," the great teacher of the Franciscan order?
4. As the result of a false science, *crystals* were so named because of their resemblance to ice, which was then supposed to have "lost its fluidity." Explain the words: jovial, saturnine, mercurial, disastrous, ill-starred.
5. According to an ancient theory of medicine, the disposition of mind and body depends upon a proper proportion of four principal moistures (humors). Explain: good humor, bad humor, dry humor.
6. Occasionally a name will embody some original error. Explain the words: America, turkey, dinde (French), gypsies, Indian.
7. On being asked of what city he was, Diogenes replied that he was a cosmopolite. How must this reply have sounded to Greek ears?
8. By which one of the causes enumerated in this chapter would you explain the change in the following words from their primary to their secondary meaning: caprice, halcyon, voluble, temper, spirit, prude, loyalty, journal, minion, knave, roué?
9. What facts may be inferred from the history of the words: thrall, paper, stipulation, expend, calculate?

CHAPTER V

SEMI-LOGICAL AND MATERIAL FALLACIES

The term **fallacy** in the narrow or technical sense is used to designate some breach of the rules of correct inference; in the broader, more popular sense it denotes any one of the numberless ways in which men may fall into error. **Sophism, paradox,** and **paralogism** are inexact synonymes of this word and have a great variety of meaning. Thus paradoxical may be applied to an argument, which merely appears to be an offense against logic, or to a point of view which seems to offend common sense, or which only is beyond common apprehension. A sophism denotes ordinarily an argument which deceives not the author of it, but his opponent only, or which places the burden of the proof upon the latter. A paralogism may mean a particular error which the mind shows a special tendency to adopt, and in this sense it is employed by the German philosopher Kant. Fallacies commonly hinge upon an ambiguity in the meaning of terms or relationships. Thus it may appear *paradoxical* that this word and its synonymes should exhibit on their own part such a wide range of ambiguity.

PARADOX REMOVED BY EXTENDING THE MEANING OF TERMS OR RELATIONSHIPS

It is well known to the mathematician that he has often to extend the sense of his terms or of his symbols of relationship in order to take account of cases not suspected, when his definitions were set down. Thus, the primitive sense attaching to quantity will not permit of the interpretation of imaginaries. This, one of the prime difficulties which the beginner in algebra has to overcome, depends upon his instinctively holding to the *original* meaning of quantity, while striving to grasp the *new*. Many of the paralogisms of the older logicians depend upon their inability to generalize their conceptions, when they have met with special cases that resist interpretation. One would like, for example, to be able to say: If *a* is a class and *b* is a class, then what is *a* and *b* is a class, and what is either *a* or *b* is a class. But this will involve the conception of a class that has no members, and such a class has *peculiar* (*i.e., paradoxical*) properties. Thus the members of the class squares and the members of the class circles have no members in common and these latter are contained in and are also excluded from the members of any other class whatsoever. The corresponding notion of a proposition, that is true under no circumstances, is one with which the older logicians could not deal, and hence for them paradoxical, although it is habitual and conscious in popular usage and is, therefore, recognized by common sense;

> "I will not be afraid of death and bane
> Till Birnam forest come to Dunsinane"

means: "I will not be afraid till what must forever remain untrue comes true; till an impossibility is possible; an emphatic way of saying, 'without qualification, I will not be afraid.'" Again:

> "Nay, had I power, I should
> Pour the sweet milk of concord into hell,
> Uproar the universal peace, confound
> All unity on earth,"

the meaning being, in part: if the consequent be false (I *shall* pour, etc.), then so is the antecedent false (I *have* the power), for an asserted implication is taken to be true. This sense is here rendered unambiguous by the use of the conditional.

EQUIVOCATION AND AMPHIBOLOGY

Many fallacies that arise because of the ambiguity of terms, and which are listed in the manuals of logic, are not intended seriously. Such are jests or puns. A famous case of punning is found in the conversation of Hamlet with the gravedigger:

Ham.—Whose grave's this, sirrah?
Clo.—Mine, sir.
Ham.—I think it be thine indeed, for thou liest in't.
Clo.—You lie out on't, sir, and therefore 'tis not yours: for my part, I do not lie in't, and yet it is mine.

Franklin's well-known remark (on the occasion, I imagine it was, of the signers attaching their names to the Declaration) might or might not be taken seriously, except for its logic, "If we do not hang together, gentlemen, we may expect, each one of us, to hang separately." There are cases in which one cannot be quite sure whether an ambiguity is intended or not. Suppose, for example, the assertion: "Shakespeare did not create the plays. They were conceived by another of the same name." If Mark Twain be the author of this saying, it is at once clear that an ambiguity is intended, but another author may intend to say that Bacon published them under an assumed name. A fallacy which depends upon the ambiguity of a single word is commonly called **equivocation;** if it consists in the ambiguity of a sentence or phrase, it is termed **amphibology.** As an illustration of the fallacy of equivocation consider the following argument for a protective tariff: "When we buy in a foreign country, we get the goods and they get the money, and when we sell in a foreign country, they get the goods and we get the money. How much better, then, to buy and sell in our own country, for in that case we retain both the goods and the money." The following would constitute an equally good (*and* an equally bad) retort: "When we buy in our own country, the producer loses the goods and the consumer loses the money. But the consumers and the producers make up the entire community. Therefore, when we buy in our own country, we lose both the goods and the money."

An historical dispute as to whether logic is a science or an art depends, probably, upon an ambiguity in the original meaning of the word. Logic is from a Greek word, λογική, an adjective with some substantive understood, which in turn is derived from λόγος. This word possessed a twofold sense, denoting both man's thought and his expression of it, a distinction exactly rendered by the two Latin words, *ratio* and *oratio*. The same equivocation was carried by the derivative λογική, and hence arose, no doubt, the dispute as to whether logic deals with the laws of thought or with the laws of the expression of thought.

FALLACY OF ACCENT

Whenever the sense of an assertion is changed, because the emphasis is thrown on some particular word, the **fallacy of accent** occurs. A certain barber is supposed to have invited patronage by placing before his shop a sign which contained the following (unpunctuated) statement:

"What do you think
I'll shave you for nothing
And give you a drink"

The visitor, once shaved and having demanded his drink, would be taken outside before the sign, which the barber would then read:

"What! do you think
I'll shave you for nothing
And give you a drink?"

Another instance, which has even led to sectarian controversy, is the meaning of the phrase, "Drink ye all of it." Shall we say, "Drink ye *all* of it," or rather, "Drink ye all *of* it"?

FALLACY OF MANY QUESTIONS

To the fallacy of **many questions** are usually referred all cases in which too many meanings are contained, or in which the issue on that account is generally confused. A good example is the conversation in Hamlet between the grave-diggers. Here the first remark is not in the form of a question, but calls, none the less, for a reply. The fallacy might be termed equally well the fallacy of **many statements.** The example will illustrate, too, what is called in logic a case of **non sequitur.**

> *First Clo.*—If the man go to this water and drown himself, it is, will he, nill he, he goes; mark you that; but if the water come to him and drown him, he drowns not himself: argal, he, that is not guilty of his own death, shortens not his own life.
> *Second Clo.*—Will you ha' the truth on't? If this had not been a gentlewoman, she should have been buried out of Christian burial.

Other fallacies are committed without the intention that they be taken seriously. Polonius conveys a sly hint to Hamlet when he says:

> "If you call me Jeptha, my lord,
> I have a daughter that I love passing well";

and Hamlet as slyly escapes by pretending that the remark contains a formal fallacy, for he rejoins:

"Nay, that follows not."

Nothing brings a conversation more abruptly to an end or more quickly disarms an opponent than the habit of taking him literally, for, arguing as it does a lack of imagination and even a lack of intellect, he is at once aware that the discussion cannot be maintained on the projected level. Nor does this habit characterize the unlettered only. Many an excellent scholar will betray the essential poverty of his mind by traits which point the same moral, by his attachment to words rather than meanings, or by his scorn of a style that is elegant because elevated, or, again, let us say, by his liking for what he calls the *impersonal* (*i.e.* literal) narration of history. De Morgan remarks that "the genius of uncultivated nations leads them to place undue force in the verbal meaning of engagements and admissions, independently of the understanding with which they are made. Jacob kept the blessing which he obtained by a trick, though it was intended for Esau; Lycurgus seems to have fairly bound the Spartans to follow his laws till he returned, though he only intimated a short absence, and made it eternal."

FALLACY OF ACCIDENT

The same writer recounts the following tale from Boccaccio, a tale in which the man appears to have possessed more of *esprit* than his master, but who could hardly have expected to be taken too literally: "A servant who was roasting a stork for his master was prevailed upon by his sweetheart to cut off a leg for her to eat. When the bird came upon table, the master desired to know what had become of the other leg. The man answered that storks had never more than one leg. The master, very angry, but determined to strike his servant dumb before he punished him, took him next day into the fields where they saw storks, standing each on one leg, as storks do. The servant turned triumphantly to his master; on which the latter shouted, and the birds put down their other legs and flew away. 'Ah, sir,' said the servant, 'you did not shout to the stork at dinner yesterday; if you had done so, he would have shown his other leg, too.'" The servant was here guilty of what logicians call the **fallacia accidentis,** of predicating of roasted storks what can only be predicated generally of storks. But the fallacy of accident may easily involve us in serious difficulties. The illustration below, taken from De Morgan, deserves to be quoted in full:

"The law in criminal cases demands a degree of accuracy in the statement of the *secundum quid* which many people think is absurd. . . . Take two instances as follows: Some years ago, a man was tried for stealing a ham, and was acquitted upon the ground that what was proved against him was that he had stolen a portion of a ham. Very recently, a man was convicted of perjury, "in the year 1846," and an objection (which the judge thought of importance enough to reserve) was taken, on the ground that it ought to have been "in the year *of our Lord* 1846." . . . In the two instances, which by many will be held equally absurd, a great difference will be seen by anyone who will imagine the two descriptions, in each case, to be put before two different persons. One is told that a man has stolen a ham; another that he has stolen a part of a ham. The first will think he has robbed a provision warehouse, and is a deliberate thief; the second may suppose that he has pilfered from a cook shop, possibly from hunger. As things stand, the two descriptions may suggest different amounts of criminality, and different motives. But put the second pair of descriptions in the same way. One person is told that a man perjured himself in the year 1846; and another, that he perjured himself in the year of our Lord 1846. As things stand, there is no imaginable difference; for there is only one era from which we reckon."

CONSCIOUS AMBIGUITIES

There is another large and important class of fallacies rather neglected, I think, by the logicians; arguments, which are not to be taken literally, but for a reason very different from the one that applies to the illustrations enumerated above. These are statements which are formally correct, but in which an ambiguity of terminology is *intended,* it may be for rhetorical purposes. Even the unlettered will not take you literally, if you remark that "business is business." The formal correctness of the phrase tends to force its acceptance, but it is quite evident that more is meant than meets the ear. Again, if I assert that "man is a vertical animal," it will be clear that more than a mere tautology is meant. It is said of Lincoln, while making a tour of the trenches after a brisk fight, that he remarked, with evident disgust of the whole affair (I quote from memory), "Anyone who likes this sort of thing must enjoy it very much." If it be said that "a man is a man for all that," it is to call attention to the fact that a tautology is not always true; that rather "a man is not himself *sometimes*." When the king and the others have left the play and Hamlet is left alone with Horatio, he says:

"For if the king like not the comedy,
Why then, belike, he likes it not, perdy,"

meaning that the burden of a bad conscience is the king's and not his.

Professor Stratton in an article appearing in the *Atlantic Monthly* says: "It is a prevailing belief that the mind is a convenient name for countless special operations or functions" and that these are independent. "When you have trained one of these you have trained that limited function and none other. What you do to the mind by way of education knows its place; it never spreads. *You train* *what you train.*" Here the formal correctness of the tautology seems to reinforce the argument. But this view of the character of mind ignores many important facts. "The psychological experiments which have so troubled the waters of education prove that normally *you train what you do not train.*" And, again, the conscious fallacy, the deliberate offense against logic, is correctly employed in favor of the opposite view.

In those verses of Lewis Carroll, which he calls "The Three Voices," the man in the piece, who has been accused by the lady of giving himself over to the exclusive instincts of his gourmandizing self, urges in his own defense that

"Dinner is dinner, tea is tea."

His defense is undermined, however, by her resolve to take his statement only at its face value, for she replies:

". . . Yet wherefore cease,
Let thy scant knowledge find increase;
Say men are men, and geese are geese."

Here the intent is not only to overthrow the opponent's argument, to render his contention impotent by refusing his implied ambiguity, but also to make a joke at the expense of logic itself, which is thus charged with giving us in its implications no information that we did not have before.

IGNORATIO ELENCHI

Again, an opponent may be disarmed for the moment by a statement that, while true, is irrelevant; what is called in logic the **ignoratio elenchi.** Thus Hamlet avoids telling his secret to Horatio and Marcellus by a reply that, to them at least, seems to have no bearing on the case. He says:

> "There's ne'er a villain dwelling in all
> Denmark
> But he's an arrant knave,"

and Horatio replies, quite properly:

> "There needs no ghost, my lord, come from
> the grave,
> To tell us this";

but Hamlet escapes again by simple agreement with this statement in its literal sense and by refusing to seek its implications. He rejoins:

> "Why, right; you are in the right;
> And so, without more circumstance at all,
> I hold it fit, that we shake hands, and part."

Agreement on the part of disputants is the end of discussion. It is said of the Frenchman Fontenelle, that he so far detested all forms of argument that he would refuse to differ with his opponent, employing habitually the phrase, *tout est possible,* whenever debate threatened. Another case in which an admission by an opponent may be taken unfairly, or, it may be, ironically, so that argument is effectually ended, is cited by De Morgan. "A writer disclaims attempting a certain task as above his powers, or doubts about deciding a proposition as beyond his knowledge. A self-sufficient opponent is very effective in assuring him that his diffidence is highly commendable, and fully justified in the circumstances."

AMBIGUITIES OF COMMON WORDS

Some of the ambiguities to which very common words are liable give rise to misunderstandings that are sometimes serious. Such words as the adjectives of quantity, *all* and *some,* the definite article, *the,* and the copula, *is,* come under this head. The assertions, "all of the angles of a triangle equal two right angles" and "all of the angles of a triangle are less than two right angles," are both true, if *all* is taken collectively in the first instance and distributively in the second. The phrases, "all of these twelve men are a jury" and "all of these men are liable to be prosecuted" illustrate the same ambiguity. In Rousseau's conception of the social contract it is said that each member of society is called upon to surrender *all* his rights in order that the rights of *all* may be preserved—a result that may appear paradoxical at first blush, until the equivocation in the use of the italicized word is noticed. In the phrase,

Chacun se donnant a tous ne se donne a personne,

the meaning is quite clear and unambiguous. The Latin *omnis* preserves the same double meaning. Thus the state of savage man is described as the *bellum omnium contra omnes.* The word, *both,* is similarly equivocal. If I say, "Both this man and his wife are either male or female," the case is true in the distributive but untrue in the collective sense of *both;* and the opposite will hold if I say, "This man can walk on both legs." But, "A man can hop on both legs" is true in either sense.

One of the chief difficulties of the logic of Hamilton depends on the ambiguity of the meaning of *some.* Of this word he says: "A remarkable uncertainty prevails in regard to the meaning of particularity and its signs. Here *some* may mean *some only—some, not all,"* and is "definite in so far as it excludes omnitude." Thus, "Some Greeks are Athenians." "On the other hand, *some* may mean *some at least—some, perhaps all."* Thus, "Some men are rational animals" where man is defined as a rational animal. If it be argued that "Socrates is a man and man is a class, therefore Socrates is a class," there are those who will find here an ambiguity in the meaning of the copula or else in the meaning of the singular term. Either Socrates is regarded as a class of one member, they will say, or the relation of an individual to a class is to be distinguished from the relation of a class to a class. The definite article in English sometimes generalizes and sometimes individualizes. "The animal" is general when we speak of "the animal in man," but otherwise when we say, "Have no fear, the animal will find his way home." Generally it is remarked, "*Man* is unfaithful," but "*The dog* is faithful to man." In Greek, in French, in German, on the contrary, the definite article is required before *man,* when the word is intended in the universal sense.

PETITIO PRINCIPII

It is generally true, and is, indeed, set down as one of the axioms of logic, that, if two propositions are true together, then either one of them may be assumed separately to be true. The statement will perhaps appear trivial, but a serious fallacy frequently arises in connection with it, and in the following way: Suppose that, in order to prove a given proposition, we should assume two others, such that one or both of the two assumed ones should be merely a disguised expression of the given one. If we should suppress the premises and assert the conclusion by itself, we should then virtually assume the conclusion—that is, by assuming the right to suppress the premise equivalent to it. Such a fallacy is known as a **petitio principii.**

Serious instances of this fallacy are not uncommon in the history of science. Thus, most of the apparently successful efforts to demonstrate the so-called parallel axiom of Euclid are breaches of this rule. The demonstrator commonly takes for granted, intuitively, some principle which is equivalent to the result he seeks. The great geometer Gauss, writing on his efforts to effect this proof to his friend Wolfgang von Bolyai in the year 1799, says: "Certainly I have come upon much that for the majority would pass as a proof, but in my eyes demonstrates nothing." He then goes on to enunciate equivalent propositions that would be covertly taken for granted by many, but whose assumption would constitute a *petitio principii.* If the student will realize that for two thousand years mathematicians had struggled with this proof, the same fallacy being committed again and again, he will appreciate the seriousness of the difficulty. De Morgan relates that the mathematician Lagrange once wrote a memoir on the theory of parallels. While presenting it to the members of the French Academy, he withdrew the manuscript in the middle of the reading with the remark, *"Il faut que j'y songe encore."*

ACHILLES AND THE TORTOISE

In concluding this chapter on fallacies we shall include the case of a famous sophism which Zeno the Eleatic employed, in order to prove that motion is impossible. It is known as the paradox of Achilles and the tortoise. De Quincey gives the following account in one of his essays, and, as it cannot be better related, we shall quote him in full:

"Achilles, most of us know, is celebrated in the 'Iliad' as the swift-footed (ποδας ωκυς 'Αχιλλευς); and the tortoise, perhaps all of us know, is equally celebrated among naturalists as the slow-footed. In any race, therefore, between such parties, according to the equities of Newmarket and Doncaster, where artificial compensations as to the weight of riders are used to redress those natural advantages that would else be unfair, Achilles must grant to the tortoise the benefit of starting first. But if he does *that,* says the Greek sophist, then I, the sophist, back the tortoise to any amount, engaging that the goddess-born hero shall never come up with the poor reptile. Let us see. It matters little what exact amount of precedency is conceded to the tortoise; but say that he is allowed a start of one-tenth part of the whole course. Quite as little does it matter by what ratio of speed Achilles surpasses the tortoise; but suppose this ratio to be that of ten to one, then, if the racecourse be ten miles long, our friend the slow-coach, being by the conditions entitled to one-tenth of the course for his starting allowance, will have finished one mile as a *solo* performer before Achilles is entitled to move. When the *duet* begins, the tortoise will be entering on the second mile precisely as Achilles enters on the first. But, because the Nob runs ten times as fast as the Snob, whilst Achilles is running his first mile, the tortoise accomplishes only the tenth part of the second mile. Not much, you say. Certainly not very much, but quite enough to keep the reptile in advance of the hero. This hero, being very little addicted to think small beer of himself, begins to fancy that it will cost him too trivial an effort to run ahead of his opponent. But don't let him shout before he is out of the wood. For, though he soon runs over that tenth of a mile which the tortoise has already finished, even this costs him a certain time, however brief. And during that time the tortoise will have finished a corresponding subsection of the course—*viz.*, the tenth part of a tenth part. This fraction is a hundredth part of the total distance. Trifle as that is, it constitutes a debt against Achilles, which debt *must* be paid. And whilst he *is* paying it, behold our dull friend in the shell has run the tenth part of a hundredth part, which amounts to a thousandth part. To the goddess-born what a flea bite is that! True, it is so; but still it lasts long enough to give the tortoise time for keeping his distance, and for drawing another little bill upon Achilles for a ten-thousandth part. Always, in fact, alight upon what stage you will of the race, there is a little arrear to be settled between the parties and always *against* the hero. 'Vermin, in account with the divine and long-legged Pelides, Cr. by one billionth or one decillionth of the course,' much or little, what matters it, so long as the divine man cannot pay it off before another installment becomes due? And pay it off he never will, though the race should last for a thousand centuries."

It may be argued (as indeed it has been) that we can easily calculate the exact spot where Achilles will overtake the tortoise. But such a solution clearly misses the point. "Of course . . . it becomes easy, upon assuming a certain number of feet for the stride of Achilles, to mark the precise point at which that 'impiger' young gentleman will fly past his antagonist like a pistol shot, and being also '*iracundus, inexorabilis, acer,*' will endeavor to leave his blessing with the tortoise in the shape of a kick (though, according to a picturesque remark of Sidney Smith, it is as vain to caress a tortoise, or, on the other hand, to kick him, as it is to pat and fondle, or to tickle, the dome of St. Paul's)."

It is often said, somewhat patronizingly, that had Zeno grasped the modern notion of a differential coefficient, the limiting value of the ratio of two infinitesimals, there would have been no paradox.

This is the solution of Leibnitz and it is the one which De Quincey accepts. "The infinity of space in this race of subdivision is artfully run against a *finite* time; whereas, if the one infinite were pitted, as in reason it ought to be, against the other infinite, the endless divisibility of time against the endless divisibility of space, there would arise a reciprocal exhaustion and neutralization that would swallow up the astounding consequences, very much as the two Kilkenny cats ate up each other."

It must be remarked, however, that this solution is equally beside the point. The real difficulty, when the argument is properly stated, is to come to the end of an infinite series—that is, to come to the end of something that has no end by definition. The real fallacy, I believe, lies in an ambiguity in definition. The arguer defines an infinite series as one which has no *last term,* and later revokes the condition for his opponent, reinserting the last term as something that has to be passed through *for him.* Briefly the steps are these: "Achilles, in order to catch the tortoise, must pass through the last term of an infinite series. But an infinite series *has no* last term. Accordingly, in order to catch the tortoise, Achilles must do that which (by definition) he cannot do." The solution is to reject the major of this syllogism. If the series has no last term there is no need to pass through *a last term* in order to reach the limit. The last term, which is excluded as a *possible* obstacle in the original definition, is reinvoked as a *real* obstacle for him against whom the argument is directed.

Exercises

1. Given an original illustration of each one of the fallacies specifically named in this chapter.
2. Examine the following statements and set forth clearly the sophism or paradox therein contained:

 A man accustomed to put his trust in dreams, one night dreamed that all dreams are vain. (From Jeremy Taylor's sermon on "The Deceitfulness of the Heart.")

 The Cretan Epimenides says that "all the Cretans are liars."

 The riches of a producer depend on the scarcity of his commodity. (From Bastiat, *Sophismes Économiques.*)

 Wealth consists in the abundance of things. The many who have little, combine to protect the few who have much.

3. Select from the following such as contain a "circle" in definition:

 Exceptive propositions affirm a predicate of all the subject with the exception of certain defined cases.

 An *affirmative* proposition is one in which an agreement is affirmed between the subject and the predicate.

 A *number* is anything which is the number of some class.

By the *mass* of a body is meant the quantity of matter contained in the body.

Force is that which tends to modify motion.

4. What ambiguities are implied in the following expressions:

"With respect to the appearance of this work (Fichte's *Characteristics of the Present Age*), I have nothing further to say to the Public than that I have nothing to say."

"Let anyone who reads this work without understanding it, assume no more than this: that he does not understand it."

"I have no other but a woman's reason,
I think him so, because I think him so."
—*Two Gentlemen of Verona,* Act I, Sc. ii.

"Non amo te, Sabidi, nec possum dicere quare;
Hoc tantum possum dicere, non amo te."
—MARTIAL, Ep. I., xxxiii.

5. Examine the argument given below in order to determine its validity or invalidity:

Differences cannot be expressed, for suppose two things, *x* and *y* to be different. Then we should say "*x* is not *y*. Now *y* is a way of existing and, consequently, not-*y* is a way of not existing; so that, in trying to express a difference, we have only said that *x* is a way of not existing."

IMMEDIATE INFERENCE

CHAPTER VI

THE UNIVERSE OF THE CATEGORICAL FORMS

We have seen that the categorical forms, A, E, I, and O, are composed of the terms (a and b), an adjective of quantity (*all, no, some, not all*) and the copula (*is*). In previous chapters we have noticed many cases in which the terms and their relations (*all is, no is, some is, not all is*) take on ambiguous meanings. In particular it has been said that the word *some* is to be taken in the sense of *some at least, possibly all.*

The student may well inquire by what right it is that we are allowed to understand this word in any sense we please. We reply that this meaning of the word is unambiguously forced upon us by the propositions which we say shall be true or untrue in our science. For example, we are going to say that

$$\mathrm{A}(ab) \text{ implies } \mathrm{I}(ab)$$

is a true, or valid, implication, and this would not be the case if *some* meant *some, not all.* We are going to say, too, that

$$\mathrm{I}(ab) \text{ implies } \mathrm{O}(ab)$$

is an untrue, or invalid, implication. But such an implication would *follow,* would be true, if the word *some* were taken in this latter sense. We intend that the propositions which are valid or invalid in our system shall be confirmed as true or untrue by common sense. The meaning first given, *some, at least, possibly all,* is the interpretation which will be always verified in experience, when we come to apply our theory practically.

MEANING OF THE TRUE AND UNTRUE

There are two other words with which we shall have to deal, whose sense is best rendered unambiguous at the outset. Thus, *true* means *necessarily true,* true in all cases, true for all meanings of the terms. If the student were to represent the sense of "some *a* is *b*" by means of the diagram (Fig. 5) below

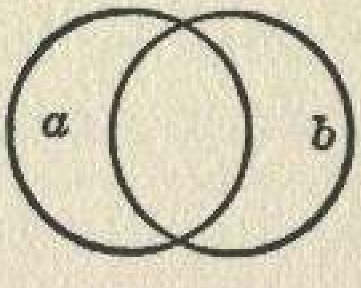


FIG. 5

it might then appear to him that the implication,

I (*ab*) implies O (*ab*),

is true. That this is only an *apparent* truth, would have been manifest at once, if he had employed instead either one of the diagrams of Fig. 6.

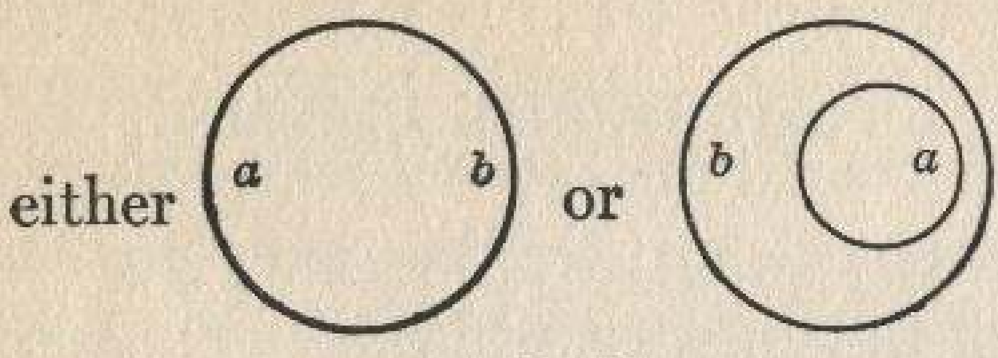


FIG. 6

as his representation of "some *a* is *b*."

The word *untrue,* in turn means *not necessarily true,* not true in every instance, not true for all meanings of the terms; that is, there is at least one set of meanings of the terms which will invalidate the proposition in question. Thus, in order to become aware that the implication,

$$\mathbf{I}(ab) \text{ implies } \mathrm{O}(ab),$$

is not generally true, it would be enough to point to *either one* of the diagrams of Fig. 6, or to assign appropriate concrete meanings to the terms. Thus if *a* stands for metals and *b* stands for elements,

> "If some metals are elements,
> then some metals are not elements,"

the untruth of the general statement is at once manifest. If the illustration selected had been

> "If some red substances are elements,
> then some red substances are compounds,"

the untruth of the original implication would not have been shown.

THE PROPOSITIONAL UNIVERSE

These matters being clear, we may pass to the chief topic of this chapter, which is to explain what is meant by the *universe* of the categorical forms and to define certain technical expressions. In the last chapter, when we represented the categorical propositions by means of diagrams, we assumed that there is a certain analogy between the manner in which closed areas overlap and the manner in which classes overlap. This analogy was first pointed out by the mathematician Euler in a popular work on "natural philosophy" entitled *Letters Addressed to a German Princess,* and such diagrams are, accordingly, referred to as Euler's diagrams. If the student will examine the figures set down at the end of Chapter III, it will be intuitively clear to him, if he assumes this analogy, that any two classes, *a* and *b*, must be related in one, and cannot be related in more than one, of the following five ways (Fig. 7):

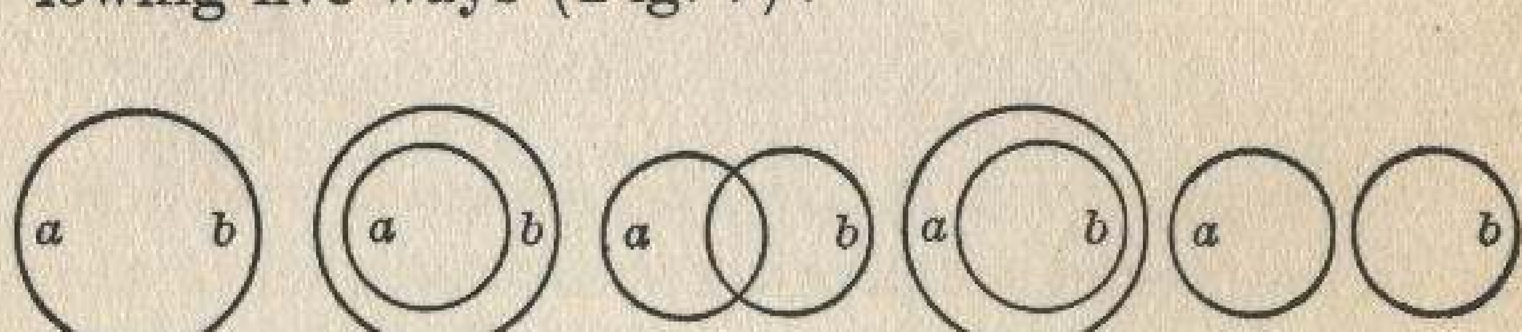


FIG. 7

These five cases may be conceived as five possibilities, one and only one of which can be realized in any particular case. That is, for any pair of concrete meanings of *a* and *b* (triangles and trilaterals, elements and compounds, Shakespearean scholars and Englishmen, etc.) one of the five possibilities is realized and the others remain unrealized. If I assert that one of these five representations is the true one, no matter what meanings the terms may take on, I assert something that is true. In the form of a disjunction the assertion would be, "Either the first, or the second, or the third, or the fourth, or the fifth possibility must be realized" for every meaning of *a* and *b*. This disjunction is called the **propositional universe,** or the **universe of the categorical forms,** or, again, the **logical sum** of all the possibilities.

If it is clear that at least one of these diagrams represents the true relation of *a* to *b,* it will be equally manifest that the relation of *a* to *b* cannot be represented in more than one of these five ways. If I assert the contrary to this last condition, I say something that is false. In the form of a conjunction this assertion would be, "Both the first (say) and the last (say) possibility are realized" at once. Such a conjunction is called the **propositional null,** or, again, the **logical product** of two or more of the possibilities.

CONTRADICTORIES, CONTRARIES, SUBCONTRARIES AND SUBALTERNS

By means of this conception of a propositional universe the more fundamental relations connecting the categorical forms may be established at once. Since "all *a* is *b*" asserts that one of the first two possibilities is realized and "some *a* is not *b*" asserts that one of the last three possibilities is realized, the disjunction,

"Either all *a* is *b* or some *a* is not *b*,"

—that is, the assertion that "either A is true or O is true," is precisely equivalent to the *propositional universe,* and is, therefore, of necessity a true statement. Similarly, the conjunction,

"All *a* is *b* and some *a* is not *b*,"

—is equivalent to the *propositional null,* and is false, for A and O contain no possibilities in common. Accordingly, A and O cannot both be true and cannot both be false.

In general, whenever two categorical propositions contain one or more possibilities in common, they may both be true, but not otherwise. Thus, I and O may both be true, but A and E cannot both be true. Whenever two categorical propositions, taken together, do not make up the universe of possibilities, they may both be false, but not otherwise.

Thus, A and E may both be false, but I and O cannot both be false.

We proceed to set down the following definitions, which will prove to be of great importance for our subsequent theory:

Two propositions which cannot both be true and cannot both be false are called **contradictories.**

Two propositions which cannot both be true but which may both be false are called **contraries.**

Two propositions which may both be true but which cannot both be false are called **subcontraries.**

Two propositions which may both be true and which may both be false are called **subalterns.**

The student who has made only a slight progress in his mathematical studies will still realize that we cannot classify the four assertions, A, E, I, and O, under these heads by a mere reference to a set of Euler's diagrams. The truths that we assume without demonstration must appear among our axioms, however "self-evident" they may otherwise seem. We accordingly assume:

Postulate 1.—A and O cannot both be true and cannot both be false.

Postulate 2.—E and I cannot both be true and cannot both be false.

From these assumptions it follows that A and O and that E and I are contradictory pairs. In the next chapter postulates and principles will be set down and theorems will be deduced from them, by means of which it will be possible to say that A and E are contraries, that I and O are subcontraries, and that A and I, and E and O, are subalternate pairs. As a preliminary exercise, however, the student should scrupulously verify these facts for himself, employing the diagrams of Fig. 7 with this end in view. In order to facilitate this verification as well as to emphasize its importance, the figure is again reproduced on this page below.

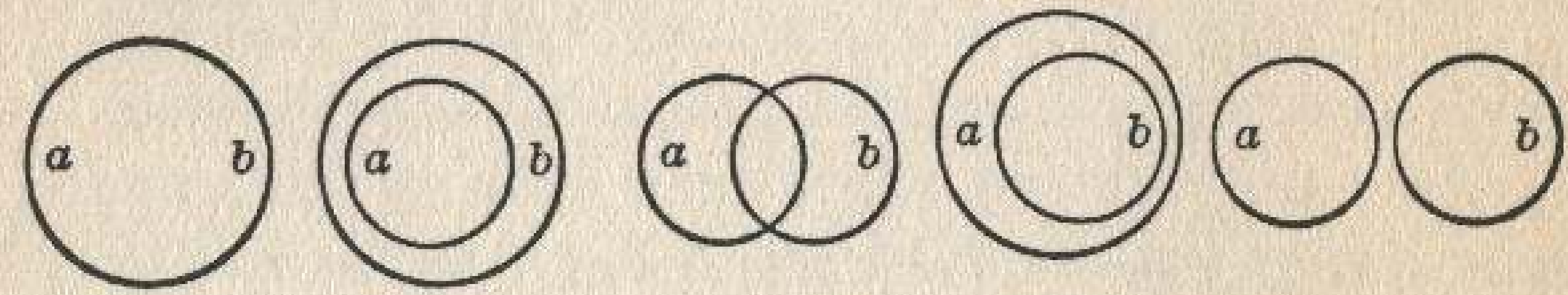


FIG. 8

EXERCISES

1. What propositions must be true and what ones must be false, when we take A, E, I, and O to be true in succession?
2. What propositions must be true and what ones must be false, when we take A, E, I, and O to be false in succession?
3. Draw a set of diagrams which will represent A, E, I, and O when the terms, *a* and *b*, have been reversed. In what cases does this alteration leave the original meaning unchanged?
4. In an argument your opponent has managed to establish the truth of a proposition, which is subcontrary to the one which you are maintaining yourself. How would you retort his contention?
5. Is it more desirable in an argument to establish the contrary of your opponent's view or the contradictory?
6. Assume that the first four of Euclid's axioms must be affirmed, but that either of the remaining two may be denied. If the fifth and sixth axioms are subcontraries, how many geometries different from Euclid's are possible?

CHAPTER VII

THE MOODS OF IMMEDIATE INFERENCE

In the hypothetical proposition, x implies y (if x is true, then y is true), the part x to the left of the word *implies* is called the **antecedent** and the part y to the right is called the **consequent.** Here x and y may stand for any sort of proposition, but if each one is a single categorical form, then we should replace x and y by a more definite notation, for example, "A(ab) implies I(ba)," "if E(ab) is true, then A(ab) is false," etc. Any implication of this latter specific type is known as **immediate inference.**

It will be recalled that the comma in the bracket between the terms is used in order to indicate that the term-order is not settled. Thus, just as O(a, b) may represent either O(ab) or O(ba), so all propositions like "A(a, b) implies A(b, a)" may have either one of two term arrangements. A difference between two forms of inference which is dependent on term-order alone is known as a difference of **figure.**

FIGURES OF IMMEDIATE INFERENCE

If the term-order in the antecedent is the same as the term-order in the consequent—that is, if, for example, "A(*a*, *b*) implies A(*a*, *b*)" be written:

either "A(*ab*) implies A(*ab*),"
or "A(*ba*) implies A(*ba*),"

then "A(*a*, *b*) implies A(*a*, *b*)" is said to be expressed in the *first figure* of immediate inference. If the term-order in the antecedent is the reverse of the term-order in the consequent—that is, if, for example, "A(*a*, *b*) implies A(*a*, *b*)" be written,

either "A(*ab*) implies A(*ba*),"
or "A(*ba*) implies A(*ab*),"

then "A(*a*, *b*) implies A(*a*, *b*)" is said to be expressed in the *second figure* of immediate inference. It is clear that this implication will be true in the first, but untrue in the second figure, so that a difference of figure may very possibly involve a difference in the *truth-values* of the two cases.

ARRAY OF IMMEDIATE INFERENCE

It is evident that all the variants of immediate inference are to be gotten by permuting the four letters A, E, I, and O, two at a time, and by taking each letter once with itself. We should thus obtain sixteen distinct propositions of the type we are considering, as follows:

A (a, b) implies A (a, b)
A (a, b) implies E (a, b)
A (a, b) implies I (a, b)
A (a, b) implies O (a, b)
E (a, b) implies A (a, b)
E (a, b) implies E (a, b)
E (a, b) implies I (a, b)
E (a, b) implies O (a, b)

I (a, b) implies A (a, b)
I (a, b) implies E (a, b)
I (a, b) implies I (a, b)
I (a, b) implies O (a, b)
O (a, b) implies A (a, b)
O (a, b) implies E (a, b)
O (a, b) implies I (a, b)
O (a, b) implies O (a, b)

It will be convenient from time to time to leave unexpressed the word *implies* and the (a, b) and to write down the same set of sixteen implications in the following more abbreviated fashion:

AA	EA	IA	OA
AE	EE	IE	OE
AI	EI	II	OI
AO	EO	IO	OO

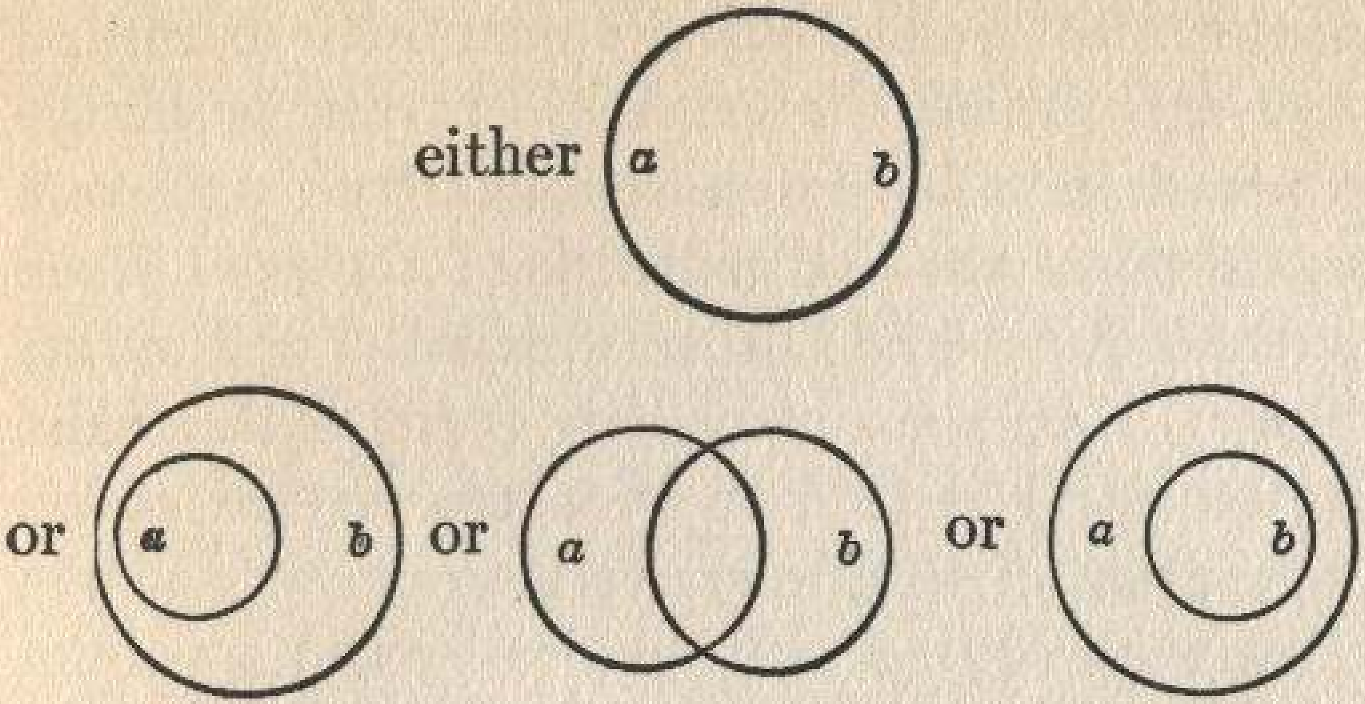
Each proposition of the set may be expressed in either the first or the second figure and there are, consequently, thirty-two possible forms of immediate inference. The entire set of thirty-two is said to constitute the **array** of immediate inference. Each member of the array is called a **mood** of the array. The true propositions of the array are called **valid moods** of the array. The remainder are called **invalid moods** of the array.

VALIDITY AND INVALIDITY DETERMINED BY EULER'S DIAGRAMS

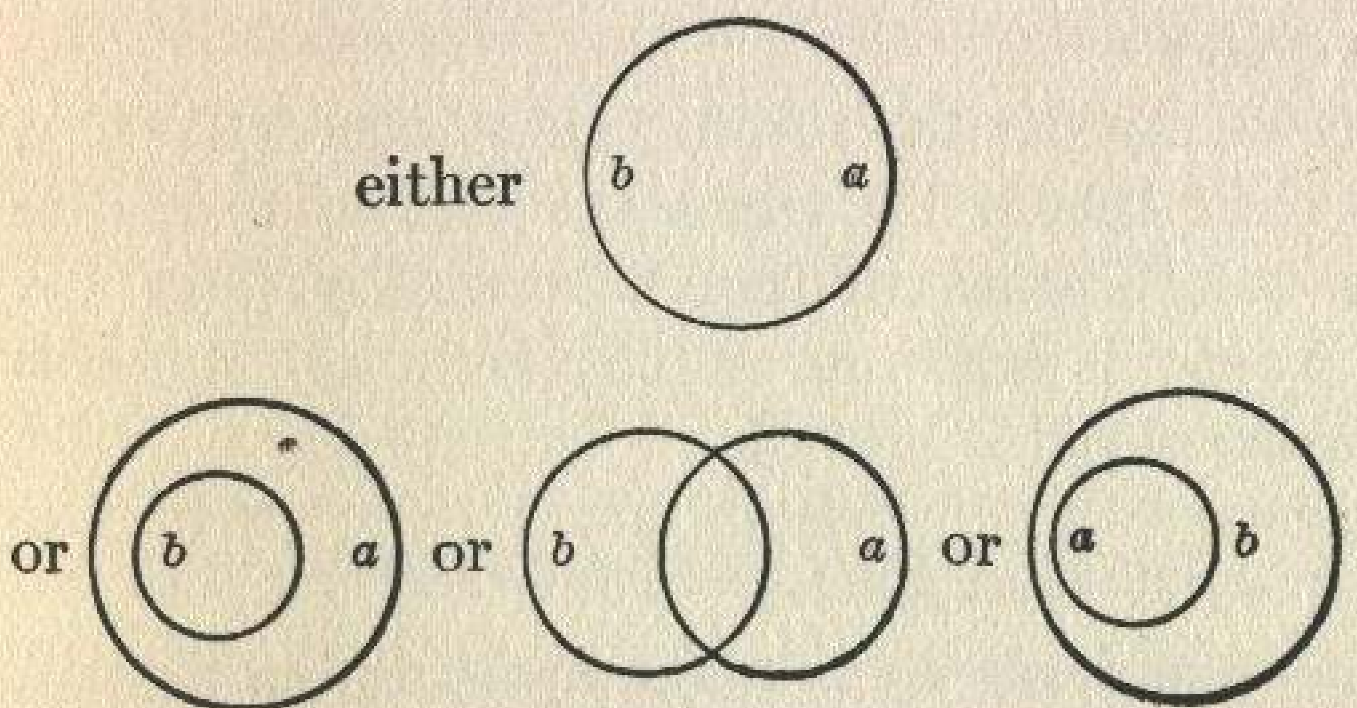
In order to determine precisely what are the valid and what are the invalid moods of this set, let us employ the method of inspection by means of diagrams which was explained in the last chapter. A few examples will suffice to illustrate this method. The student in completing the exercise which is here proposed, will do well to direct his attention to Fig. 7 of the last chapter.

(1) Consider the mood E(*ab*) implies O(*ab*), or, in our abbreviated notation, EO in the first figure. This asserts that if the fifth possibility (Fig. 7) is realized, then at least one of the last three possibilities is realized. It is intuitively evident then that EO in the first figure is a valid mood.

(2) Consider the mood I(*ab*) implies I(*ba*), or, in our abbreviated notation, II in the second figure. I(*ab*) is represented by



and **I** (*ba*) is represented by



These two modes of representation are identical, except that the diagrams do not appear in the same order. But, since the order in which the diagrams appear is irrelevant, it is intuitively clear that the mood is a valid one. If we had chosen to consider the mood AA in the second figure, it would have appeared at once that not all of the possibilities contained in the antecedent are contained as well in the consequent, and the empirical invalidity of the mood would have been manifest at once.

In general, if the antecedent is of the *same form* as the consequent and the mood is valid in the second figure, then the antecedent (or, indifferently, the consequent) is said to be a **convertible** form. The operation of **simple conversion** consists in the interchange of subject and predicate and the proposition in question is said to be **simply convertible.** The student will discover that this operation is permissible in the case of E and I, but not in the case of A and O.

(3) Consider the mood, I(*ab*) implies A(*ab*). It is clear that if the terms are such as to realize either the third or the fourth possibility (Fig. 7), then I(*ab*) is true and A(*ab*) is false. Accordingly, the mood is invalid.

DEDUCTION OF THE VALID MOODS

This case leads us to make again the important observation: Since true means *necessarily true* and untrue means *not necessarily true,* it is enough to point to one diagrammatic representation of the antecedent which at the same time is not a diagrammatic representation of the consequent, in order to become aware of the invalidity of the mood. It is said of certain treatises of the Hindus on geometry, that the master, instead of offering a proof of the separate theorems, was content, after stating the proposition, to draw the figure and write under it a word like *ecce*. The pupil was thus expected to gather intuitively the abstract or general truth from the observation of a single illustration. The student is well aware that the ideal of the Greek geometers was to deduce the theorems of the science from the fewest possible number of initial assumptions. Whether this ideal be a mistaken one or not, it has at least inspired the procedure of all science down to the present day. If we were to apply this historical contrast of the Greek and the Hindu geometers to ourselves, we might say that up to now our study of logic has been carried out on the Indian plan. Up to now we have been Hindu logicians, for we have been content merely to write *ecce* underneath our diagrams—a sort of Cartesian test, an application of the *clare et distincte percipio*. But from this moment forth we shall fashion our doctrine after the Helladian model. We shall deduce all the true and all the untrue variants of immediate inference by the aid of certain principles from the fewest possible number of initial postulates.

Definition.—Two propositions that cannot both be true and cannot both be false are said to be **contradictory.** By the postulates of the preceding chapter it follows that A(ab) and O(ab) and that E(ab) and I(ab) are contradictory pairs.

Principle I.—If in any valid mood antecedent and consequent be interchanged and each be replaced by its contradictory, a valid mood will result.

Postulate 1.—A(*ab*) implies A(*ab*),
Postulate 2.—A(*ab*) implies I(*ab*),
Postulate 3.—I(*ab*) implies I(*ba*),
Theorem 1.—O(*ab*) implies O(*ab*),
(from Postulate 1, by Principle I),
Theorem 2.—E(*ab*) implies O(*ab*),
(from Postulate 2, by Principle I),
Theorem 3.—E(*ba*) implies E(*ab*),
(from Postulate 3, by Principle I).

These are as many results as can be gotten from our assumptions. We therefore proceed with the introduction of an additional principle and with a definition which will make its application possible.

Definition.—If x implies y is a valid implication. then x is said to be a **strengthened** form of y and y is said to be a **weakened** form of x.

This definition of the meaning of strengthening and weakening is not to be taken in the traditional way, but in a more general sense. If it happens to be true, for example, that y implies x in addition to the fact that x implies y, then y will not only strengthen to x, but will also weaken to x. Thus, by postulate 3, I(*ab*) weakens to I(*ba*) and I(*ba*) strengthens to I(*ab*). But since postulate

3 can also be written I(*ba*) implies I(*ab*), this latter expression being only another way of writing II in the second figure, it follows that I(*ba*) is also a strengthened form of I(*ab*) and that I(*ab*) is also a weakened form of I(*ba*). Again A(*ab*) weakens to I(*ab*) and I(*ab*) strengthens to A(*ab*) by postulate 2. In applying the principle about to be given, it must be noticed that the theorems just established give us the right to strengthen or to weaken in the same sense as do the postulates.

Principle II.—If in any valid mood the antecedent be strengthened or the consequent be weakened, a valid mood will result.

Theorem 4.—A(*ab*) implies I(*ba*),
(for Postulate 3 gives us the right to weaken the consequent of Postulate 2).

Theorem 5.—E(*ba*) implies O(*ab*),
(for Theorem 3 gives us the right to strengthen the antecedent of Theorem 2).

Theorem 6.—I(*ab*) implies I(*ab*),
(for Postulate 3 may be written in either of two ways, as explained above. Accordingly, Postulate 3 gives us the right to strengthen its own antecedent or to weaken its own consequent).

Theorem 7.—E(*ab*) implies E(*ab*),
(for, similarly, Theorem 3 gives us the right

to strengthen its own antecedent or to weaken its own consequent. Or we might have arrived at the same result by applying Principle I to Theorem 6).

DEDUCTION OF THE INVALID MOODS

We have, accordingly, by postulating the validity of three of the moods of immediate inference deduced the remaining seven by the aid of two principles. These two, as well as the two which are given below, will have to be assumed later on in any case, but in a more general form. The deduction of the invalid moods is left as an exercise for the student. Since it will be necessary to postulate four of these moods as invalid, he will have eighteen theorems to deduce. The postulates and the principles of deduction are given below. It is only necessary to add that the additional results of theorems 4-7 (above) must be kept in mind when he comes to apply Principle IV (below).

Postulate 4.—A (*ab*) does not imply A (*ba*),
Postulate 5.—A (*ab*) does not imply O (*ba*),
Postulate 6.—A (*ab*) does not imply O (*ab*),
Postulate 7.—E (*ab*) does not imply I (*ab*).

Principle III.—If in any invalid mood antecedent and consequent be interchanged and each be replaced by its contradictory, an invalid mood will result.

Principle IV.—If in any invalid mood the antecedent be weakened or the consequent be strengthened, an invalid mood will result.

Theorems.—The other (18) invalid moods.

Exercises

1. The process by which we infer I from A or O from E in the second figure, is called *conversion by limitation* or *per accidens*. Cast the following into categorical form and convert by limitation:

   "A favorite has no friend."

2. When the terms of a proposition are simply converted, the resulting proposition is called the *converse* and the original proposition is called the *convertend*. What is the converse of the following:

   "No man e'er felt the halter draw,
   With good opinion of the law."

3. Immediate inference by *privative conception* consists in passing from an affirmative to a negative equivalent to it, or *vice versa*. Thus, "all metals are elements" is the same as "no metals are compounds"; "some elements are not metals" is the same as "some elements are non-metals." Effect this transformation in the statements of Exercises 1, 2, 4, and 5.
4. Conversion by *contraposition* consists in replacing the terms by their negatives and interchanging them. It is not permissible in the case of E and I. Convert by contraposition:

   "All that glisters is not gold."

5. Transform the following by privative conception and convert the result by contraposition and by limitation:

"No man can eat his cake and have it, too."

CHAPTER VIII

THE RULES OF IMMEDIATE INFERENCE

It is the custom of the traditional logic to formulate certain rules by whose aid the invalid moods of immediate inference may be detected immediately. These rules all turn upon the meaning of a *distributed* term. We begin, therefore, with an explanation of the sense of this conception, giving the definition at the outset and setting forth its application in the sequel.

DISTRIBUTED TERMS

Definition I.—Distributed terms are those modified, either implicitly or explicitly, by the quantitative adjectives "all" or "no." All others terms are **undistributed.**

In the first place, it is to be noticed that before the predicate of A and the predicate of I, the word "some" is unexpressed but understood. When we assert "all *a* is *b*," we mean: "all *a* is some (it may be all) *b*"; and the same remark holds of I. This fact may be more easily seen to hold of I, if we appeal to the property of simple convertibility of this form. When "some *a* is *b*" is written equivalently, "some *b* is *a*," the quantitative adjective which is implicit before the predicate in the first case becomes explicit before the subject in the second case. The predicate of A and the predicate of I are, therefore, according to our definition, undistributed terms. It is equally clear that the subject of A is distributed, since it is modified by "all," and that the subject of I is undistributed, since it is modified by "some."

That the subject of E, "no *a* is *b*," is distributed, is at once apparent from our definition. But the distributed character of the predicate will be manifest as well, if we appeal to the property of the simple convertibility of E, established in the last chapter. Thus, "no *a* is *b*" being logically equivalent to "no *b* is *a*," the quantity of the predicate-term becomes explicit when it is made the subject. The same result would appear in another way if we were to assume the right to change E into A by privative conception, expressing it in the form "all *a* is non-*b*," and then in the form, "all *b* is non-*a*," the terms being then explicitly modified by the adjective "all." Accordingly, E distributes both its subject and its predicate by definition.

As regards the O-form, "some *a* is not *b*," it is apparent at once that the subject is an undistributed term, for it is modified by "some" and not by "all." But it is not so easy to see that the predicate is distributed. In order to become aware of this fact, imagine the part "some *a*" to represent a fixed part of the *a* class. We may then imagine the contradictory of this part and designate it by the phrase, "non-some *a*." This part will constitute everything that is not "some *a*." The student will then be able, perhaps by the aid of a diagram which he may construct for himself, to understand that "some *a* is not *b*" is exactly rendered by the phrase, "all *b* is non-some *a*." The meaning which was implicit before the predicate in the first form has become explicit before the same term appearing as the subject in the new but equivalent expression. We conclude, then, that O distributes its predicate, but does not distribute its subject. These results appear in the following scheme, the distributed terms being printed in black letter:

A(***a*** *b*)	E(***a*** ***b***)
I(*a* *b*)	O(*a* ***b***)

We shall now state the first rule for the immediate detection of the invalid moods of immediate inference and we shall only introduce additional ones when it shall have been shown that this one is not in itself sufficient to effect our purpose.

Rule 1.—A form in which a given term appears undistributed does not imply a form in which that same term appears distributed.

Consider the mood A (*ab*) implies A (*ba*), or, in our abbreviated notation, AA in the second figure. The predicate of the antecedent is an undistributed term, but it appears distributed as the subject of the consequent. The mood is, consequently, declared invalid by the first rule. Again, in the mood AE in the first figure the subject term is distributed in both antecedent and consequent, but the predicate of the antecedent is distributed in the consequent. The mood is therefore declared invalid by the rule.

The student would now do well to construct the array of immediate inference for himself and to determine precisely just which moods in each figure come under the rule in question. He will find that some moods remain whose invalidity is not declared. We proceed, accordingly, to formulate two additional rules, which will prove exactly enough to effect our purpose. These will have to be preceded by definitions which will render them applicable.

AFFIRMATIVE FORMS AND NEGATIVE FORMS

Definition 2.—A form whose predicate is undistributed is called an **affirmative** form. By results already established it follows that A and I are affirmative. This definition will very possibly bewilder the student upon first consideration, for he will miss the motive which prompts it. He will rather have expected us to define an affirmative form by means of a *synonyme,* after the fashion of the dictionary. Instead of that we have followed a procedure which is usual in science; we have selected a *property* which is characteristic of affirmative forms, but which does not characterize the others, and we have used this property in order to define them. Our next definition is:

Definition 3.—A form which distributes its predicate is called a **negative** form. By results already established it follows that E and O are negative forms. The two rules which remain to be stated are:

Rule 2.—An affirmative form does not imply a negative form.

Rule 3.—A negative form does not imply an affirmative form.

These three rules will be found *sufficient* for the purpose in hand, for it will be discovered that by means of them all the moods previously found to be invalid are declared untrue. That they are also *necessary*—that is, that no one of them can be dispensed with—will appear at once from the following consideration: Suppose that an invalid mood has been found that is declared invalid by the first rule and by no other rule. It is clear that this rule could not then be omitted from our list; and the same remark applies to the other two. The three rules are all necessary because we can point to at least one example which falls uniquely under each rule.

COROLLARY TO THE RULES

In addition to the rules there is a corollary which follows upon their assumption, and whose application depends upon the following:

Definition 4.—A form which distributes its subject is said to be **universal**. By results already established A and E are universal forms.

Definition 5.—A form which does not distribute its subject is said to be **particular.** By results already established I and O are particular forms. The facts that have now been made matters of definition are conveniently remembered by means of the mnemonic scheme which is given below, the distributed terms being printed in black letter.

	Affirmative	Negative
Universal	A (**a** *b*)	E (**a** **b**)
Particular	I (*a* *b*)	O (*a* **b**)

Corollary.—A particular form does not imply a universal form.

This theorem will be proven by showing that every one of the moods in question are declared invalid by one or more of the rules. Thus, OA in both figures is thrown out by the third rule and IE, IA and OE in both figures by the first rule.

A generalization which is based upon an examination of specific instances is said to be arrived at by **induction.** If the instances examined are all the instances that there are, as in the case of our three rules and the corollary, it is said to be **complete,** and the general truth arrived at is said to be based upon a **perfect** induction.

Exercises

1. Is the proof of the binomial theorem in ordinary algebra based upon a complete induction?
2. Does a generalization founded upon a single instance possess any degree of probability? Compare in this connection the following arguments:

   This box contains a dozen buttons; therefore, every box contains a dozen buttons.

   This solar system has eight planets; a fact that may well be true of every solar system.

   I am aware of the existence of my own mind by direct introspection; other men's behavior is apparently purposive and rational and like my own; therefore, other men, too, have minds.

3. Construct the array of immediate inference and place after each invalid mood the number of a rule or corollary that declares it to be invalid.
4. Prove that there is only one invalid mood which illustrates the second rule uniquely.
5. Make a list of examples which fall uniquely under the first and under the third rule.
6. Why is it that there is no unique illustration of the corollary?

SYLLOGISM

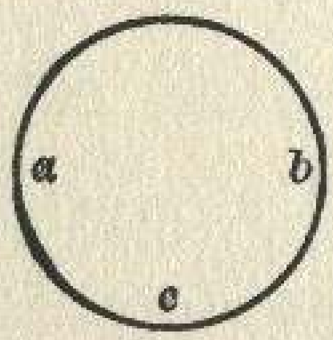
CHAPTER IX

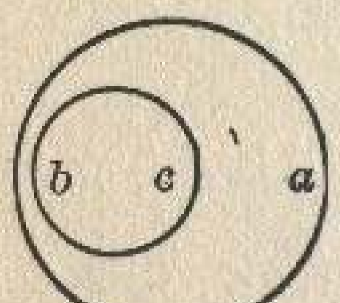
MOODS AND FIGURES OF THE SYLLOGISM

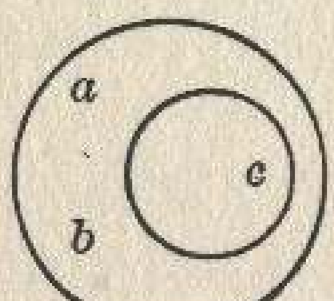
We have now to study an array of a somewhat more general character than the one of immediate inference, and we may begin, not by describing it in the abstract, but by directing attention to a few specific examples. Consider the proposition:

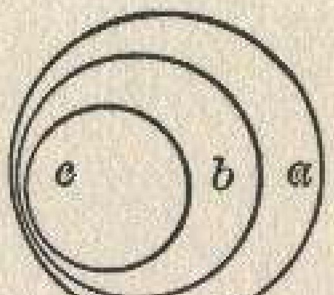
$$\mathbf{A}(ba) \text{ and } \mathbf{A}(cb) \text{ implies } \mathbf{A}(ca),$$

and suppose that it is our desire to represent the antecedent as a whole. The diagrams below will evidently exhaust all the modes of representation that are possible.

either 

or 

or 

or 

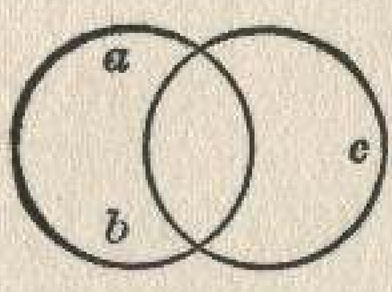
It will be observed that each one of the four ways of representing the antecedent is at the same time a way of representing the consequent. Accordingly, if *a, b* and *c* are related as in the antecedent, then it follows that *a* and *c* are related as in the consequent, so that the original proposition is a valid implication.

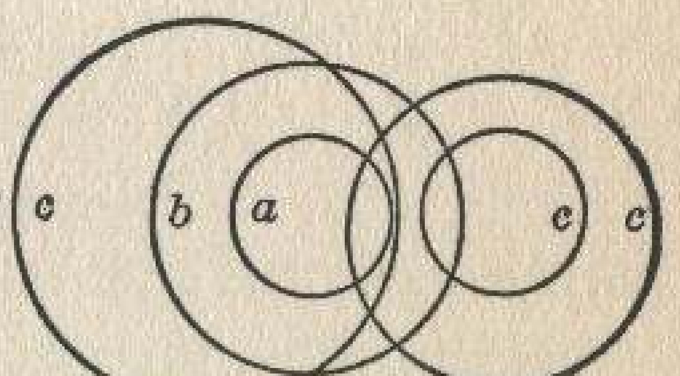
RULE FOR CONSTRUCTING THE DIAGRAMS

The rule for constructing the diagrams which will represent the antecedent as a whole, is this: if the second form in the antecedent has (say) three modes of representation, then represent the first form completely three times (on three separate lines) and add to the first line the first way of representing the second form in the antecedent, to the second line the second way and to the third line the third way. The antecedent will then be completely represented as a whole. For example, consider the implication:

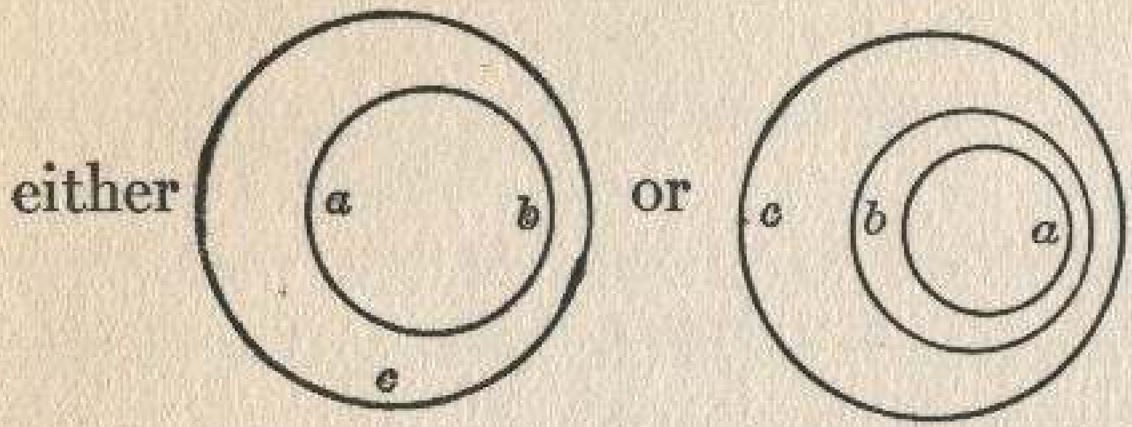
$$\text{A}(ab) \text{ and } \text{O}(cb) \text{ implies } \text{O}(ca).$$

Since O(*cb*) is represented in three ways (see Chap. VI, Fig. 7), we represent A(*ab*) completely three times. Now supply to the first line the first way of representing O(*cb*), that is,

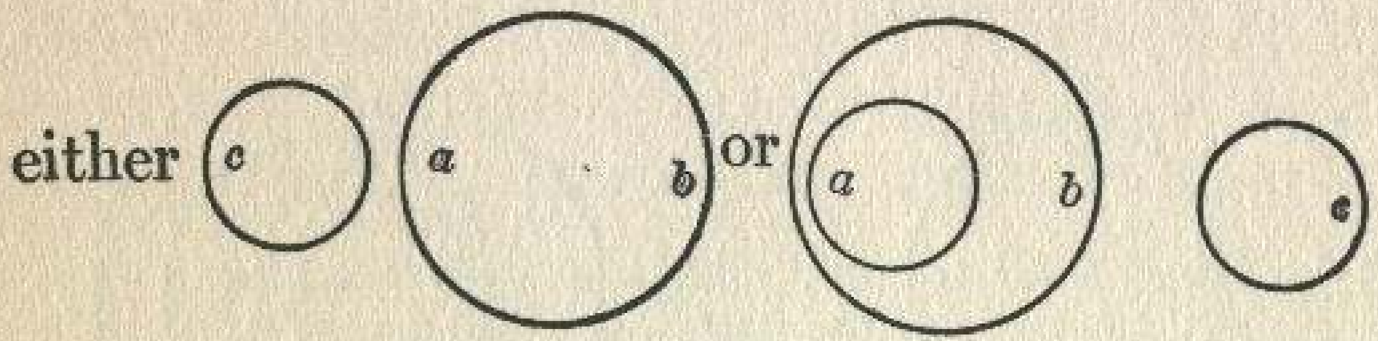
either 

or 

and to the second line the second way of representing O(cb),



and, finally, to the third line the third way of representing O(cb),



It will be perceived at once that each separate manner of denoting *a, b* and *c,* as related in the antecedent, is also a manner of denoting *a* and *c* as related in the consequent. It is intuitively evident, then, as in the last illustration, that the implication is validly drawn. It will be necessary, perhaps, for the student to examine closely the more complicated diagram which appears in the first line, in order to satisfy himself that, together with the others, it exhausts all the possibilities there are.

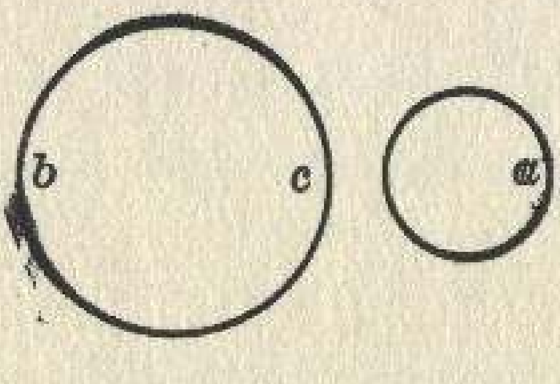
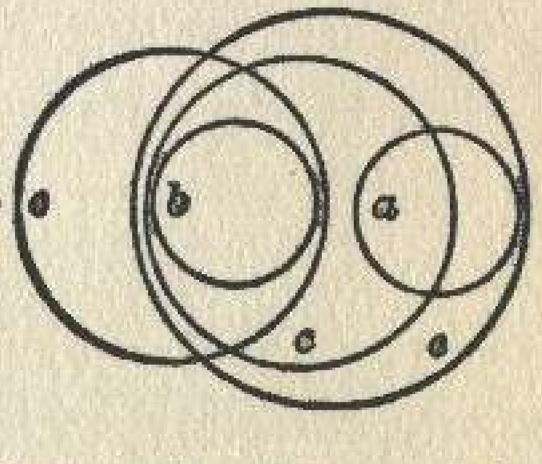
INVALIDITY DETERMINED BY DIAGRAM

Since we have understood *untrue* to mean not necessarily true, it is clear that in order to perceive the invalidity of any proposition of the form under consideration it will be enough to point to a single representation of the antecedent which at the same time is not a representation of the consequent. When such a case is at hand, we are at once made aware of the implication's untruth.

Let us consider, then, a further case:

E(*ab*) and A(*bc*) implies E(*ca*).

The complete expression of the antecedent is:

either  or 

But in the last diagram we have two separate instances of the untruth of E(*ca*). Consequently, the implication is invalid.

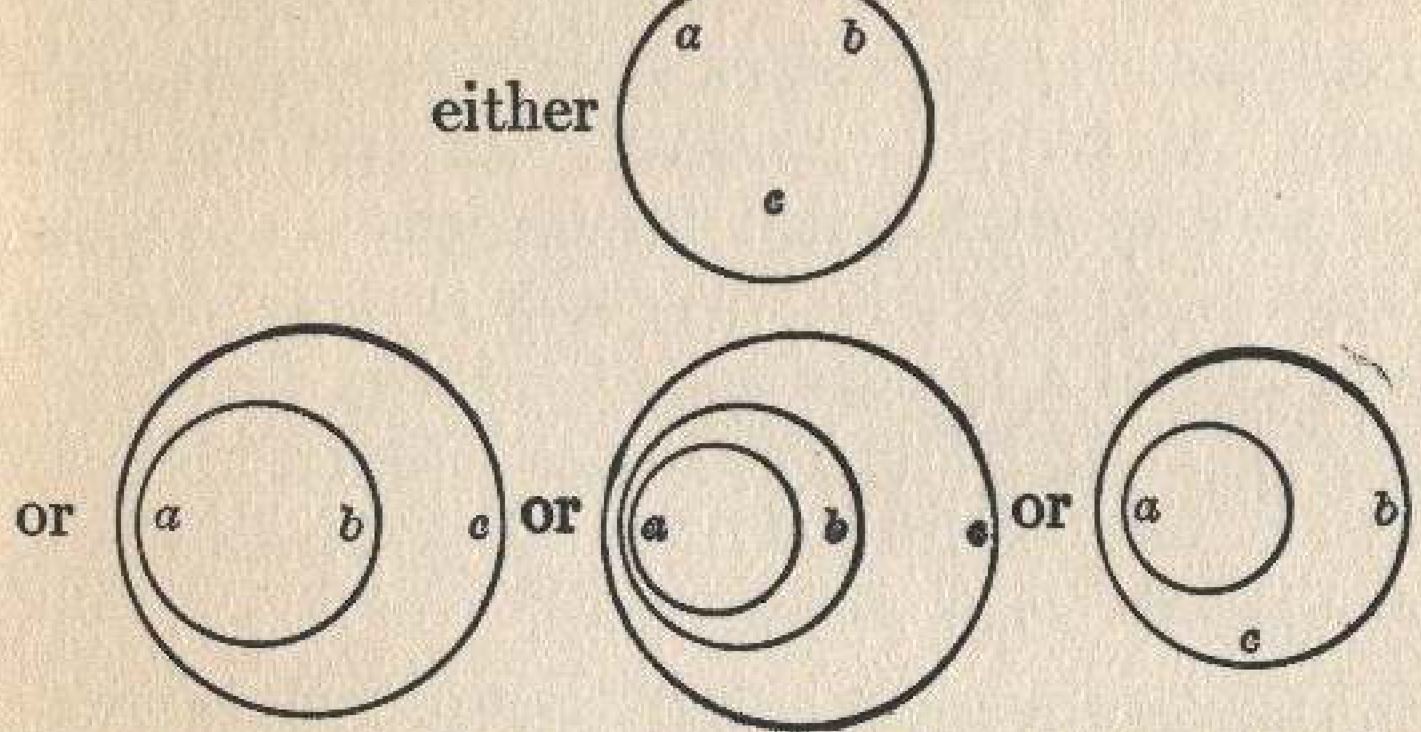
The first example which we examined above was,

A(*ba*) and A(*cb*) implies A(*ca*).

Consider now the following variation:

A(*ab*) and A(*bc*) implies A(*ca*),

and observe that the two cases differ in regard to their term-order. The complete representation of the antecedent is:



and three distinct instances will be observed in these diagrams of the untruth of the consequent, so that the implication is invalid. It is to be remarked, then, that the validity of an implication of the type under consideration depends not only upon the particular categorical forms which enter into it, but also upon the particular manner in which the terms are arranged.

DETERMINATION OF FIGURE

We shall now determine all the possible ways of arranging the terms. These will evidently be not more than eight in number, *viz.,*

ba	*ab*	*ba*	*ab*
cb	*cb*	*bc*	*bc*
ca	*ca*	*ca*	*ca*

ba	*ab*	*ba*	*ab*
cb	*cb*	*bc*	*bc*
ac	*ac*	*ac*	*ac*

It is clear that the two forms conjoined in the antecedent may be written in either order that we choose. In technical language this fact would be expressed by saying that the conjunctive relation of logic is commutative. We may then, if we wish, always write a specific one of the two first. *We agree as a matter of convention, always to write first the form which contains the predicate of the consequent.* Thus, we write

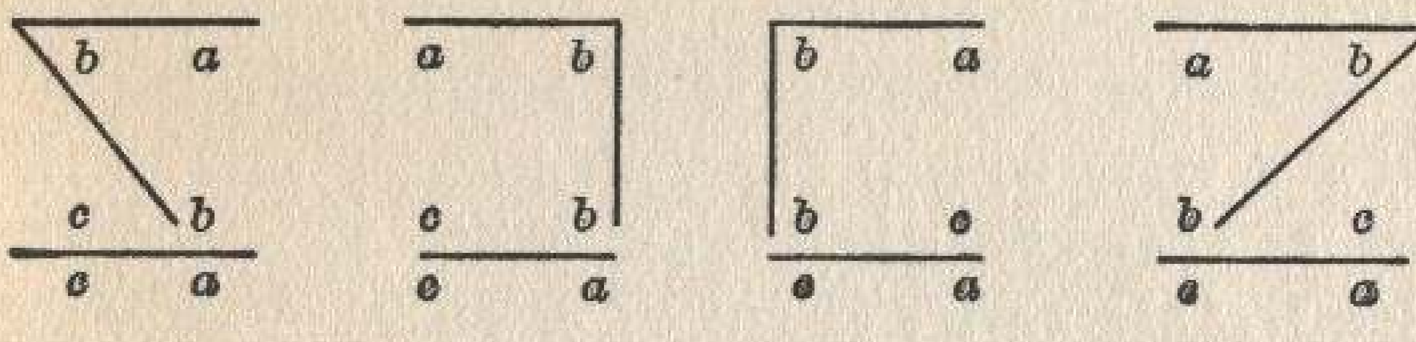
A(*ba*) and A(*cb*) implies A(*ca*),
rather than A(*cb*) and A(*ba*) implies A(*ca*).

To accord with this convention, the second set above will have to be rearranged, thus:

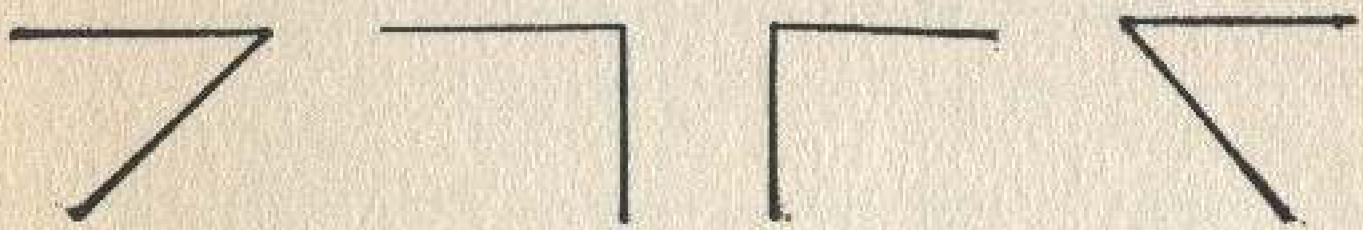
cb	*cb*	*bc*	*bc*
ba	*ab*	*ba*	*ab*
ac	*ac*	*ac*	*ac*

We shall now show that the term-arrangements in this set are only a repetition of those in the first set above, but in a different order, so that it will turn out that there are only four distinct ways of arranging the terms.

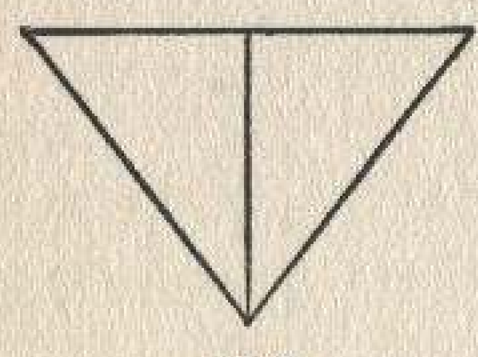
Suppose that we were to draw two lines, one connecting the terms in the categorical form written first in the antecedent and another connecting the term which does not appear in the consequent. Then the first four varieties of term-order will appear thus:



and the second set will present the same varieties of figure, but with the first and last case reversed. The figures



will in each instance give a very clear geometrical image of the number of possible term-orders. The student will do well to commit to memory at once the four variations of the first set, which we shall constantly refer to as figures 1, 2, 3, and 4, respectively. The four figures are easily remembered as combined in an isosceles triangle standing on its vertex (see below).

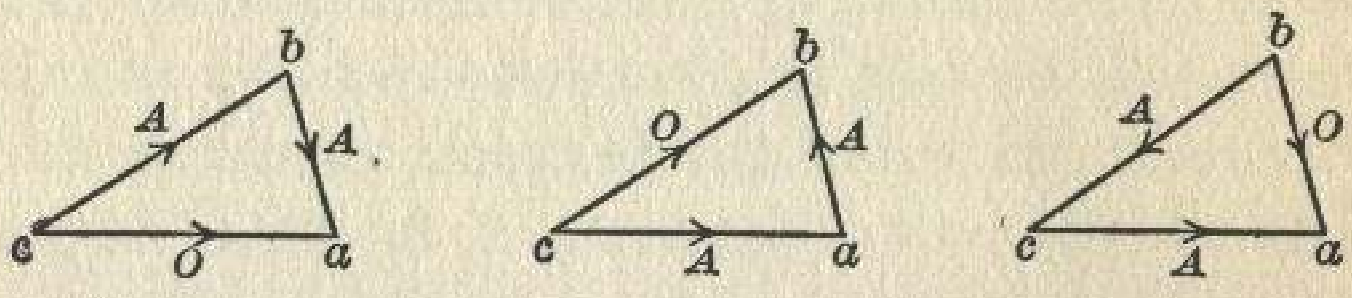


OTHER DIAGRAMMATIC METHODS OF DETERMINING FIGURE

While this method of determining the term-order will prove quite sufficient for all purposes, it is by no means the only device that might be constructed. Form a triangle with the term *a* at the end of the base to the right, the term *c* at the left, and the term *b* at the vertex above. Let an arrow indicate the direction of "flow" from subject to predicate, or, the order subject-predicate. Then if we choose

A(*ba*) and A(*cb*) implies O(*ca*),
A(*ab*) and O(*cb*) implies A(*ca*),
O(*ba*) and A(*bc*) implies A(*ca*),

these three propositions, whose invalidity the student may confirm for himself, would be represented by the diagrams given below:



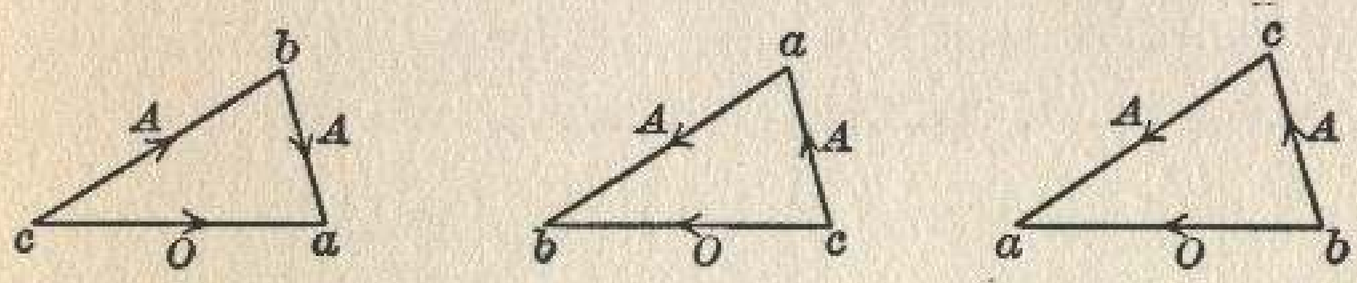
In the sequel the student will have to accustom himself to cases which do not at first appear to belong to any one of the conventional figures. Consider the three term-orders:

ba	*ab*	*ca*
cb	*ca*	*bc*
ca	*cb*	*ba*

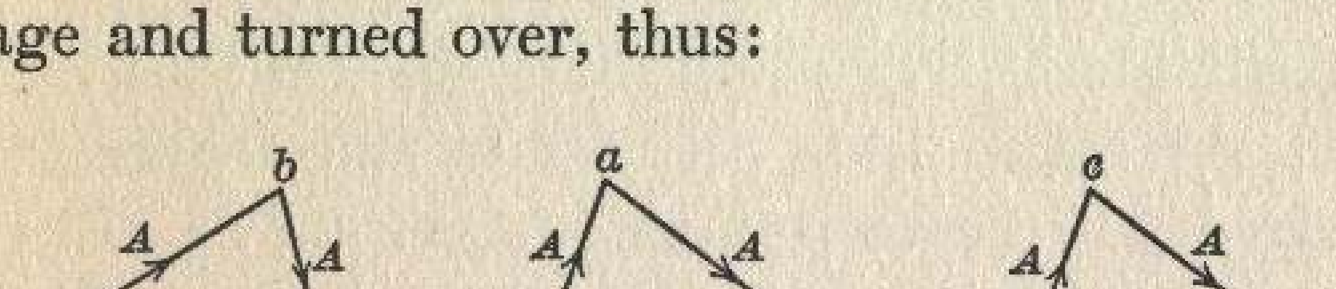
By following the original directions for determining the figure, it will be easy to recognize these as all variations of the first case. If we were to write these out, say in the form:

A(*ba*) and A(*cb*) implies O(*ca*),
A(*ab*) and A(*ca*) implies O(*cb*),
A(*ca*) and A(*bc*) implies O(*ba*),

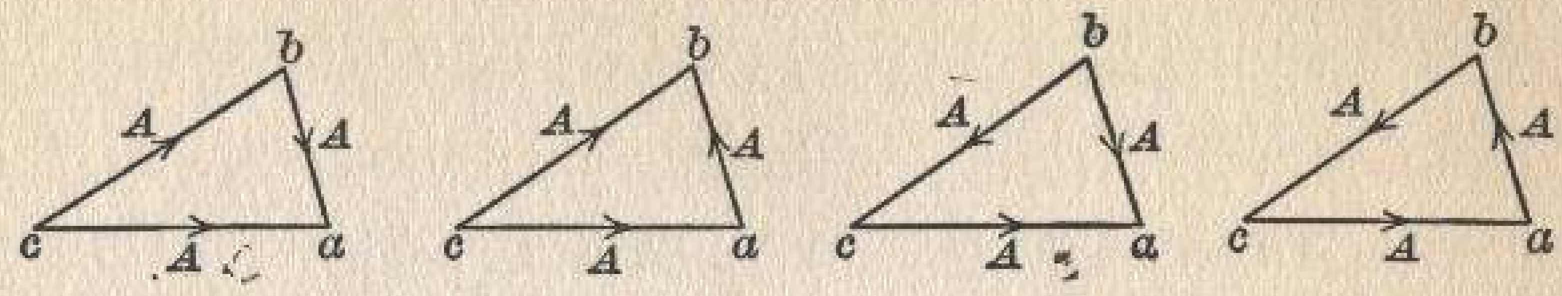
these three equivalent statements would be represented by means of our triangles as follows:



It will be noticed that in each instance the direction of "flow" as indicated by the arrows is continuous and in one direction from the subject of O to the predicate of O. The formal identity of the three cases will appear more clearly if the second and third figures be taken out of the plane of the page and turned over, thus:



Suppose, finally, that we should wish to represent by means of the triangles a single combination of letters (AAA say) in each one of the four figures. Our diagrams would then appear as follows:



PARTS OF THE SYLLOGISM

We proceed now to summarize these results and to define a certain number of technical terms. If we imagine that the A's below may be replaced in any way by any one of the other letters, E, I, O, then the **syllogism** is a form of implication belonging to one of the following types:

1. A(*ba*) and A(*cb*) implies A(*ca*),
2. A(*ab*) and A(*cb*) implies A(*ca*),
3. A(*ba*) and A(*bc*) implies A(*ca*),
4. A(*ab*) and A(*bc*) implies A(*ca*).

These differences are known as the first, second, third, and fourth **figures** of the syllogism, respectively. The two forms conjoined in the antecedent are called the **premises** and the consequent is called the **conclusion.** The predicate of the conclusion is called the **major term** and points out the **major premise,** which by convention is written first in the antecedent. The subject of the conclusion is called the **minor term** and points out the **minor premise,** which by convention is written second in the antecedent. The term which is common to the premises and which does not appear in the conclusion is called the **middle term.**

ARRAY OF THE SYLLOGISM

There will evidently be sixty-four syllogistic variations, obtained by taking the permutations of the four letters, A, E, I, O, three at a time. Each one of these may be expressed in each one of the four figures, so that we shall have two hundred and fifty-six cases in all to consider. These are known as the **moods** of the array of the syllogism. True propositions of the array are known as **valid moods** of the array. The remainder are known as **invalid moods** of the array.

In representing the array of the syllogism it will prove convenient, as in the case of immediate inference, to omit the word *and* and the word *implies,* as well as the parts (*a, b*), (*b, c*), (*c, a*) and to exhibit each mood as a simple combination of the three letters. In constructing the array, the best method to employ will be to add to each one of the sixteen permutations of the four letters, A, E, I, O, taken two at a time, each one of the four letters in succession. The array under each figure will then appear thus:

AAA	EAA	IAA	OAA
E	E	E	E
I	I	I	I
O	O	O	O

AEA	EEA	IEA	OEA
E	E	E	E
I	I	I	I
O	O	O	O
AIA	EIA	IIA	OIA
E	E	E	E
I	I	I	I
O	O	O	O
AOA	EOA	IOA	OOA
E	E	E	E
I	I	I	I
O	O	O	O

The student should now construct the array for himself and examine a great number of the moods in each one of the four figures, in order to determine the validity or invalidity of each one, by the method of inspection explained at the beginning of this chapter. We remark that six true propositions will be found under each figure, but in different positions in the array.

We shall assume now that the student has made a list of the valid moods of the syllogism, having applied the method of inspection to the two hundred and fifty-six possible cases. In order that he may verify his results, the six that are valid under each figure are set down below:

1.	2.	3.	4.
A A A	A E E	A A I	A A I
A A I	A E O	A I I	A E E
A I I	A O O	E A O	A E O
E A E	E A E	E I O	E A O
E A O	E A O	I A I	E I O
E I O	E I O	O A O	I A I

THE MNEMONIC LINES

The valid moods of the Aristotelian syllogism are conveniently remembered by means of the following mnemonic lines, the vowels in each separate word standing for the mood in question,

*Barbara, Celarent, Darii, Ferio*que prioris;
Cesare, Camestres, Festino, Baroko, secundæ;
Tertia, *Darapti, Disamis, Datisi, Felapton,*
Bokardo, Ferison, habet; quarta insuper addit
Bramantip, Camenes, Dimaris, Fesapo,
Fresison.

This mnemonic early appears in the *Summulæ Logicales* of Petrus Hispanus, who was afterward Pope John XXI, but without the line which records the fourth figure. He does not, however, profess to be the author of it. Several other versions are found in later writers. A Greek mnemonic of the same kind is inserted in early editions of the *Organon* of Aristotle. The moods not listed are gotten by weakening the universal conclusions where they occur, to particular conclusions. Jevons remarks: "This device, however ingenious, is of a barbarous and wholly unscientific character; but a knowledge of its construction and use is still expected from the student of logic, and the verses are therefore given and explained." What Jevons overlooks is the fact, to be explained immediately, that this is really a first crude attempt at a deduction of the moods and, consequently, a matter of historical interest.

Besides the vowels in each word which stand for the mood in question, the initial consonants, as well as *m, p, s,* and *k,* have each a special significance. Thus, *s* means: convert simply in the preceding premise or conclusion. The letter *p* requires a conversion *per accidens* and *m* indicates an interchange of premises. By performing the indicated operations a mood in the second, third, or fourth figure will be reduced to the first, the initial consonant indicating to which mood of the first figure the reduction will be effected. Two illustrations will suffice to indicate the method.

(1) The mood *Cesare* is E A E in the second figure, or

$$E(ab) \text{ and } A(cb) \text{ implies } E(ca).$$

Here only one change is indicated, that of simple conversion in the major, the letter *s* following the first vowel. There results, evidently,

$$\text{E}(ba) \text{ and } \text{A}(cb) \text{ implies } \text{E}(ca),$$

and this is the mood *Celarent* recorded in the first line.

(2) The mood *Bramantip* is A A I in the fourth figure, or

$$\text{A}(ab) \text{ and } \text{A}(bc) \text{ implies } \text{I}(ca).$$

Converting the conclusion *per accidens,* which is required by the letter *p,* we get A(ac), and transposing the premises, as indicated by the letter *m,* the mood becomes

$$\text{A}(bc) \text{ and } \text{A}(ab) \text{ implies } \text{A}(ac).$$

The student will have no difficulty, if he resorts to the method of triangular representation already explained, in recognizing this as A A A in the first figure, the mood recorded as *Barbara* in the first line of the mnemonic verses.

REDUCTIO AD IMPOSSIBILE

The moods *Baroko* and *Bokardo* require a special treatment known as indirect reduction, or *reductio ad impossibile,* and indicated by the letter *k.* In the old logic the premises were taken to be true by presumption. Consider, now, the mood *Baroko* and suppose the conclusion does not follow from the premises. If O(*ca*) is false, then A(*ca*) is true. Combine this latter form with the major premise and we have

$$\text{A}(ab) \text{ and } \text{A}(ca) \text{ implies } \text{A}(cb),$$

a mood in *Barbara* whose conclusion contradicts the minor of *Baroko*. Our supposition, then, that the conclusion of *Baroko* does not follow, turns out to be impossible, and the proof depends upon the validity of *Barbara*.

INDIRECT REDUCTION BY OTHER PROCESSES

Indirect reduction may also be effected by other processes, either by attaching the negative particle to the predicate term, or else by the process of conversion by contraposition (see the exercises at the end of Chapter VII and the account of negative terms in Chapter II). Thus, converting the major of *Baroko* by contraposition and attaching the negative particle in minor and conclusion to the predicate term, we have

All *non-b* is *non-a*,
Some *c* is *non-b*,
Some *c* is *non-a*.

If *non-a* and *non-b* be represented by *a′* and *b′*, respectively, this becomes

$$A(b' a') \text{ and } I(cb') \text{ implies } I(ca')$$

which will be readily recognized as A I I in the first figure, a mood recorded in the first line of the mnemonic verses as *Darii*.

A similar process may be applied to *Bokardo* as follows: Attach the negative particle to the predicate terms and we should have:

$$I(ba') \text{ and } A(bc) \text{ implies } I(ca').$$

This is the mood *Disamis*, which reduces to *Darii* in the usual way by following out the operations indicated by the letters.

Other moods besides *Baroko* and *Bokardo* are susceptible to this method of indirect reduction. Thus, *Camenes*,

$$A(ab) \text{ and } E(bc) \text{ implies } E(ca),$$

reduces to *Barbara*, if we employ the two operations of conversion by contraposition and by negation—that is,

$$A(b' a') \text{ and } A(cb') \text{ implies } A(ca'),$$

and the mood, *Ferison*—that is,

$$E(ba) \text{ and } I(bc) \text{ implies } O(ca),$$

reduces to *Darii*, when we have converted simply in the minor premise—that is,

$$A(ba') \text{ and } I(cb) \text{ implies } I(ca').$$

EXERCISES

1. Would any of the geometrical images employed in the determination of figure be changed for *Alice Through the Looking Glass?*
2. Reduce the moods of the other figures to those of the first, following the operations indicated in the mnemonic verses.
3. By indirect reduction change *Camestres* to *Barbara, Felapton* to *Darapti,* and *Fresison* to *Datisi.*
4. Reduce *Bokardo* by the method of *reductio ad impossibile.*
5. Remembering that contraries cannot both be true, reduce *Camestres* by the method of the last example.
6. If (*a b c*) is taken in cyclical order—that is, read *ab, bc, ca*—will any other figures result on permuting the three letters?
7. If *ab, bc, ca* stands for the fourth figure, what will be the effect of permuting these pairs in every possible way, regard being had to the proper order of the premises.

CHAPTER X

DEDUCTION OF THE SYLLOGISTIC MOODS

We shall now, as in the case of immediate inference, by postulating the truth of the smallest possible number of the valid moods of the syllogism. deduce the remainder by the aid of two principles. The assumptions which we shall have to make, are as follows:

Postulate 1.—A (ba) and A (cb) implies A (ca),
Postulate 2.—E (ba) and A (cb) implies E (ca),
Principle I.—If in any valid mood either premise and the conclusion be interchanged and each be replaced by its contradictory, another valid mood will result.
Principle II.—If in any valid mood a premise be strengthened or the conclusion be weakened, another valid mood will result.
Theorems.—The remaining (22) valid moods.

EXAMPLES OF DEDUCTION

When the student has carefully studied the examples which are set down below, he should be able to carry out the entire deduction without further aid and the work of doing this should have been completed before reading the remainder of the chapter.

(1) Suppose that we were to combine the first postulate and the first principle. Interchanging minor premise and conclusion and replacing each by its contradictory we obtain

A(*ba*) and O(*ca*) implies O(*cb*).

Here the major term is *b*, so that the premises are in the normal order, the minor term is *c* and the middle term has become *a*. The figure is now determined in one of the ways already described, *viz.*,

b a
c a
c b

so that if our first theorem, AOO in the second figure, be written according to the original convention, our result becomes

Theorem 1.—A(*ab*) and O(*cb*) implies O(*ca*).

Similarly, by interchanging the major premise and the conclusion and replacing each by its contradictory, we should have obtained OAO in the third figure and this mood, if we employ the original convention of the third figure, becomes

Theorem 2.—O(*ba*) and A(*bc*) implies O(*ca*)

(2) The mood AOO in the second figure being now established as valid, we may apply to it either one of the principles in the same sense as to the

postulates. Let us begin by writing the mood with the terms ordered as in the original convention and, applying Principle II, let us strengthen the minor premise O(cb) to E(bc). This will be possible by applying a result of immediate inference already established, *viz.,*

$$\text{E}(bc) \text{ implies } \text{O}(cb).$$

Accordingly, our next result becomes

Theorem 3.—A(ab) and E(bc) implies O(ca), or, AEO in the fourth figure is a valid mood.

(3) Suppose that we were to return now to the first principle and apply it to the result which has just been obtained. Contradicting major and conclusion and interchanging, we obtain immediately

$$\text{A}(ca) \text{ and } \text{E}(bc) \text{ implies } \text{O}(ab).$$

It is important that the student should not fail to observe that the premises are no longer in the normal order and that the normal order must be restored before the figure can be ascertained. Failure to make this change might result, as he will readily see, not only in a mistake in the figure but also in the mood. Our result is, accordingly, EAO in the fourth figure, or, if the terms be ordered as in the original convention,

Theorem 4.—E(ab) and A(bc) implies O(ca).

Had we chosen to contradict and interchange minor and conclusion of AEO in the fourth figure, we should have obtained in the same way

Theorem 5.—A(*ab*) and A(*bc*) implies I(*ca*). Or A A I in the fourth figure is a valid mood.

It will now be observed that the application of Principle I to any mood in the fourth figure places the premises out of the normal order, but leaves the figure unchanged. Employing a more technical language, we should say that the fourth figure is invariant under Principle I.

Having deduced the twenty-two theorems, the student should set himself the exercise of deriving the valid moods under each figure separately, and he should try to arrive at each result by the fewest possible number of steps. In deducing those under the fourth figure, it will economize steps and so add to the elegance of his demonstration, if he will keep in mind the rule stated in the last paragraph. The following rules, which the student will do well to verify for himself, show the effect on mood and figure of contradicting and interchanging either premise and the conclusion.

Rules of Contradiction and Interchange

Contradicting and interchanging major and conclusion, we should have:

(1) The first figure changes to the third and conversely, and the premises remain in normal order;

(2) The second figure changes to the third with the normal order of the premises reversed;

(3) The fourth figure remains invariant with the normal order of the premises reversed.

Contradicting and interchanging minor and conclusion we should have:

(1) The first figure changes to the second and conversely, and the premises remain in normal order;

(2) The third figure changes to the second, with the normal order of the premises reversed;

(3) The fourth figure remains invariant, with the normal order of the premises reversed.

It will also be found advantageous to state in the form of rules the effect of simple conversion in either premise or in the conclusion.

Rules of Simple Conversion

(1) Simple conversion in the major premise changes the first figure to the second and conversely, the third figure to the fourth and conversely.

(2) Simple conversion in the minor premise changes the first figure to the third and conversely, the second figure to the fourth and conversely.

(3) Simple conversion in the conclusion changes the first figure to the fourth and conversely and the second and third figures remain invariant.

DEDUCTION OF THE INVALID MOODS

It remains in order to complete the solution of the syllogism, to deduce all of the two hundred and thirty-two invalid variants from the fewest possible number of initial assumptions. The most elegant way to proceed will be to begin with a single postulate and a single principle and to introduce further assumptions only when we are compelled to do so. We state, accordingly,

Postulate 3.—E (*ba*) and E (*cb*) does not imply I (*ca*),

Principle III.—If in any invalid mood a premise be weakened or the conclusion be strengthened, another invalid mood will result.

Let us begin by weakening the major to E (*ab*), since E (*ba*) implies E (*ab*); and secondly, by weakening the minor to E (*bc*). Finally, let us weaken the premises to E (*ab*) and E (*bc*) respectively. We shall then have established by postulate and theorem the invalidity of EEI in all four figures. If, now, the E- premises be weakened to O-premises and the I- conclusion be strengthened to an A- conclusion in every possible way, the untruth of

E E I	E O I	O E I	O O I
E E A	E O A	O E A	O O A

will have been established in each one of the four figures. The invalidity of thirty-one moods has, accordingly, been made to depend upon the invalidity of E E I in the first figure alone. It should be noted in this connection that the application of Principle IV (below) to any mood in this set of thirty-two will yield no mood that is not already contained in the set and that Postulate 4 (below) will yield no mood of the set by either principle. We now introduce the second postulate and the second principle.

Postulate 4.—A(*ab*) and A(*cb*) does not imply I(*ca*).

Principle IV.—If in any invalid mood either premise and the conclusion be interchanged and each be replaced by its contradictory, another invalid mood will result.

The application of this principle will offer no difficulty that has not been already overcome, and no doubt the practice which the student has had in the derivation of the valid moods, will enable him to dispense with further illustrations here. Thus we should obtain at once the theorems:

a. A A A (second figure) by 4, iii,
b. A E O (first and third figs.) by 4, iv,
c. A E E (first and third figs.) by b, iii,
d. A I I (second and fourth figs.) by c, iv,
e. I A I (first and second figs.) by d, iii,
f. E A E (third and fourth figs.) by e, iv.

Other moods which follow from this postulate and whose invalidity may be easily established in all four figures are:

EIE IEE IEO III IIA

and of this set of theorems it can be said that each one is independent of the original set of thirty-two.

In order to deduce the invalid moods that remain, it will be necessary to assume five other postulates, of whose independence the student will be able to satisfy himself by considerations similar to those set forth above. These are:

A A O (first fig.)
A A O (fourth fig.)
O A O (first fig.)
A A A (fourth fig.)
E E O (first fig.)

THE RULES OF SYLLOGISM

It was a part of the traditional treatment of the syllogism to formulate certain rules for the immediate detection of the invalid moods. We shall state these and we shall then prove that they are necessary and sufficient for the purpose which they effect.

Rule 1.—Two negative premises do not imply a conclusion. E E I₁ unique rules & examples.

Rule 2.—Two affirmative premises do not imply a negative conclusion. A A O₄

Rule 3.—An affirmative premise and a negative premise do not imply an affirmative conclusion.

Rule 4.—Two premises in neither of which the middle term is distributed do not imply a conclusion.

Rule 5.—Two premises in which a given term occurs undistributed, do not imply a conclusion in which that same term occurs distributed.

These rules are *sufficient*, for they declare invalid all moods already recognized as invalid. They are all *necessary*, for we can point to at least one example that falls uniquely under each rule. In seeking unique illustrations of each rule, the student will do well to write out in full the mood to be examined and to underline the distributed terms.

Exercises

1. Beginning each time with *Barbara* and *Celarent*, deduce the valid moods of the second, third, and fourth figures.
2. From A A A (fourth fig.) deduce twenty-six other invalid moods of the syllogism.
3. Construct the array of the syllogism and place after each invalid mood the number of a rule that declares it to be invalid.
4. Make a list of examples that fall uniquely under each one of the five rules.
5. Prove that there can be only one invalid mood that will illustrate the second rule uniquely.
6. Show that, as the result of a complete induction of all of the moods in question, it follows from the rules as a corollary that two particular premises do not imply a conclusion.
7. Show similarly, that it follows from the rules that a universal premise and a particular premise do not imply a universal conclusion.

8. Prove that all moods of the first figure are invalid, wherein the minor is not affirmative or the major is not universal.
9. Prove that all moods of the second figure are invalid, wherein the major is not universal or both premises affirmative.
10. Prove that all moods of the third figure are invalid, wherein the minor is not affirmative or the conclusion is not particular.
11. Establish the following rules of the fourth figure:

    If the major is affirmative, the minor must be universal.
    If the minor is affirmative, the conclusion must be particular.
    Neither premise can be a particular negative nor can the conclusion be a universal affirmative.
    If one premise be negative, the major must be universal.

CHAPTER XI

THE HYPOTHETICAL SYLLOGISM

The forms of implication to which I shall now invite the attention of the reader are generally known as **conditional arguments.** They possess a peculiar attraction for minds whose interest is not readily aroused, save by the applications of a science, or by the enumeration of specific cases, for they conform to certain modes of rhetorical expression which seem natural, because they have become habitual. We cannot do better than begin with the citation of a few examples. The first we shall borrow from the logical compendium of Archbishop Whately. Suppose some one were to argue for the reality of miracles in the following way: "If no miracles had been displayed by the first preachers of the Gospel, they could not have obtained a hearing; but they did obtain a hearing; therefore, some miracles must have been displayed by them." He would, then, employ a form of argument known as the **hypothetical syllogism.** Or, again, suppose a certain theologian of opposite faith were to say: "If the doctrines of Calvin conform with the word of God, it is not necessary to publish them separately, and if they are at variance with the word of

God, they are wicked; but either they conform or they do not; accordingly, either these doctrines are wicked or their publication is unnecessary." An argument of such a form is known as a **dilemma.**

It would be well to examine these cases with especial care, regarding the form and the content of each and endeavoring to settle, without any of the apparatus of logic, the question as to whether the argument is good or bad. After the reading of this chapter it may be profitable to return to consider them again in the light of what will have been learned. The difference between our first judgments, together with the grounds on which we base them, and our later more sophisticated ones, which will then be founded on developed theory, will yield a rough measure of the practical value of this undertaking. The word *dilemma* is from the Greek δι- *two* and λῆμμα *assumption,* and the two parts of the disjunction appearing after the first semicolon are known as the *horns* of the dilemma. Some of the dilemmatic arguments which will be set down below are historically famous, and deservedly so, for it will not always be easy to put our finger on the "screw that is loose in our logical conundrum."

DILEMMA OF THE CROCODILE

The following account of the well-known dilemma denominated "The Crocodile" we shall quote in full from one of the essays of De Quincey:

"I recall at this moment a little metrical tale of Southey's, in which the *dramatis personæ* are pretty nearly the same, *viz.,* a crocodile, a woman and her son. In that case, however, the crocodile is introduced as a person of pattern morality, for the woman says of him—

" 'The king of the crocodiles never does wrong:
He has no tail so stiff and strong
Petitioners to sweep away,
But he has ears to hear what I say.'

Not so the crocodile known to the Greek dialecticians. *He* bore a very different character. If he has no tail to interfere with Magna Charta and the imprescriptible right of petitioning, he had, however, teeth of the most horrid description for crushing petition and petitioner into one indistinguishable pulp; and, in the particular case contemplated by the logicians, having made prisoner of a poor woman's son, he was by her charged with the same purpose in regard to her beloved cub as the Cyclops in the 'Odyssey' avows in regard to Ulysses, *viz.,* that he reserved him in his larder for an *extra bonne bouche* on a gala day. The crocodile, who, generally speaking, is the most uncandid of reptiles, would not altogether deny the soft impeachment, but, in order to sport an air of liberality which was far from his heart, he protested that, no matter for any private views which he might have dallied with in respect to the young gentleman, he would abandon them all on one condition (but, observe, a condition which he privately held to be impossible for a woman to fulfil), *viz.,* that she should utter some proposition which was incontrovertibly true. The woman mused upon this; for though she knew of propositions that no neutral party could dispute—as this, for instance, that crocodiles are the most odious of vermin—it was evident that her antagonist would repel *that* as an illiberal and one-sided personality. After some consideration, therefore, she replied thus—'You will eat my son.' There and then arose in the crocodile's brain a furious self-conflict, from which it is contended that no amount of Athenian chicanery would ever deliver him; since, if he *did* eat her son, then the woman had uttered a plain truth, which the crocodile himself could not have the face to deny, in which case (the case of speaking truth), he had pledged his royal word *not* to eat him; and thus he had acted in a way to make the word of a crocodile, or his bond, or even the tears of a crocodile, a mere jest among philosophers. On the other hand, if in contemplation of these horrid consequences he did *not* eat her son, then the woman had uttered a falsehood in asserting that he would, and it became a royal duty in *him,* as a guardian of morality, to exact the penalty of her wickedness. . . . Truth absolute was provided for; in that case the son was to be spared.

Absolute falsehood was also provided for; in that case the son was to die. But truth conditional was *not* provided for. Supposing the woman to say something contingent on a case that might or might not be realized, then it became necessary to wait for the event. But here there was no use in waiting, since, whichever of the two possible events should occur, either equally and irretrievably landed the crocodile in a violation of his royal promise."

CONSTRUCTIVE HYPOTHETICAL SYLLOGISM

The **constructive hypothetical syllogism** is of the following general form:

> If x is true, then y is true;
> but x is true;
> therefore y is true.

This argument is called *modus ponens,* or the mood which affirms. Here the minor asserts the antecedent of the major; or, otherwise, what is asserted only hypothetically in the major is asserted absolutely or without qualification in the minor. This condition, of course, allows us to *suppress* the antecedent altogether and to assert the consequent by itself. The general rule is: whenever the antecedent is verified, the consequent is verified itself, provided the implication is a valid one. Thus suppose that I say: if the spring is late, the fruit crop will be abundant; but the spring *is* late; therefore, the fruit crop will be abundant. The conclusion may be asserted by itself, if the premises are granted.

But it is important to observe that the conclusion *follows* whether the premises are true or not. Let us suppose the possible cases:

(1) x true and y true;
(2) x true and y false;
(3) x false and y true;
(4) x false and y false.

In the first case the major and minor are true and the conclusion may be asserted without qualification; in the second case the major is false and the minor is true and the conclusion cannot be asserted by itself; in the third case the major is true and the minor is false and the conclusion cannot be asserted by itself; in the fourth case the major is true and the minor is false and, again, the conclusion cannot be asserted by itself. Observe, however, that in the third case the conclusion, while it cannot be asserted because of the argument, because the premises cannot be suppressed, may yet be asserted on *extra-logical* grounds—*i.e.*, because it is true in point of fact; and the same remark applies to the first case.

PARADOX OF TRISTRAM SHANDY

In order to illustrate the "common-sense" tendency to suppress the antecedent when the implication seems to be a valid one, we shall quote an argument of Mr. Bertrand Russell's, an argument known as the paradox of Tristram Shandy: Mr. Russell says: "Tristram Shandy, as we know, employed two years in chronicling the first two days of his life, and lamented that, at this rate, material would accumulate faster than he could deal with it, so that, as years went by, he would be farther and farther from the end of his history. Now I maintain that, if he had lived forever, and had not wearied of his task, then, even if his life had continued as eventfully as it began, no part of his biography would have remained unwritten. For consider: the hundredth day will be described in the hundredth year, the thousandth in the thousandth year, and so on. Whatever day we may choose as so far on that he cannot hope to reach it, that day will be described in the corresponding year. Thus any day that may be mentioned will be written up sooner or later, and therefore no part of the biography will remain permanently unwritten. This paradoxical but perfectly true proposition depends upon the fact that the number of days in all time is no greater than the number of years."

Leaving out of account the question as to whether or not there is a paradox involved in the comparison of the two infinites, let us examine the following hypothetical syllogism: "If Tristram Shandy should live forever, and should not weary of his task, he will complete his biography; but it is agreed that he shall live forever and shall not weary of his task; accordingly, he will complete his biography." The argument being formally correct, the only question is whether or not the premises may be suppressed and the conclusion asserted by itself. But in the minor it seems that we have agreed that an impossibility shall transpire; not because one cannot live forever for biological reasons, but because one cannot pass through each term of an infinite series, because one cannot come to the end of something that has no end by definition. We have agreed in the minor to regard an impossibility as possible, to regard a false proposition as true. There is no paradox, as soon as we have seen that the premises cannot be suppressed and that, therefore, the conclusion cannot be asserted by itself.

FALLACY OF AFFIRMING THE CONSEQUENT

A familiar fallacy, known as the **fallacy of affirming the consequent,** may be conveniently cited in this connection. Thus, suppose one were to argue:

"If the study of logic furnishes the mind with a multitude of useful facts it will deserve cultivation; now, we agree that it deserves cultivation; accordingly, it must furnish the mind with a multitude of useful facts."

This argument would be formally fallacious. The premises might easily be regarded as true and the conclusion as false under the same circumstances, for the consequent might follow from any one of a great number of antecedents. The study of logic might deserve cultivation as an aid to forensics or to legal studies; or, because its history is connected with the general history of philosophical speculation, or, because its technical vocabulary has passed into common use and cannot be properly understood by the mere help of a dictionary; or, because its foundations are presupposed by those of arithmetic and geometry; or, because constituting one of the few intellectual disciplines of the Middle Ages, a knowledge of it casts a flood of light upon the workings of the mediæval mind; and so on for any number of other reasons that might be enumerated.

A similar fallacy, and one not mentioned, so far as my knowledge extends, in the manuals of logic, consists in the inference that, because the conclusion of a true syllogism may be asserted, at least one of the premises may be asserted to be true. This we shall call the **fallacy of affirming the conclusion.** Consider the following argument:

"If all the Troglodytes are virtuous and this man is a Troglodyte, then this man is virtuous; but this man is virtuous; therefore, this man is a Troglodyte, or else all of the Troglodytes are virtuous."

The Troglodytes of the *Lettres Persanes* of Montesquieu were a mythical people, for the most part so savage and evil that the notions of human justice and equity had ceased to operate practically in their relations with one another. They perished of their own injustice, with the exception of two families, who "were humane and loved virtue." The argument is evidently fallacious, but part of the antecedent of the major, which is a true syllogism, is false.

DESTRUCTIVE HYPOTHETICAL SYLLOGISM

The **destructive hypothetical syllogism** is of the following general form:

> If x is true, then y is true;
> but y is untrue;
> therefore x is untrue.

This argument is called *modus tollens,* or the mood which denies. Thus: "If perfect justice prevailed on earth, then virtue would receive its reward in this life; but virtue is not compensated for here below; therefore, perfect justice does not prevail on earth."

Voltaire, ridiculing the dictum of Leibniz that "all things are for the best in the best of all possible worlds," prefers to believe in a finite rather than a wicked God. An argument commonly given is: "If God were both omnipotent and good he would moderate the grosser evils in the world; but he does not do this; accordingly, either he is an evil being, or else his power is limited."

FALLACY OF DENIAL OF THE ANTECEDENT

The characteristic fallacy to be noticed in connection with this type of argument is the **fallacy of denial of the antecedent.** We quote from Jevons:

" 'If the study of logic furnished the mind with a multitude of useful facts like the study of other sciences, it would deserve cultivation; but it does not furnish the mind with a multitude of useful facts; therefore it does not deserve cultivation.' This is evidently a fallacious argument because the acquiring of a multitude of useful facts is not the only ground on which the study of a science can be recommended. To correct and exercise the powers of judgment and reasoning is the object for which logic deserves to be cultivated, and the existence of such other purpose is ignored in the above fallacious argument."

A valid argument, of a form more general than the destructive hypothetical syllogism, may be described as follows:

> If x and y are true, then z is true;
> but x is true and z is false;
> therefore, y is false.

Thus we might imagine Horatio to address Hamlet: "If my name is Horatio and I speak truth, it was thy father's ghost."

And Hamlet to reply: "Thy name is Horatio, but 'twas no ghost; ergo, thou speakest falsely."

The corresponding fallacy may be termed the **fallacy of denial of a premise.** A bad argument would result, if one were to reason as follows:

"The Germans may win on land, but if the British navy remains intact, they will not be able to dictate the peace. Therefore, if they win both on land and on sea, they will be able to dictate the peace."

COMPLEX HYPOTHETICAL SYLLOGISM

There is a further form of the hypothetical syllogism, to which attention may be profitably directed, but which is not specifically mentioned, I believe, in the elementary compendiums. We shall term it the **complex hypothetical syllogism.** Its general expression is:

> If x and y are true, then z is true;
> but z is untrue;
> therefore, either x is untrue or y is untrue.

The corresponding fallacy, or the one most commonly to be met with in connection with this argument, will be illustrated by a single example. It consists in interchanging minor and conclusion in the form given above.

"If all Shakespearean scholars are Englishmen and Churchill is a Shakespearean scholar, then Churchill is an Englishman; but either not all

Shakespearean scholars are Englishmen, or Churchill is not a Shakespearean scholar; therefore, Churchill is not an Englishman."

Another case would arise, if we were to infer, because the two premises of a true syllogism are false, that the conclusion is false.

Exercises

Select from the following such as are valid arguments:

1. If a thing can be conceived as nonexistent, its essence does not involve existence; but it is not of the essence of a centaur to exist; therefore, a centaur can be conceived as nonexistent.
2. The world cannot have existed always if it had a beginning in time; but it can have existed always; therefore, it had no beginning in time.
3. There would be no such thing as freedom, if all causation were according to natural law; but human freedom is a fact; therefore, not all causation is in accordance with natural law.
4. Human freedom is meaningless if God is omniscient, for he is then aware of what our decision in any case will be, before we decide; but man *is* a free agent; therefore, it cannot be that God is altogether omniscient.

CHAPTER XII

THE DILEMMA

The **dilemma** is an argument very commonly employed whenever it can be shown that any one of a number of possibilities leads to the same result. We shall introduce our theory of this mode of reasoning with a quotation from Archbishop Whately. It was "urged by the opponents of Don Carlos, the pretender to the Spanish throne; which he claimed as male heir, against his niece the queen, by virtue of the Salic law excluding females; which was established (contrary to the ancient Spanish usage) by a former king of Spain, and was repealed by King Ferdinand. They say 'if a king of Spain has a right to alter the law of succession, Carlos has no claim: and if no king of Spain has that right, Carlos has no claim; but a king of Spain either has or has not such right; therefore (on either supposition) Carlos has no claim.' " It will be a good exercise for the student, when he has finished his reading of this chapter, to undertake to retort this argument.

THE PARADOX OF GORGIAS

Another example is to be found in the contention of the Greek Gorgias that nature does not exist. Thus, if the world had a beginning in time, an infinite time must have elapsed before the moment of creation; but an infinite time never can elapse, and hence the moment of creation could never have arrived. Accordingly, the world is *uncreated*. If the world had no beginning in time (*i.e.,* has always existed), an infinite time must have elapsed before the present moment; but this is impossible; we cannot come to the end of something that has no end; therefore, the world must have been *created*.

The fact that time is infinite would have to be established by a separate proof. Thus, if past time were only finite, there must have been a *time* when there was no time (a formal contradiction); and if future time were only finite, there will come a *time* when there is no time (a formal contradiction). Consider the statement: x is both a and not-a, where x is some individual thing or class and a is a property, which it may or may not possess. It is evident that, of classes, the only values of x that will make this a true statement are classes that contain no objects. It will be untrue that triangles are both three-sided and not three-sided, but true that square-triangles are both square and not-square at the same time. Similarly, if x stands for *nothing,* the statement, nothing is both a and not-a, is true. Now, as regards the World, it is forced upon us that it is both *created* and *uncreated;* and since this is true, it is forced upon us that the world is *nothing* (*i.e.,* a nonexistent something).

THE PARADOX OF CORAX AND TISIAS

For the benefit of those who will refuse to consider seriously a philosophical dispute of this sort (and there are many such) we shall furnish an instance related to the experience of daily life. Before there was any science of logic, the Greek sophists (the traveling teachers) taught an art of argumentation, which was often applied practically within the courts of law. We cite the following well-known case from the *Lectures* of Sir William Hamilton: It is known as the *Litigiosus* or *Reciprocus.*

"Of the history of this famous dilemma there are two accounts, the Greek and the Roman. The Roman account is given us by Aulus Gellius, and is there told in relation to an action between Protagoras, the prince of the sophists, and Euathlus, a young man, his disciple. The disciple had covenanted to give his master a large sum to accomplish him as a legal rhetorician; the one half of the sum was paid down, and the other was to be paid on the day when Euathlus should plead and gain his first cause. But when the scholar, after the due course of preparatory instruction, was not in the same hurry to commence pleader as the master to obtain the remainder of his fee, Protagoras brought Euathlus into court and addressed his opponent in the following reasoning: "Learn, most foolish of young men, that however matters may turn up (whether the decision to-day be in your favor or against you), pay me my demand you must. For if the judgment be against you, I shall obtain the fee by decree of the court, and if in your favor, I shall obtain it in terms of the compact, by which it became due on the very day you gained your first cause. You thus must fail, either by judgment or by stipulation." To this Euathlus rejoined: "Most sapient of masters, learn from your own argument that whatever may be the finding of the court, absolved I must be from any claim by you. For if the decision be favorable, I pay nothing by the sentence of the judges, but if unfavorable, I pay nothing in virtue of the compact, because, though pleading, I shall not have gained my cause." The judges, says Gellius, unable to find a *ratio decidendi,* adjourned the case to an indefinite day, and ultimately left it undetermined. I find a parallel story told, among the Greek writers, by Arsenius, by the Scoliast of Hermogenes, and by Suidas, of the rhetorician Corax (*anglicè* Crow) and his scholar Tisias. In this case, the judges got off by delivering a joke against both parties, instead of a decision in favor of either. We have here, they said, the plaguy egg of a plaguy crow, and from this circumstance is said to have originated the Greek proverb, *κακοῦ κόρακος κακὸν ᾠὸν.*"

COMPLEX CONSTRUCTIVE DILEMMA

The **complex constructive dilemma** has the form: If x is true, then y is true, and if z is true, then w is true; but either x is true or z is true; therefore either y is true or w is true.

Our illustrations are taken from Whately. Thus, "If the obedience due from Subjects to Rulers extends to religious worship, the ancient Christians are to be censured for refusing to worship the heathen idols; if the obedience, etc., does not so extend, no man ought to suffer civil penalties on account of his religion; but the obedience, etc., either does so extend, or it does not; hence, either the ancient Christians are to be censured, etc., or else no man ought to suffer civil penalties on account of his religion.

Or, again: "If man is capable of rising, unassisted, from a savage to a civilized state, some instances may be produced of a race of savages having thus civilized themselves; and if man is not capable of this, then the first rudiments of civilization must have originally come from a superhuman instructor; but either man is thus capable, or not; therefore, either some such instance can be produced, or the first rudiments, etc."

COMPLEX DESTRUCTIVE DILEMMA

The **complex destructive dilemma** has the form: If x is true, then y is true, and if z is true, then w

is true; but either *y* is untrue or *w* is untrue; therefore, either *x* is untrue or *z* is untrue.

Thus: "If this man were wise, he would not speak irreverently of Scripture in jest; and if he were good, he would not do so in earnest; but he does it either in jest or in earnest; therefore, he is either not wise, or not good."

The commonest fallacy to be noticed in connection with the dilemma is the case in which the minor is not a true disjunction—*i.e.*, the case in which the minor does not exhaust all of the possibilities. The conclusion can only be asserted when the premises are true, unless, to be sure, it is true in independence of the argument. We quote from Jevons:

"Dilemmatic arguments are, however, more often fallacious than not, because it is seldom possible to find instances where two alternatives exhaust all the possible cases, unless one of them be the simple negative of the other in accordance with the law of excluded middle. Thus if we were to argue that if a pupil is fond of learning he needs no stimulus, and that if he dislikes learning no stimulus will be of any avail, but as he is either fond of learning or dislikes it, a stimulus is either needless or of no avail, we evidently assume improperly the disjunctive minor premise. Fondness and dislike are not the only two possible alternatives, for there may be some who are neither fond of learning nor dislike it, and to these a stimulus in the shape of rewards may be desirable."

A dilemma is said to be retorted, whenever an equally cogent dilemma to the contrary effect is produced. The retort of Euathlus to Protagoras is a case in point, and the argument of Gorgias contains two cogent dilemmas in the opposite sense, both of which the arguer accepts as valid. An Athenian mother, according to Aristotle, addressed her son in the following words: "Do not enter into public business, for if you say what is just, men will hate you; and if you say what is unjust, the gods will hate you." To which Aristotle retorts: "I ought to enter into public affairs; for if I say what is just, the gods will love me; and if I say what is unjust, men will love me."

PROOF OF THE DILEMMATIC ARGUMENT

In concluding this chapter we shall establish the formal validity of the dilemmatic argument, basing our proof on the principles which follow, and assuming (as is commonly done) the identity of the implication, x is untrue implies y is true and the disjunction, either x is true or y is true.

Principle I.—If antecedent and consequent of a valid implication be contradicted and interchanged, a valid implication will result.

Principle II.—If in any valid implication a factor in the antecedent be strengthened, or the consequent be weakened, a valid implication will result.

Principle III.—If in any valid implication the same factor be conjoined to both antecedent and consequent, a valid implication will result.

(1) If (major) y is untrue implies x is untrue; and (minor) x is untrue implies z is true; then (conclusion) y is untrue implies z is true.

For (Principle II) the minor allows us to weaken the consequent of the major.

(2) If (antecedent) x is true implies y is true; then (consequent) y is untrue implies x is untrue.

For (Principle I) the antecedent and consequent of the principal antecedent may be contradicted and interchanged.

(3) If (major) x is true implies y is true; and (minor) x is untrue implies z is true; then (conclusion) y is untrue implies z is true.

For the major of (1), being the same as the consequent of (2), may be strengthened (Principle II) to the antecedent of (2).

(4) If x is true implies y is true; and z is true implies w is true; and x is untrue implies z is true; then (consequent) y is untrue implies z is true and z is true implies w is true.

For (Principle III) the same factor (z is true implies w is true) may be conjoined to both antecedent and consequent of (3).

(5) If (major) y is untrue implies z is true; and (minor) z is true implies w is true; then (conclusion) y is untrue implies w is true.
For (Principle II) the minor allows us to weaken the consequent of the major.

(6) If x is true implies y is true; and z is true implies w is true; and x is untrue implies z is true; then y is untrue implies w is true.
For the consequent of (4) being the same as the antecedent of (5) may be weakened (principle II) to the consequent of (5).

(7) If x is true implies y is true; and z is true implies w is true; and either x is true or z is true; then either y is true or w is true.
For we shall assume the right (as is commonly done) to replace the last premise and the conclusion of (6) by the disjunctions that appear in (7).

The result (7) will be recognized as the complex constructive dilemma, which has, accordingly, been derived from the three principles, which we assumed at the outset. The proof is laborious, when expressed without the aid of symbols. In that case the whole derivation could be set forth in a few lines. The proof of the complex destructive dilemma is left as an exercise for the student.

Exercises

Select from the following such as are valid arguments:

1. If this man is a realist, he believes that the order of nature is independent of mind, and if he is an idealist and believes it dependent on mind, he must assume an absolute object; but either he believes it dependent or not; therefore in either case, he must profess belief either in an absolute object or in an absolute order of nature.
2. If a body moves, it must move in the place where it is or in the place where it is not; it cannot move where it is, for then it would not be there, and obviously it cannot move where it is not. Accordingly, its motion is impossible.

CHAPTER XIII

THE SORITES

The **sorites** is of the same form as the syllogism, for its terms are arranged in the same way in a cyclical order, but it is of a more general character. The number of terms is greater than three and, as in the case of immediate inference and syllogism, the number of premises is one less than the number of terms. Because there is sometimes a large number of premises, it will be more convenient to employ in place of the class-symbols, *a, b, c,* etc., the ordinal numbers, *1, 2, 3,* etc.

CONSTRUCTION OF THE VALID MOODS

We shall begin by illustrating the manner of constructing a valid mood of the sorites from a chain of valid syllogisms. Suppose that we were to be given the chain,

A(*21*) and A(*32*) implies A(*31*),
A(*31*) and A(*43*) implies A(*41*),
A(*41*) and A(*54*) implies A(*51*),

and were asked what valid moods of the sorites is thereby implied. It is clear that the major premise of the last syllogism, being the same as the conclusion of the second, may be strengthened to A(*31*) and A(*43*). The immediate result of this strengthening is a valid mood of the sorites, *viz.*,

A(*31*) and A(*43*) and A(*54*) implies A(*51*).

The major premise of this last implication may in turn be strengthened to A (*21*) and A (*32*) because of the first syllogism, and we should have

A(*21*) and A(*32*) and A(*43*) and A(*54*) implies A(*51*).

This, then, is the valid mood of the sorites whose truth is implied by the chain of generating syllogisms.

Another method of constructing a valid mood of the sorites from a chain of syllogisms depends upon another principle.

Principle.—If in any valid implication the same factor be conjoined to both antecedent and consequent, a valid implication will result.

Let our chain of syllogisms be

E(*21*) and A(*32*) implies E(*31*),
E(*31*) and I(*34*) implies O(*41*),
O(*41*) and A(*45*) implies O(*51*),

and suppose that we conjoin to antecedent and consequent of the first syllogism the minor premise of the second and so obtain

E(*21*) and A(*32*) and I(*34*) implies
E(*31*) and I(*34*).

The second syllogism allows us to weaken the consequent of this result to O(*41*) and upon carrying out this operation we should obtain

E(*21*) and A(*32*) and I(*34*) implies O(*41*).

Now conjoin to antecedent and consequent of this mood of the sorites the minor premise of the third syllogism, A(*45*)—that is,

E(*21*) and A(*32*) and I(*34*)
and A(*45*) implies O(*41*) and A(*45*),

and weaken the consequent of this implication to O(*51*) by authority of the last member of the chain. Accordingly,

E(*21*) and A(*32*) and I(*34*)
and A(*45*) implies O(*51*),

is the valid mood of the sorites which was to be generated.

THE INVERSE OPERATION

Finally, let us consider the operation which is inverse to the preceding. Suppose, being given a valid mood of the sorites, we should be asked to find the chain of syllogisms upon which it depends. Let the mood be

A(*12*) and A(*23*) and O(*43*)
and A(*45*) and A(*56*) implies O(*61*).

The premises of the first syllogism of the chain will be the same as the first two premises of the sorites and the minor of the second syllogism will be the same as the third premise of the sorites and so on. The fragment of the chain so far ascertained will be:

A(*12*) and A(*23*) implies
and O(*43*) implies
and A(*45*) implies
and A(*56*) implies

Now the conclusion of the first syllogism, whose premises appear out of the normal order, is evidently A(*13*) and this must be the major of the syllogism which follows. The conclusion of this in turn is determined as O(*41*). If we were to continue this process, each member of the chain in succession would be ambiguously determined as,

A(*12*) and A(*23*) implies A(*13*),
A(*13*) and O(*43*) implies O(*41*),
O(*41*) and A(*45*) implies O(*51*),
O(*51*) and A(*56*) implies O(*61*).

RULES OF THE SORITES

From the considerations that have gone before we conclude that certain valid moods of the sorites can be constructed from chains of valid syllogisms. A complete solution of the sorites would contain a proof that the only valid moods that exist can be built up from chains of valid syllogisms in the manner described. This proof is too advanced for a work of this character. We observe, however, that it depends upon the following truths:

1. If the conclusion is affirmative, then all the premises are affirmative.
2. If the conclusion is negative, then one and only one premise is negative.
3. If the conclusion is universal, then all the premises are universal.
4. If the conclusion is particular, then not more than one premise is particular.

Exercises

1. What valid mood of the sorites can be generated from the following chain of syllogisms?

$$A(21) \text{ and } A(32) \text{ implies } A(31),$$
$$A(31) \text{ and } A(43) \text{ implies } A(41),$$
$$\cdots\cdots\cdots$$
$$A(n\text{-}1\ 1) \text{ and } A(n\ n\text{-}1) \text{ implies } A(n\ 1).$$

2. From what chain of valid syllogisms can we build up the sorites,

$A(21)$ and $A(32)$ and . . .
. . . and $A(t\ t\text{-}1)$ and $I(t\ t+1)$ and $A(t+1\ t+2)$ . . .
. . . and $A(n\text{-}1\ n)$ implies $I(n\ 1)$?

3. Which one of the rules enunciated at the end of this chapter will establish the invalidity of the sorites,

$$E(12) \text{ and } E(23) \text{ and } E(34) \text{ implies } E(41)?$$

4. State a rule analogous to one of the rules of the syllogism which will declare to be invalid:

$$A(12) \text{ and } A(32) \text{ and } A(43) \text{ implies } A(41).$$

APPENDIX I

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NOTE ON SUBALTERNATION AND THE DISPUTED SYLLOGISTIC MOODS

Padoa in his introduction to Peano's system [1] remarks: "The untruth of the traditional moods of the syllogism, by means of which from two universal judgments one would deduce a particular judgment, has been recognized separately by Miss Ladd (1883), Schröder, Nagy, Peano, *etc.* It is one of the first and most remarkable results of the adoption of a logical ideography."

Bertrand Russell [2] says: "But with our definitions, All *S* is *P* does not imply Some *S* is *P*, since the first allows the non-existence of *S* and the second does not; thus conversion *per accidens* becomes invalid, and some of the moods of the syllogism are fallacious, *e.g., Darapti:* All *M* is *S*, All *M* is *P*, therefore Some *S* is *P*, which fails if there is no *M*."

And Couturat observes: [3] "From All *a* is *b* (Il

[1] *Revue de Métaphysique et de Morale*, Vol. 20, p. 67, note.
[2] *Introduction to Mathematical Philosophy*, p. 164.
[3] *Revue de Métaphysique et de Morale*, Vol. 21, p. 258.

n'y a pas de *a* non-*b*) one cannot infer Some *a* is *b* (Il y a des *ab*). This inference could never mislead except in virtue of the additional and tacit premise: There exist members of the *a*-class, which seems implied in the language."

These opinions, taken at random, are typical of workers in this field and have become incorporated in the tradition of the science. In the following solution of the difficulty the symbols employed will be the sign ($\angle$) for "inclusion" and for "implication"; the sign ($'$) over a class or over a proposition to indicate "negation"; and the signs of ordinary algebra for logical multiplication and addition. Let us assume the following identities:

$$A(ab) = (a \angle b) \{ (b \angle a) + (a \angle b')' (b' \angle a)' \},$$
$$E(ab) = (a \angle b') \{ (b' \angle a) + (a \angle b)' (b \angle a)' \},$$
$$I(ab) = (a \angle b')' + (b' \angle a)' \{ (a \angle b) + (b \angle a) \},$$
$$O(ab) = (a \angle b)' + (b \angle a)' \{ (a \angle b') + (b' \angle a) \}.$$

It will be seen at once that the following conditions hold *for all meanings of the terms:* (1) ***E, I*** and ***A, O*** are contradictory pairs; (2) The product of ***A*** and ***E*** vanishes, so that subalternation holds true; (3) The ordinary process of privative conception is valid, for ***A***(*ab*) is the same as ***E***(*ab'*), ***O***(*ab*) is the same as ***I***(*ab'*); (4) Conversion by contraposition holds of ***A***(*ab*) and ***O***(*ab*), but not of ***E***(*ab*) and ***I***(*ab*); (5) ***E***(*ab*) and ***I***(*ab*) are simply convertible, whereas ***A***(*ab*) and ***O***(*ab*) are not simply convertible; (6) $A(ab)$ and $I(ab)$ become true when the terms are identified, whereas $E(ab)$ and $O(ab)$ then become false; (7) $E(ab)$ and $O(ab)$ become true when the terms are contradictory, whereas $A(ab)$ and $I(ab)$ become false in the same circumstances.

It only remains to verify the syllogistic moods *Barbara* and *Celarent,* for the truth of the remainder (including *Darapti*) follow at once from these. We have:

$$A(ba) = (b \angle a)\{(a \angle b) + (b \angle a')'(a' \angle b)'\},$$
$$A(cb) = (c \angle b)\{(b \angle c) + (c \angle b')'(b' \angle c)'\},$$
$$A(ca) = (c \angle a)\{(a \angle c) + (c \angle a')'(a' \angle c)'\},$$

and the following variations of the principle of transitivity:

$$\begin{array}{ll} 1. & (c \angle b)(b \angle a) \angle (c \angle a), \\ 2. & (a \angle b)(b \angle c) \angle (a \angle c), \\ 3. & (b \angle c)(b \angle a')' \angle (c \angle a')', \\ 4. & (a' \angle b)'(c \angle b) \angle (a' \angle c)', \\ 5. & (b \angle a)(b \angle c')' \angle (a \angle c')', \\ 6. & (c' \angle b)'(a \angle b) \angle (c' \angle a)'. \end{array}$$

If, now, we multiply together both sides of 1 and 2, we obtain an implication I, whose antecedent is the first term in the product of $A(ba)$ and $A(cb)$. Multiplying together both sides of 1, 3 and 4, we obtain an implication II. Multiplying together both sides of 1, 5 and 6, we obtain an implication III. Multiplying together both sides of 1, 4 and 5 and conjoining to the resulting antecedent the factors $(b \angle a')'$, $(c' \angle b)'$, we obtain an implication IV. If, now, both sides of I, II, III, and IV be added, *i.e.*, if the sum of the antecedents be taken to imply the sum of the consequents, we establish at once the truth of *Barbara.*

Celarent follows at once from *Barbara* in virtue of the equality of $A(ab)$ and $E(ab')$.

APPENDIX II

THE THEORY OF MULTIPLE IMPLICATION AND ITS APPLICATION TO THE GENERALIZED PROBLEM OF EPIMENIDES[1]

Let us allow $p, q, r, \cdots$ to stand for propositional functions, that is, for variables whose truth-values are in general dependent on the meaning of the terms that enter into them, and let the expression $(p\,q\,r \cdots)$ have the following verbal interpretation:

"$p, q, r, \cdots$ are simultaneously true for some meanings of the terms that enter into them."

We shall speak of this expression, $(p\,q\,r \cdots)$, or its negative, as the *existential* of the *n elements*, $p, q, r, \cdots$, and we shall say that it is of the first *order* and *n*th *degree*. The function containing existentials of the first order or their negatives as elements, will be of the second order and so on. For the particular case of one variable we should write:

$(p) \;= p$ is true for some meanings of the terms,
$(p)' \;= p$ is true for no meanings of the terms,
$(p')' = p$ is true for all meanings of the terms,
$(p') \;= p$ is true for not all meanings of the terms,

the prime being used as the symbol of denial or negation.

[1] Reprinted from Bulletin of the American Mathematical Society, January-February, 1929.

The effect of the symbol ( ), regarded as an operator, will be in general to weaken the expression on which it operates. Thus,

$$p \text{ implies } (p),$$
$$(p')' \text{ implies } p,$$

but not conversely. This simple provision is important, for it will lead at once to the result that certain generally recognized principles are untrue in a logic of complete generality. Thus it will turn out that the equality,

$$p \text{ implies } q = p' \text{ or } q$$

does not hold true without limitation, namely, the limitation that the variables stand only for "propositions," for zero or one values, and not for propositional functions. This limitation is one that reduces the generality of our science and is directly related to an application of our theory which we shall make before we end. If we write,

$$p < q = p \text{ implies } q,$$

we shall be prepared to set down the following definition,

$$(pq)' = p < q',$$

and from this follows the law of expansion for the general case,

$$(p\,q\,r \cdots)' = pq < r' + \cdots = p < q' + r' + \cdots,$$

that is, the elements are conceived to be conjoined as in a product, so that the consequent becomes the sum of the contradictories of the separate parts.

We might, if we liked, derive the fundamental laws of the calculus of propositions by assuming certain very simple properties of the existential, together with the fundamental equations,

$$(pq) = pq + p'q'(p)(q) + (p)(q')' + (q)(p')',$$
$$(pq)' = (p)' + (q)' + p'q(q') + q'p(p'),$$
$$(\overline{pq}) = (p') + (q'),$$
$$(\overline{pq})' = (p')'(q')',$$

the bar over pq standing for the negation of the product, or $p' + q'$. Or, we can derive the simple properties of the existential from the laws of the calculus of propositions. Obviously we may raise the degree of the function without changing its value, by adding to it as many one-elements as we choose. This fact is important, because it will enable us to provide that the symbol of implication can always be made to appear in our development. Thus $(o' = i)$,

$$(p)' = (ip)' = i < p',$$
$$(p)' = (pi)' = p < o.$$

And it is clear, too, that the elements may be permuted in any order, for

$$(pq)' = p < q' = q < p' = (qp)',$$

and that no change will result if we add to the existential as many elements already contained in it as we like. Similarly, the appearance of a pair of contradictory elements is a condition for the vanishing of the existential, if it be not negated; otherwise a condition for its becoming unity.

We approach now the consideration of a matter of the first importance, involving as it does, the creation of an infinite series of meanings of the "true" and the "false," together with an indication of how each one of these distinctions can be uniquely defined. We assume as true, for *all* meanings of the terms and for all meanings of p,

$$p < (p);$$

and we assert, moreover, that the converse,

$$(p) < p,$$

is true for *not all* meanings of the terms.

Now by allowing p to take on the values p', (p), $(p)'$, etc. and by the contradiction and interchange of antecedents and consequents, we should obtain a series of results that express the relation of the existentials of first and second order and beyond to one another and to the free variable within the propositional universe. These results are represented diagrammatically in the following table, the longer space being understood to "contain" the smaller spaces which it extends beyond.

$((p)) \mid ((p))'$
$(p) \mid (p)'$
$((p)')' \mid ((p)')$
$p \mid p'$
$((p')') \mid ((p')')'$
$(p')' \mid (p')$
$((p'))' \mid ((p'))$

We are now in a position to define each one of these meanings of "true" and "false." If we write

$$f(p) = p \text{ is false},$$
$$t(p) = p \text{ is true},$$

it will be clear that $t = f'$, no matter what value f may have, and that, consequently, t will be unambiguous as soon as f has been defined.

Suppose, for example, we wish $f(p)$ to mean p'. It will be enough to require f to satisfy the equation

$$f(f) = f',$$

for while there is an infinite series of meanings of "false" that satisfy the implication

$$f(f) < f',$$

for example,

$$(p)', \quad ((p))', \quad ((p)'), \ldots,$$

and which do not satisfy the converse, and while there is an infinite series of distinct meanings of "false" that satisfy the implication

$$f' < f(f),$$

for example,

$$(p'), \quad ((p')), \quad ((p')')', \ldots,$$

and which do not satisfy the converse, there is only one meaning of "false," given by

$$f(p) = p',$$

which satisfies both implications at once. Accordingly $f = p'$ is unique and $t = p$.

Any meaning of "false" from among the members of the infinite series of meanings that this term can have, can be uniquely determined by requiring

f to satisfy a given equation whose form can be found at once by inspection. Thus, if

$$f(p) = p < o, \qquad t(p) = (p < o)',$$

is the meaning of "true" and "false" desired, we assume f and t to satisfy respectively the equations

$$f(f) = f < o, \qquad t(t) = (t < o)',$$

for these solutions will prove to be unique. If

$$f(p) = (p' < o)', \qquad t(p) = p' < o,$$

are the meanings we chose, we require f and t to satisfy, respectively,

$$f(f) = (f' < o)', \qquad t(t) = t' < o,$$

and so on.

Thus, if ϕ is the form of denial, so that $f(p) = \phi(p)$, then regarding ϕ and f as independent variables, the vanishing of $\phi f' + f\phi'$ is the condition that must be satisfied if f is to have the required value.

The unique solution of the equation $f(p) = o$ is $p = i$ for all meanings of f, as will appear as the result of a complete induction,

$$f(p) = p', \quad (p)', \quad (p'), \quad \text{etc.}$$

Hence p is true absolutely, that is, for all meanings of t,

$$t(p) = p, \quad (p), \quad (p')', \quad \text{etc.,}$$

if f satisfy the implication

$$f < f(f).$$

The unique solution of the equation $f\ (p) = i$ is $p = o$ for all meanings of f. Hence p is false absolutely, if f satisfy the implication[1]

$$f(f) < f.$$

An interesting application of our theory will result if we should now go on to specialize the meaning of p and understand p to mean "this proposition." We should then have

$$f(p) = \text{this proposition is false;}$$

or, if we like,

$$1.f(1).$$

It is this situation that is supposed to give rise to a paradox, for it is supposed to lead to the identity of f with its own contradictory. Thus if 1. is o, $f(1)$ is i, if 1. is i, $f(1)$ is o and 1. and $f(1)$ are taken to be the same thing. The difficulty here depends on the assumption that every meaningful assertion must be specifically either zero or one, that is, must be a proposition, within Russell's understanding of that term. The solution to the difficulty will appear as soon as this limitation has been removed. Let us begin by defining the parts that enter into our assertion. The meaning of "false" has been defined already. It remains to define "this proposition."

We have here, perhaps, a range of choice in defi-

[1] For further results that concern the theory of multiple implication see the writer's *Symbolic Logic*, (Edwards Bros., Ann Arbor, 1933). For a brief history of the problem of Epimenides as of other insolubilia see Professor E. R. Guthrie's admirable little monograph entitled *The Paradoxes of Mr. Russell.* This work, little known because privately printed, has not attracted the attention it deserves.

nition but let us require p to satisfy the equation.

$$(f)(f') = o.$$

If this condition be understood to stand for the meaning of p, it will be seen to have two unique solutions, $p = o$ and $p = i$. This, then, is the solution of the difficulty involved in the assertion

"this proposition is false,"

or, as we might indicate it,

$$1.f(1).$$

That is, if 1. is two-valued, being ambiguously zero or one, then $f(1)$ is similarly two-valued and no contradiction can arise.[1]

The same solution can be shown to hold in the more general case. Consider a "cycle" of three assertions of the form

$$1.f(2) \qquad 2.f(3) \qquad 3.f(1).$$

We may now attach to these too a determinate and consistent meaning. From the symmetry of the situation we may take it for granted that each one has the same general character and that this general character is the one that we assumed above. That is, these three assertions are to be conceived as two-valued in the sense defined. Since 1, 2 and 3, then, are two-valued, so will the following be two-valued in the same sense, viz.

$$f(1) \qquad f(2) \qquad f(3)$$

[1] We must not, of course, fall into the familiar fallacy of confusing the use of "all" in the collective and distributive sense. For all meanings (collectively) p and $f(p)$ are the same, but they are not equal for all meanings, where "all" has the meaning of "every."

and for two reasons. In the first place, because if p is two-valued so is $f(p)$, (for the condition which p is required to satisfy, may be regarded equally as the condition which $f(p)$ is required to satisfy, for all meanings of f, and there are two and only two such values, viz., $f(p) = o$ and $f(p) = i$), and in the second place, because they are repetitions of 1, 2 and 3. Consequently, no contradiction can arise from the simultaneous assertion of 1, 2 and 3. The same remarks will evidently hold of a "cycle" of any number of assertions of the same form and the generality of the method is evident.

HOW THE MIND FALLS INTO ERROR

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A BRIEF TREATMENT OF FALLACIES FOR THE GENERAL READER

BY

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PREFATORY NOTE

NOTHING has a higher value in the eyes of the mere teacher attempting to present some subject dry and forbidding in itself, than a fresh illustration; and a teacher of logic is no exception to the rule. Every student in this field who has suffered at the hands of preceptors is familiar with that ancient argument: " What you bought yesterday, you eat to-day; you bought raw meat yesterday; therefore, you eat raw meat to-day." A well-known writer observes of this case: "This piece of meat has remained uncooked, as fresh as ever, a prodigious time. It was raw when Reisch mentioned it in the *Margarita Philosophica* in 1496: and Doctor Whateley found it in just the same state in 1826." Fresh illustration is as much of a necessity for teacher or student in this field as in any other and it is this need for which the present essay hopes in some sort to provide. The writer, encouraged by certain favorable attention which the chapter on fallacies in his *First Book in Logic* has aroused, has tried to give that material through expansion and amplification a somewhat wider appeal.

H. B. S.

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CHAPTER I

PARADOX AND ITS REDUCTION

LOGIC, it is generally supposed, is a term that is employed to designate a definite body of doctrine. But such is by no means the case. There are many differing views as to what logic is presumed to deal with, differing ideas as to the nature of the problems, which logic is supposed to solve. To understand this divergence of view as to what constitutes the proper domain of the science, it would be essential to have some acquaintance with the general history of ideas. The work of Mill, for example, was a culmination of the English tradition which begins with Bacon; and the "New Organon," a supposed emancipation from the "Organon" of Aristotle, was taken to set up a new instrument of investigation which would make unnecessary the use of Aristotle's own. It is largely because of this Baconian tradition that the study of formal logic has fallen into disrepute.

It is sometimes amazing to mark the effect of an intellectual tradition in the demolition of alien points of view. When the Lockian philosophy was imported into France and supported by the authority of Voltaire's name, the rationalistic tradition, already weakened, was swept away. In the seventeenth century it dominated the thought of continental Europe. The empirical tradition as it has established itself in England and America, together with effects that cannot be too highly valued, has carried in its wake a certain blight. The study of formal logic, never quite at home on such soil, maintains itself with difficulty. De Morgan says: "We live in an age in which formal logic has long been nearly banished from education: entirely, we may say, from the education of the habits." What we propose in this essay is a classification of the fallacies of the reason, following in the main, but not strictly, the classical tradition; and we shall illustrate each heading or division of the subject by illustrations drawn from the sciences and from belles-lettres.

FALLACY, SOPHISM AND PARADOX

Fallacy is a word that is used in a restricted sense to designate an offense against some one of the principles of correct inference; but it is loosely employed as well to denote any one of the ways by which men come to adopt erroneous opinions. A fallacy may be termed a **sophism**, a **paradox** or a **paralogism** whenever we have in mind the result, rather than the false method by which the result is gained. Thus, any view which runs counter to our common apprehension of things, or which seems to offend our common sense, is said to be paradoxical. We may call an opponent's argument sophistical, whenever it has placed us in some awkward position from which we find no rational escape. The German philosopher Kant employs the term paralogism to designate a particular error which the mind shows a special tendency to adopt.

What is simple and self-evident to a man of one viewpoint may appear paradoxical to another whose viewpoint is different. The following anecdote concerning Agassiz might easily pass for a case in point for the merchant or for the financier. The directors of a lyceum lecture course had endeavored to include one of the popular addresses of the famous naturalist, but their offer was rejected. "We will give you double your price," they said, "if you will accommodate us." "Ah, sirs," he returned, "I should be glad to help you, but I cannot afford to waste time in making money." Having no sense of the paradox, Agassiz could never understand why this remark enjoyed such popularity.

AN OPINION IN THE TEETH OF GENERAL FAME

The man whose logical faculty is not stirred by paradox and by the subsequent reduction of paradox to the plain status of intelligible fact, must possess an intellect that is indeed soft and unadventurous. The following passage from De Quincey will yield us a further example. He says:

"One fact, which struck me by accident, and not until after a long familiarity with Kant's writings, is this, that in all probability Kant never read a book in his life. This is paradoxical, and undoubtedly in the very teeth of general fame, which represents him to have been a prodigious student in all parts of knowledge, and therefore, of necessity it may be thought, a vast reader."

Having stated the case thus boldly, this writer now prunes away the implications that mislead, and softens the hard note of logical discord with reconciling distinctions:

"What! none? No, none at all; no book whatsoever. The books of which he read most were, perhaps, books of voyages and travels; for he himself gave lectures on what he called Physical Geography. . . . But whenever the business of the writer was not chiefly with facts, but with speculations built on facts, Kant's power of thought gave him a ready means of evading the labor of reading the book. Taking the elemental principles of the writer as stated by himself or another . . . he would then *integrate* these principles for himself. . . . In this way he judged of Plato, Berkeley, and many others. . . . Yet these were writers in his own department; and if he would not read *them,* it may be presumed that . . . he would read nobody."

THE SOPHISM OF BOSSUET

The author of a sophism is commonly held to be convicted not only of a breach of the rules of correct thinking, but consciously convicted. In this sense Socrates defined a sophist, not as a wise man (the original connotation of the term), but as one who makes the worse appear the better reason. The term may thus be applied by one's opponent to any logical dilemma in which he discovers himself and from which he finds no rational escape.

In illustration of the manner in which a sophism may lurk among current opinions we may cite the following instance. Even the great Kant, whose dialectical powers will hardly be questioned, precipitately placed upon this argument the stamp of his approval. It is this: If the Roman Church, as they say, admits no possibility of salvation except within its own pale, and the Protestant admits that the Romanist has still a chance of salvation, then, as Bossuet remarks, a wise man will make a safer choice by indorsing the Catholic faith.

This argument is older than Bossuet and was answered by the Protestants again and again. Besides the slight value in the eyes of God of a faith based on mere expediency, other still more convincing considerations might be urged. Coleridge remarks of this case: "The Protestant does not say that a man can be saved who chooses the Catholic religion, not as true, but as the safest; for this is no religion at all, but only a pretense to it." Jeremy Taylor, in a letter to a lady who had been converted to Catholicism, observes:

"I wish that you would consider that, if any of our men say salvation may be had in your church . . . it is only because you do keep so much of that which is our religion, that upon the confidence of *that* we hope well concerning you. And we do not hope anything at all that is good of you or your religion, as it *distinguishes* from us and ours: we hope that the good which you have in *common* with us may obtain pardon, directly or indirectly, or may be an antidote of the venom, and an amulet of the danger, of your very great errors. So that, if you derive any confidence from our concession, you must remember where it takes root; not upon anything of yours, but wholly upon the excellence of ours." Moreover, "whatever we talk, things are as they are, not as we dispute," and it would be small consolation to the lady, becoming ultimately aware of her mistake, that the Protestants in charity and in tolerance of opinion had conceded her a chance of the salvation of which a sophism too precipitately endorsed had deprived her.

A LITTLE LOGIC IS A DANGEROUS THING

Doctor Johnson said of a man, who had cleverly maintained that the difference between virtue and vice is illusory: "When he leaves our house, let us count our spoons." During a discussion regarding the existence of God, Voltaire put the servants out of the room, with the remark: "I do not care to have my throat cut in the night"—the difference between a little logic and none at all, operating in the domain of morals, being measured by the difference between theft and murder, if we accept the estimates of these two authorities. "A little skill in antiquity inclines a man to Popery," says another writer. "But depth in that study brings him about again to our religion."

An intellect which has developed neither the habit nor the power of logical distinction, will be prone to subscribe to any half truth that is speciously expressed. A man is declared to possess no talent because he does not write well; whereas, his talents, it may be presumed, are contained in his original nature, while his ability to give expression to his thought may depend on the circumstances of his training. Who shall say how much original genius has failed on this account! Even untruths, and untruths that are only plausible because of failure to review the most ordinary scientific facts, found themselves upon this weakness of the logical faculty. It was recently urged by a correspondent to a daily paper that the daylight-saving law (by which the clocks are set forward an hour at a specified time of the year) has a tendency to weaken the respect for truth in the minds of the young, as if the time of day were some unalterable fact revealed to man on tables of stone. In truth, exactly the opposite tendency might be urged by the informed or intelligent opponents of this view; for the lesson is taught that certain things are only true by agreement, and taught in such a manner that no single member of the state may evade that knowledge, that there is a difference between natural fact, that resists any whim of man that it be otherwise, and human conventions that are altered at will.

> "A little learning is a dangerous thing;
> Drink deep, or taste not the Pierian spring;
> There shallow draughts intoxicate the brain,
> And drinking largely sobers us again."

COMMON MEASURES FOR SOLVING PARADOX

The commonest means employed by the logician in the solution of a paradox is that of extending the meaning of the terms that enter into the statement of the difficulty or by the introduction of some clarifying distinction. Many illustrations of this procedure, were they not too technical for our present purposes, might be drawn from the history of the sciences. No experience is more familiar to the mathematician than the need of extending the meaning of his indefinables—that is, the terms or relationships of which he treats—in order to account for limiting cases, whose crucial character was not suspected, when his original definitions were set down. Thus, one of the characteristic obstacles which the student of algebra has to overcome, is the meaning which he is called upon to attach to imaginary quantity. The primitive meaning of quantity, the only one with which he is familiar, will not suffice for the purpose in hand. Instinctively he holds to the *original* meaning of the term, while striving to grasp the *new*. The older logicians encountered many paralogisms because they were unable to generalize their conceptions and redefine their terms. In this connection a simple illustration may be given. One would like to be able to say of any two classes, that their *logical product* is also a class. But this will compel the introduction into the science of the notion of a class that has no members, and such a class has paradoxical properties. Again, the idea of a proposition that is never true, is one with which the older logicians could not deal, although it is common in popular usage and recognized by common sense:

> "I will not be afraid of death and bane
> Till Birnam forest come to Dunsinane,"

means: I will not be afraid till an impossibility becomes possible; that is, without qualification, I will not be afraid. Again:

> "Nay, had I power, I should
> Pour the sweet milk of concord into hell,
> Uproar the universal peace, confound
> All unity on earth,"

which means: if what follows is false, then what implies it is false, for an implication that is asserted, is true by supposition. The use of the conditional renders the meaning unambiguously.

A popular illustration of the reduction of paradox to the status of intelligible fact is contained in the following anecdote: The story is told of a well-known teacher that he on one occasion received a visit on the part of the parent of a student whom he had refused to pass in his course. "Sir," he said, addressing his visitor, "your son's work was both good and original. But"—and he proceeded to reduce the paradox, for this man was known to dote on logical distinctions—"the original work was not good, was, in effect, bad; whereas, the remainder, though otherwise good, was not original. As the case stood, therefore, it was out of the question to pass him." Daniel Webster in one of his speeches once made a similar distinction: "I have read their platform; but I see nothing in it both new and valuable. What is valuable is not new and what is new is not valuable."

The following examples, which further illustrate the meaning of paradox, are taken from the writings of De Quincey and, since they cannot be better related, we shall quote this author in full:

"A great philosopher pronounces the people of Crete, one and all, liars. But this great philosopher, whose name is Epimenides, happens himself to be a Cretan. On his own showing, therefore, Epimenides is a liar. But if so, what he says is a lie. Now, what he says is, that the Cretans are liars. This, therefore, as coming from a liar, is a lie; and the Cretans as is now philosophically demonstrated, are all persons of honor and veracity. Consequently, Epimenides is such. You may depend on everything he says. But what he says most frequently is, that all the Cretans are liars. Himself, therefore, as one amongst them, he denounces as a liar. Being such, he has falsely taxed the Cretans with falsehood, and himself amongst them. It is false, therefore, that Epimenides is a liar. Consequently, in calling himself by implication a liar, as one amongst the Cretans, he lied. And the proof of his veracity rests in his having lied. And so on *da capo* for ever and ever.

"A more pleasant example of the same logical see-saw occurs in the sermons of Jeremy Taylor. That man, says the inimitable bishop, was prettily and fantastically troubled, who, having used to put his trust in dreams, one night dreamed that all dreams were vain; for he considered, if so, then *this* was vain, and the dreams might be true for all this. (For who pronounced them *not* true except a vain dream?) But if *they* might be true then *this* dream might be so upon equal reason. And dreams *were* vain, because this dream, which told him so, was true; and so round again. In the same circle runs the heart of man. All his cogitations are vain, and yet he makes especial use of this—that that thought which thinks so, *that* is vain. And if *that* be vain, then his other thoughts, which are vainly declared so, may be real and relied upon. You see, reader, the horrid American fix into which a man is betrayed, if he obeys the command of a dream to distrust dreams universally, for then he has no right to trust in this particular dream, which authorizes his general distrust. No; let us have fair play. What is sauce for the goose is sauce for the gander. And this ugly gander of a dream, that notes and protests all dreams collectively, silently and by inevitable consequence notes and protests itself.

"So natural, indeed, to the morbid activity of man are these revolving forms of alternate repulsion, where flight turns suddenly into pursuit, and pursuit into flight, that I myself, when a schoolboy, invented several: this, for instance, which once puzzled a man in a wig, and I believe he bore me malice to his dying day, because he gave up the ghost by reason of fever, before he was able to find out satisfactorily what screw was loose in my logical conundrum; and thus, in fact, 'all along of me' (as he expressed it) the poor man was forced to walk out of life *re infecta,* his business unfinished, the one sole problem that had tortured him being unsolved. It was this. Somebody had told me of a dealer in gin, who, having had his attention roused to the enormous waste of liquor caused by the unsteady hands of drunkards, invented a counter which, through a simple set of contrivances, gathered into a common reservoir all the spillings that previously had run to waste. St. Monday, as it was then called in English manufacturing towns, formed the jubilee day in each week for the drunkards; and it was now ascertained (*i.e.* subsequently to the epoch of the artificial counter) that oftentimes the mere 'spilth' of St. Monday supplied the entire demand of Tuesday. It struck me, therefore, on reviewing this case, that the more the people drank, the more they would *titubate,* by which word it was that I expressed the reeling and stumbling of intoxication. If they drank abominably, then of course they would titubate abominably; and, titubating abominably, inevitably they would spill in the same ratio. The more they drank, the more they would titubate; the more they titubated, the more they would spill; and the more they spilt, the more, it is clear, they did *not* drink. You can't tax a man with drinking what he spills. It is evident, from Euclid, that the more they spilt, the less they *could* have to drink. So that, if their titubation was excessive, then their spilling must have been excessive, and in that case they must have practiced almost total abstinence. Spilling nearly all, how could they have left themselves anything worth speaking of to drink? Yet, again, if they drank nothing worth speaking of, how could they titubate? Clearly they could not; and, not titubating, they could have had no reason for spilling, in which case they must have drunk the whole—that is, they must have drunk to the whole excess imputed, which doing, they were dead drunk, and must have titubated to extremity, which doing, they must have spilt nearly the whole. Spilling the whole, they could not have been drunk. *Ergo,* could not have titubated. *Ergo,* could not have spilt. *Ergo,* must have drunk the whole. *Ergo,* were dead drunk. *Ergo,* must have titubated. And so round again, as my Lord the bishop pleasantly expresses it, *in secula seculorum.*"

GUTHRIE ON THE INSOLUBILIA

Professor Guthrie says in his work on the *Paradoxes of Mr. Russell:* "Rüstow has collected a large number of references to similar paradoxes in ancient writings, notably in the works of Plato, Aristotle, and Diogenes Laertius. Among these ancient writers the paradox of the liar was the one to attract the most attention. As a rule the paradox was accepted as final, and the fact that it existed was used in support of an attack on the validity of human knowledge, as Montaigne used it later on. What solutions were proposed were crude attempts to place it under one of the Aristotelian forms of fallacy. No analysis of the paradoxes was made, nor were they recognized as a class. It was not until the time of the Scholastic logicians that they were presented in a form which offers interesting parallels to our present statement and analysis. During the Scholastic period the interest which the Paradoxes or Insolubilia aroused was so great that many of the text-books of logic written from the fourteenth to the sixteenth centuries devoted lengthy chapters to them and there were a number of separate treatises in which their solution was attempted.

"Anything like a careful analysis of the Scholastic doctrines concerning the Insolubilia would require an extensive knowledge of medieval logic and might not be justified in a study of the logical side of the problem. But it is of some interest to see the form in which the difficulties arose and the types of solution offered, and in particular the close analogies which these bear to the solutions of more modern writers." One method of dealing with the problem (*pars propositionis non potest supponere pro toto*) "appealed to a number of later writers, Johannis Majoris Scotus, Olkot, and Rosetus among them. It is much the same as Russell's device of the theory of types which depends on the principle of the vicious circle, namely, that no term in a proposition can presuppose the proposition or have it as one of its possible values. The only Scholastic to make a serious criticism of this view was Wycliffe and the difficulty which he points out is much the same as that contained in the theory of types. If we are not to allow a proposition to refer to itself we make a general proposition like "All propositions are true or false" exceptive. It becomes "All propositions are true or false except this proposition." We would seem to do away with all general propositions about propositions and there are some of these which we do not wish to reject. . . . Wycliffe's theory is really of the type of another theory termed by an unknown author of a Paris manuscript *cassatio,* which involved denying that the propositions in question were propositions at all." Professor Guthrie at the end of his work offers a solution of the difficulty, which, however, is of too advanced a character to be given here.

CHAPTER II

EQUIVOCATION AND AMPHIBOLOGY

FALLACIES most frequently hinge upon an ambiguity in the meaning of a term or proposition. Those of the first sort are called fallacies of **equivocation**; those of the second sort **amphibology.** The first expression may, however, be employed in a more general sense so as to include both cases.

AMBIGUITIES OF COMMON WORDS

Let us consider in the first instance the ambiguities which certain very common words take on, such as the adjectives of quantity *all, some,* and *no,* the definite article, and the copula. To assert that "all the angles of a triangle equal two right angles" and "all the angles of a triangle are less than two right angles" is in each case to say something true, if *all* is regarded collectively in the first example and distributively in the second. In the two expressions, "all of these twelve men are a jury" and "all of these men are opinionated," the same ambiguity is apparent. In the *Social Con-*

tract of Rousseau a man is said to surrender *all* his rights that the rights of *all* may be preserved:

"Chacun se donnant à tous ne se donne à personne."

In this manner of phrasing the distinction the meaning is unambiguous. Civilized society is sometimes said to be only a refinement of what is taken to be characteristic of the state of savage man, the *bellum omnium contra omnes.* The Latin word thus preserves the same double meaning. A similar distinction applies to the word *both.* "This man can walk on both legs" is not true in the distributive sense of *both;* whereas, "This man can hop on both legs" is true in both the distributive and in the collective sense of *both.*

An ambiguity attaches to the word *some* which has led to no end of trouble among the logicians themselves. Regarding the use of this term the Scottish logician Hamilton has this to say: "A remarkable uncertainty prevails in regard to the meaning of particularity and its signs. Here *some* may mean some only—some, not all, and is definite in so far as it excludes omnitude. On the other hand *some* may mean some at least—some, perhaps all." An ambiguity in the use of the copula or else in the use of the singular term would appear if it were argued that "Socrates is a man and man is a class" and that "therefore Socrates is a class." Sometimes the definite article in English individualizes. This is true in the phrase, "Have no fear *the* animal will find his way home." But when we speak of "*the* animal in man" or when we say "*the* dog is faithful to man," the case is reversed, for in these cases the effect of the definite article is to generalize. In English we remark that "man is unfaithful," but in Greek, in French, and in German the definite article is required before man, when we employ this word in the universal sense.

Ambiguities appear as often in our beliefs and attitudes as in our use of words and phrases:

"Seven wealthy towns contend for Homer dead,
Through which the living Homer begged his bread."

And another case as proverbial in its own way may be cited:

"Le mariage est comme une forteresse assiégée; ceux qui sont dehors veulent y entrer, et ceux qui sont dedans veulent en sortir."

CASES OF EQUIVOCATION

An excellent illustration of equivocation is given by De Morgan. He says: "Governments permit what would otherwise be equivocation to take a strong air of truth, by legislating in detail against the principles of their own measures. The window tax is a special instance. A newspaper calls it a tax upon the light which God's beneficence has given to all. The answer would be plain enough, namely, that it is an income tax levied upon the use of that light which (how truly matters not here) is asserted to be a fair criterion of income. But this answer is destroyed by the permission to block up windows and thereby evade the tax, which is thus made to fall upon the light used, and not upon the means of using it which the size of the house affords. According to the criterion of this impost, the blocked window is as fair a criterion of the income of the occupant as the open one, and should have been so considered."

The following argument in favor of a protective tariff is not unusual. It has been said: "When we buy abroad, the domestic consumer will obtain the goods beyond doubt, but the foreign producer obtains the money. On the other hand, when we sell abroad the producer at home, while he gains the money, loses his goods. It will be better then to buy and sell at home, for in that case we retain both the goods and the money." As well might one argue: "When we buy at home, the producer loses the goods and the consumer loses the money. But every man is either a consumer or a producer or both. When we buy at home, it is clear, then, that we lose both the goods and the money." In both arguments the ambiguity of our terminology is patent enough.

In the history of logic there is an oft-recurring dispute as to whether logic is a science or an art. Its force depends in part, it may be, on an ambiguity in the sense of the word as employed by the Greeks. The Greek word λογική is an adjective with some substantive (like study, problem, art) understood. The noun λόγος denoted either man's thought or his manner of expressing his thought (the *ratio* and *oratio* of the Romans), and hence originated the dispute first mentioned, for the derivative bore the same equivocation as the substantive from which it was derived.

The force of a very common remark that "we cannot *conceive* the infinite," often depends upon an equivocal use of the italicized term. Thus, we can readily conceive the series of natural numbers which is infinite, for we know how each term is formed from its predecessor. But we could not conceive it, if by this it is meant that each term is to be brought before the mind in succession *until the last term is reached.* Only an imprudent imagination would seek to accomplish what by definition it cannot do.

Taine in his work *De l'Intelligence* cites an equivocation of the same sort: "A name is general because it is abstract; it corresponds to a whole class, because the object it denotes, being but a fragment, may be found in all the individuals of the class, which, similar in this point, remain nevertheless dissimilar in other points. . . . Can we have experience, perception, or sensible representation of this detached and isolated fragment? Assuredly not, for this would be a contradiction. . . . I have not mentally a sensible representation of a pure or abstract polygon; for the pure polygon is a figure with several sides, but whose sides do not make up any particular number. . . . To tell one to see or imagine many sides, and at the same time not to see or imagine three, four, or any definite number of sides, is, in one breath, to order and forbid the same operation."

ACHILLES AND THE TORTOISE

No treatment of fallacies could well be regarded as complete without reference to a famous paradox, the one known as the race of Achilles and the tortoise. The case may be stated in a variety of ways. In most instances the solution turns upon an ambiguity in definition. De Quincey gives the following account in one of his essays, and, as it cannot be better related, we shall quote him at length:

"Achilles, most of us know, is celebrated in the Iliad as the swift-footed; and the tortoise, perhaps all of us know, is equally celebrated among naturalists as the slow-footed. In any race, therefore, between such parties, according to the equities of

Newmarket and Doncaster, where artificial compensations as to the weight of riders are used to redress those natural advantages that would else be unfair, Achilles must grant to the tortoise the benefit of starting first. But if he does *that,* says the Greek sophist, then I, the sophist, back the tortoise to any amount, engaging that the goddess-born hero shall never come up with the poor reptile. Let us see. It matters little what exact amount of precedency is conceded to the tortoise; but say that he is allowed a start of one-tenth part of the whole course. Quite as little does it matter by what ratio of speed Achilles surpasses the tortoise; but suppose this ratio be that of ten to one, then, if the race course be ten miles long, our friend the slow-coach, being by the conditions entitled to one-tenth of the course for his starting allowance, will have finished one mile as a *solo* performer before Achilles is entitled to move. When the *duet* begins, the tortoise will be entering on the second mile precisely as Achilles enters on the first. But, because the Nob runs ten times as fast as the Snob, whilst Achilles is running his first mile, the tortoise accomplishes only the tenth part of the second mile. Not much, you say. Certainly not very much, but quite enough to keep the reptile in advance of the hero. This hero, being very little addicted to think small beer of himself, begins to fancy that it will cost him too trivial an effort to run ahead of his opponent. But don't let him shout before he is out of the wood. For, though he soon runs over that tenth of a mile which the tortoise has already finished, even this costs him a certain time, however brief. And during that time the tortoise will have finished a corresponding subsection of the course—*viz.*, a tenth part of a tenth part. This fraction is a hundredth part of the total distance. Trifle as that is, it constitutes a debt against Achilles, which debt *must* be paid. And whilst he *is* paying it, behold our dull friend in the shell has run the tenth part of a hundredth part, which amounts to a thousandth part. To the goddess-born what a flea-bite is that! True it is so, but still it lasts long enough to give the tortoise time for keeping his distance and for drawing another little bill upon Achilles for a ten-thousandth part. Always, in fact, alight upon what stage you will of the race, there is a little arrear to be settled between the parties and always *against* the hero. Vermin, in account with the divine and long-legged Pelides, Cr. by one billionth or one decillionth of course, much or little, what matters it, so long as the divine man cannot pay it off before another installment becomes due? And pay it off he never will, though the race should last for a thousand centuries."

HISTORICAL ATTEMPTS AT SOLUTION

It may be argued (as indeed it has been) that we may calculate the point where the race will end. Leibnitz remarks in a letter to M. Foucher that "P. Gregoire de St. Vincent has shown by mathematics the spot where Achilles must have caught the tortoise." But such a comment clearly misses the point. "Of course . . . it becomes easy, upon assuming a certain number of feet for the stride of Achilles, to mark the precise point at which that *impiger* young gentleman will fly past his antagonist like a pistol shot, and being also *iracundus, inexorabilis, acer,* will endeavor to leave his blessing with the tortoise in the shape of a kick (though according to a picturesque remark of Sidney Smith, it is as vain to caress a tortoise, or, on the other hand, to kick him, as it is to pat and fondle, or to tickle, the dome of St. Paul's). . . . It is precisely *because* Achilles will in practice go ahead of the tortoise, when, conformably to a known speculative argument, he ought *not* to go ahead . . . which constitutes our perplexity, or, to use a Grecian word still more expressive, which constitutes our *aporia,* that is, our resourcelessness." The same objection was made to Zeno himself, the author of the dilemma, for on one occasion, while expounding the argument, one of his listeners, without taking the trouble to reply, arose, walked a short distance, and then resumed his seat. By this gesture he imagined that he had refuted the argument *in practice*. But Zeno quite properly replied, "You do not refute me, sir, but only illustrate the paradox," for if bodies did not *seem* to move, the difficulty would not exist.

It is often said that had Zeno grasped the modern notion of the limiting value of the ratio of two infinitesimals, the paradox would have been reduced. The solution which Leibnitz offers in this same letter is: "Ne craignez point, monsieur, la tortue que les Pyrrhoniens faisaient aller aussi vite qu'Achille. Un espace divisible sans fin se passe dans un temps aussi divisible sans fin," and this is the one which De Quincey accepts: "The infinity of space in this race of subdivision is artfully run against a *finite* time, whereas, if the one infinite were pitted, as in reason it ought to be, against the other infinite, the endless divisibility of time against the endless divisibility of space, there would arise a reciprocal exhaustion and neutralization that would swallow up the astounding consequences, very much as the two Kilkenny cats ate up each other."

A CASE OF AMPHIBOLOGY

However, the solution here given equally misses the essential point, for it depends on a weak statement of Zeno's case. The real obstacle, the one upon which the difficulty hinges, depends upon the impossibility of reaching the last term of an infinite series. Upon examination it will be found that the fallacy is really a concealed case of amphibology. An infinite series is defined as one that has no last term, and this condition is later revoked, the last term being reinvoked as a real obstacle for him against whom the argument is directed. Summarized, the stages of the proof are these: "To pass the tortoise Achilles must reach the *last term* of the series. But the series *has* no last term. Accordingly, the hero cannot come up with the reptile." The solution lies quite simply in the rejection of the major premise. If there is no last term, what justice in logic can require of Achilles that he pass through the last term in order to reach the limit? If the original definition excludes the last term as a *possible* obstacle, by what right is it later reinvoked as a *real* obstacle for him who denies the compulsion of the major premise?

CHAPTER III

SATIRE, EXAGGERATION, AND FALSE ASSOCIATION

NOT all cases of equivocation are seriously intended, as when the ambiguity takes the form of a jest or pun. But these cases, too, have their serious side, and are effectively employed at times as rhetorical weapons in debate. A famous case is the conversation of Hamlet with the grave-digger:

> ***Ham.***—Whose grave's this, sirrah?
>
> *Clo.*—Mine, sir.
>
> ***Ham.***—I think it be thine indeed, for thou liest in't.
>
> *Clo.*—You lie out on't, sir, and therefore 'tis not yours: for my part, I do not lie in't and yet it is mine.

A PUN MAY HAVE A SERIOUS INTENT

A remark of Franklin's, "If we do not hang together, gentlemen, we may expect, each one of us, to hang separately," has a compulsion of its own, which is not of the logical kind. Suppose one were to say: "Shakespeare is not the author of certain well-known plays; another writer of the same name planned and executed them." We are then left in doubt as to whether an ambiguity is intended or not. If Mark Twain were known to be responsible for the remark our doubt would be removed; but the authorship of Bacon, on the other hand, may be what another writer would have us infer.

Oliver Wendell Holmes says: "People that make puns are like wanton boys that put coppers on the railroad tracks. They amuse themselves and other children, but their little trick may upset a freight train of conversation for the sake of a battered witticism." The story is told of Madame de Staël that she once showed her foot at a masked ball to Antoine Rivarol, for her vanity deemed this a sufficient means of recognition, and that he exclaimed, *Quel vilain piédestal.* Doctor Johnson remarked that if he had been aware that Boswell intended to write *his* life, he would have prevented it by *taking* Boswell's. Wordsworth once said, if he had a mind, he could write like Shakespeare. Concerning this case Charles Lamb suggested: "It is only the mind which is lacking." Thus, an ambiguity which deceives no one may yet serve as an effective retort.

PARODY AND CARICATURE

Another class of ambiguities which lack a serious intent is the case of parody, and the implications of caricature are closely akin. The well-known lines,

> "Man wants but little here below,
> Nor wants that little long,"

has been parodied in a dozen ways, of which

> "Man wants but liquor here below,
> But wants that liquor strong."

is one of the best. Doctor Johnson said: "Sir, a woman's preaching is like a dog's walking on his hind legs. It is not done well, and you are surprised to see it done at all." A great deal of fun can be crowded into a simple parody. Here is an unusual example that is worthy of being quoted. De Quincey says: "Amongst the peculiar opinions which he [a certain Doctor Maginn] professed was this—that no man, however much he might *tend* toward civilization, was to be regarded as having absolutely reached its apex until he was drunk. . . . But as such an odiously long word [civilization] must ever be distressing to a gentleman taking his ease of an evening, unconsciously perhaps, he abridged it always after 10 P.M. into *civilation.* Such was the genesis of the word. And I, therefore, upon entering it in my neological dictionary of English, matriculated it thus: *Civilation* by ellipsis, or more properly by syncope, or rigorously speaking by hiccup, from *civilization.*

The phrase, "He enjoys good health," is sometimes parodied in a form that is not without rhetorical advantage. Thus we say of So-and-so that "he enjoys *poor* health," for there are many who make of their misery a comfort, either because of added attention on the part of others, or because they may at such times indulge their sentiment of pity for themselves. In cases akin to this, therefore, an ambiguity may serve effectively to point a criticism. To this class of ambiguities might be added the misuse of learned words, conscious or unconscious, as well as humorous affectations, such as are contained in the sayings of Sheridan's Mrs. Malaprop. Where the use of learned expressions is unnecessarily labored, precise, or merely overdone, the pedant is revealed. Goldsmith says of his village schoolmaster:

> "In arguing, too, the parson owned his skill,
> For, e'en though vanquished he could argue still,
> While words of learned length and thund'ring sound
> Amazed the gazing rustics ranged around." . . .

SATIRE, IRONY, AND INNUENDO

An able speaker knows that his audience may often be won over to his own view by satire or irony, when his best logic has failed. The philosopher Renouvier used to remark that proof of the most rigorous kind may fail to convince. A famous example of innuendo is recorded of Whistler. He spoke of Ruskin's "flow of language that would, could he hear it, give Titian the same shock of surprise that was Balaam's, when the first great critic proffered his opinion." And when a correspondent observed, somewhat literally, that, after all, "the ass was right," Whistler rejoined, "But I fancy you will admit that this is the only ass on record that ever was right, and that the age of miracles is past." Leo X declared that "Erasmus injured us more by his wit than Luther by his anger." What could be more effective than Voltaire's comment when shown the poet Rousseau's ode on "Immortality"? "Voilà une lettre qui n'arrivera jamais à son addresse." But this "man of the century" was capable of turning the same weapon against himself. He was once showing to visitors his tomb built out from the outer wall in the church he had erected at Ferney and dedicated not to any saint, but *Deo solo*. "The wicked will say," he observed, "that I am neither inside nor yet outside of it." When a friend charged him with inconsistency on one occasion when he saluted the passing Host, Voltaire escaped with the remark, "We bow but we do not speak."

A sense of the more delicate kinds and gradations of humor is often an attribute of those who possess the logical faculty developed to a high degree; perhaps for this reason, that humor is allied to paradox.

"A little nonsense now and then
Is relished by the best of men."

Most notably was this true of the late Josiah Royce, a fact that will be confirmed by the opinion of anyone who may have listened to this philosopher's informal speech. The classic instance is the well-known trait in the character of Socrates, and finding expression as Socratic irony. That humor is related to matters of serious or sober intent was recognized by Boileau, when he wrote:

"Heureux qui, dans ses vers, sait d'une voix légère
Passer du grave au doux, du plaisant au sévère."

THE FALLACY OF ACCENT

An ambiguity directly related to the cases that have just been cited, gives rise to what is termed the fallacy of accent. De Morgan says: "A person who quotes another, omitting anything which serves to show the *animus* of the meaning; or one who without notice puts any word of the author he cites in italics, so as to alter its emphasis; or one who attempts to heighten his own assertions, so as to make them imply more than he would openly say, by italics, or notes of exclamation, or otherwise—is guilty of the **fallacia accentus.**"

Patronage is sometimes invited, or the attention of a patron aroused by means of an ambiguous sign or advertisement:

What do you think
I'll shave you for nothing
And give you a drink

a certain barber is supposed to have placed on the sign above his shop. A visitor enticed within by the prospect of such superlative benefits would not suspect the true rendering of the lines:

What! do you think
I'll shave you for nothing
And give you a drink?

Another case is the meaning of the command, "Drink ye all of it," giving rise as it has to semi-serious sectarian controversy, although the original Greek is unambiguous.

CONSCIOUS EXAGGERATION

Fallacies often result when the rhetorical device of conscious exaggeration is employed. Such instances may be serious or they may lack a serious intent. As a case in point consider a story of Fontenelle's from his *Dialogues of the Dead*. Dido has complained to one of her companion spirits that the poet Virgil had misrepresented her relations with Æneas, nay, had constructed the entire plot of the poem out of whole cloth:

"If my alleged flirtation were only plausible, I should not object to the suspicion; but he has given me for a lover, Æneas, a man who was dead three hundred years before I was born."

"What you suggest is, it must be allowed, in some sort an obstacle. But Æneas and yourself seem to have been made for each other. Both of you had been forced into exile; the destiny of each was set in foreign lands; you were a widow, he a widower; what a world had you not in common! It is true that you were born three hundred years later than he, but Virgil had reasons enough for bringing you together. The matter of three hundred years, when all is said, was not his affair."

"What sort of argument is this? What! Three hundred years are not three hundred years? And in spite of this obstacle two persons can meet and love?"

"It is certain that on this one point Virgil understood fine distinctions. He was certainly a man of the world. He would have it known, that in matters that concern the heart, one must not judge by appearances; that what is the least plausible in such circumstances is often the most true."

It is here impossible to recover adequately the grace and playfulness of the original text. The serious note is not struck until the final phrase is reached. It has been said that the French manner in the treatment of satire, irony, wit, and humor, is the result of a felicitous union of the Celtic temperament and the scholastic logic; the playful nature of the Gaul reacting to the discipline of intellectual rules. The enormous influence of logic in the formation of language and of literary taste, especially in the middle ages, has not always been understood and truly valued by historians.

Sometimes conscious exaggeration is to be taken seriously, as when its expression rises to poetic pitch. Compare in this connection the following passage from that extraordinary work the *Ecce Homo:* "With it (my Zarathustra) I bequeathed to my fellowmen the richest boon that has ever been conferred upon them. This book which lifts its voice above the centuries, is not alone the most sublime of all books, the book of mountain air—mankind as an actuality lies in the infinite depths beneath it—but it is as well the most profound of all, born in the innermost kingdom of truth, an inexhaustible well, into which no vessel can be lowered without coming up laden with gold and with goodness."

It is not, however, that such flights of fancy are always to be justified by mere poetic license. "They tell me," Nietzsche once observed, "that I am the event of the century. Nay, I may even constitute the necessary and fateful link that joins ten thousand centuries."

FALSE ASSOCIATION

One thinks of many instances, wherein one idea may be distorted and so misrepresented by mere association with another. This linking of two things which have nothing, or next to nothing in common, is very constantly employed. "Words, words, words," is sometimes an effective reply, whose success depends upon this same rhetorical trick of juxtaposition. Mr. Shaw likes to associate his own name with that of Shakespeare. Nietzsche exposes a case of this sort, when he exclaims indignantly, "Goethe *and* Schiller," and proceeds to enumerate other *ands* that are no less objectionable. But Nietzsche himself is a past master in the art of the ironical juxtaposition of terms. Compare in this context the following paragraph, which, however, is quoted only in part:

"My Impossibles. —— Rousseau: or the return to nature *in impuris naturalibus.* —— Dante: or the hyena who writes poetry among the tombs. —— Hugo: or the lighthouse on the sea of nonsense. —— George Sand: or *lactea ubertas,* in plain English: the ink-cow with plenty of beautiful ink."

THE EXAMPLE OF COURNOT

A fallacy analogous to the one that depends upon a false association of the sort described is mentioned by the philosopher Cournot and consists in a vulgar tendency to attach a providential or it may be a cosmic significance to coincidences that are merely fortuitous. He supposes the case in which two brothers serving in different armies, the one on the northern frontier, the other at the foot of the Alps, die in battle at the same hour. The circumstance is fortuitous because the causal chains, on which each happening depends are independent of one another. "It is because chance effects such combinations that they are rare, and it is because they are rare that they cause surprise." It would surprise no one if the two, members of the same corps, should die in the same battle on the same front. This does not exclude the possibility, on Cournot's views, that such fortuitous coincidences, are oftentimes determining causes. The shock to a parent, fatal we will say in the case supposed, might have been survived if the two deaths had been separated by an interval of time.

MANY QUESTIONS

To the fallacy of **many questions** are usually referred all cases in which too many meanings are contained, or in which the issue on that account is generally confused. A good example is the conversation in Hamlet between the grave-diggers. Here the first remark is not in the form of a question, but calls, none the less, for a reply. The fallacy might be termed equally well the fallacy of **many statements.** The example will illustrate, too, what is called in logic a case of **non sequitur.**

First Clo.—If the man go to this water and drown himself, it is, will he, nill he, he goes; mark you that; but if the water come to him, and drown him, he drowns not himself: Argal, he, that is not guilty of his own death, shortens not his own life.

Second Clo.—Will you ha' the truth on't? If this had not been a gentlewoman, she should have been buried out of Christian burial.

Other fallacies are committed without the intention that they be taken seriously. Polonius conveys a sly hint to Hamlet when he says:

"If you call me Jeptha, my lord.
I have a daughter that I love passing well;"

and Hamlet as slyly escapes by pretending that the remark contains a formal fallacy, for he rejoins:

"Nay, that follows not."

Nothing brings a conversation more abruptly to an end or more quickly disarms an opponent than the habit of taking him literally, for, arguing as it does a lack of imagination and even a lack of intellect, he is at once aware that the discussion cannot be maintained on the projected level. Nor does this habit characterize the unlettered only. Many an excellent scholar will betray the essential poverty of his mind by traits which point the same moral, by his attachment to words rather than meanings, or by his scorn of a style that is elegant because elevated, or, again, let us say, by his liking for what he calls the impersonal (i.e., literal) narration of history.

CONSCIOUS AMBIGUITIES

There is another large and important class of fallacies rather neglected, I think, by the logicians; arguments, which are not to be taken literally, but for a reason very different from the one that applies to the illustrations enumerated above. These are statements which are formally correct, but in which an ambiguity of terminology is intended, it may be for rhetorical purposes. Even the unlettered will not take you literally, if you remark that "business is business." The formal correctness of the phrase tends to force its acceptance, but it is quite evident that more is meant than meets the ear. Again, if I assert that "man is a vertical animal," it will be clear that more than a mere tautology is meant. It is said of Lincoln, while making a tour of the trenches after a brisk fight, that he remarked, with evident disgust of the whole affair (I quote from memory), "Anyone who likes this sort of thing must enjoy it very much." If it be said that "a man is a man for all that," it is to call attention to the fact that a tautology is not always true; that rather "a man is not himself sometimes." When the king and the others have left the play and Hamlet is left with Horatio, he says:

"For if the king like not the comedy,
Why then, belike, he likes it not, perdy,"

meaning that the burden of a bad conscience is the king's and not his.

Professor Stratton in an article appearing in the *Atlantic Monthly* says: "It is a prevailing belief that the mind is a convenient name for countless special operations or functions" and that these are independent. "When you have trained one of these you have trained that limited function and none other. What you do to the mind by way of education knows its place; it never spreads. You train what you train." Here the formal correctness of the tautology seems to reinforce the argument. But this view of the character of mind ignores many important facts. "The psychological experiments which have so troubled the waters of education prove that normally you train what you do not train." And, again, the conscious fallacy, the deliberate offense again logic, is correctly employed in favor of the opposite view.

In those verses of Lewis Carroll, which he calls "The Three Voices," the man in the piece, who has been accused by the lady of giving himself over to the exclusive instincts of his gourmandizing self, urges in his own defense that

"Dinner is dinner, tea is tea."

His defense is undermined, however, by her resolve to take his statement only at its face value, for she replies:

. . . "Yet, wherefore cease,
Let thy scant knowledge find increase;
Say men are men, and geese are geese."

Here the intent is not only to overthrow the opponent's argument, to render his contention impotent by refusing his implied ambiguity; but also to make a joke at the expense of logic itself, which is thus charged with giving us in its implications no information that we did not have before.

CHAPTER IV

SPECIAL CASES OF NON SEQUITUR

"IF facts do not agree with the theory, so much the worse for the facts." In common thinking this utterance is typical of any man who stands ready to pervert the truth. Facts constitute a last court of appeal to which theory must conform. The philosopher, however, does not altogether share this view of common sense. He is apt to point out that a fact derives its meaning in relation to some hypothesis, consciously or unconsciously assumed; he does not altogether indorse the view that facts are the hard and fast things that naïve reflection takes them to be. But this view of the case by no means implies that there is no such thing as misstatement of fact, which is, after all is said, the matter that concerns us here. Whistler used to take delight in making a mystery of the date and place of his birth, a "part of his habitual indifference to the sober requirements of those solemn metaphysical entities, time and space." "I never was born," he would say, "I came from on high."

MISSTATEMENT OF FACT

Misstatements of fact are commonly made under conditions of extreme provocation. In the heat of argument even obvious matters of fact may be denied, when a disputant sees that their admission would be damaging, or, it may be, fatal to his point of view. A Cambridge tutor, on being asked if he would admit that "the whole is equal to the sum of its parts," replied: "Not until I know what use you propose to make of the admission." Beaten in an argument, a disputant has even been known to take the ground:

"Certum est, quia impossibile est,"

a defense that yields to no direct attack. "Germany was amazed," said a well-known philosopher in 1914, "to find that suddenly all Germans were called Bernhardisten." In his "large circle of acquaintances," he said, he "knows not one who ever read Bernhardi, Nietzsche, or Treitschke."

"On peut dire que son esprit brille aux dépens de sa mémoire."

Adolph Lasson, a Privy Councillor and Professor of Philosophy in the University of Berlin, remarked at the time of the German invasion of Belgium in a much-quoted letter printed in an Amsterdam *Review:* "Our law is reason, our strength the strength of the spirit, our victory the victory of thought. We are truthful, our characteristics are humanity, gentleness, and conscientiousness, the real Christian virtues. In a wicked world we represent love and God is with us." Here, indeed, is the case of an old man whose view of facts is retrospective, who was thinking of a Germany of the past. This interpretation is well proven, for his attitude was hotly resented by some of the Germans themselves. Thomas Mann (*Neue Rundschau*, November, 1914) accepts the accusations of the Allies, declaring that the war is a war of German *Kultur* "against civilization":

"Denn der Mensch verkümmert im Frieden,
Müssige Ruh ist das Grab des Muts.
Das Gesetz ist der Freund des Schwachen,
Alles will es nur eben machen,
Möchte gern die Welt verflachen,
Aber der Krieg lässt die Kraft erscheinen" . . .

and announcing that German thought has no other ideal than that of militarism. On the other side a Frenchman declared that the Prussians do not belong to the Aryan race; that they descend in the direct line from the men of the Stone Age called Allophyles and that

"Le crâne moderne dont la base, reflet de la vigueur des appétits, rapelle le mieux le crâne de l'homme fossile de la Chapelle-aux-Saints, est celui du prince de Bismarck."

But the tradition of the great Goethe was not altogether dead even in those days. While these ragings were going on, Hermann Hesse composed his "prayer to the peace."

> "Jeder hat's gehabt,
> Keiner hat's geschaetzt,
> Jeden hat der süsse Quell gelabt,
> O wie klingt der Name Friede jetzt!"
>
> "Klingt so fern und zag,
> Klingt so tranenschwer,
> Keiner weiss und kennt den Tag,
> Jeder sehnt ihn voll Verlangen her." . . . .

IGNORATIO ELENCHI

The fallacy of **ignoratio elenchi**, or ignorance of the refutation, occurs whenever one is convicted of arguing to the wrong point, or whenever the inference to be drawn is generally confused. "I am not ridiculed," said Diogenes in reply to certain ones who derided him. One cannot be ridiculed unless the ridicule applies. Cases in which it is not easy to say what fact is most plausibly inferred from a given utterance and which are really concealed forms of the **ignoratio elenchi** might readily be given. Richard Porson on one occasion was invited to stay to dinner. "Thank you," he answered, "I dined yesterday." Only a person conversant with the scholar's habits of routine might safely have divined his meaning. Porson dined and fasted at irregular intervals. De Morgan cites the following instance:

"If a man were to sue another for debt, for goods sold and delivered, and if the defendant were to reply that he had paid for the goods furnished, and plaintiff were to rejoin that he could find no record of that payment in his books, the fallacy would be palpably committed. The rejoinder supposed true, shows that either the defendant has not paid, or plaintiff keeps negligent accounts, and is a dilemma, one horn of which only contradicts the defense. It is plaintiff's business to prove the sale from what *is* in his books, not the absence of payment from what is *not;* and it is then defendant's business to prove the payment by his vouchers." The same author remarks:

"A great deal of what is called evasion belongs to his head, or to that of the *ignoratio elenchi* as the sophist answers. The advocates, for instance, of the absolute unlawfulness of war never tell, unless pressed, what they think of the case of resistance to invasion. Is the country to be given up to the first foreigner who comes for it? Sometimes the extreme case comes into play: sometimes the assertion that no one will come; which is irrelevant as to the question what would be right if he did come."

Another example, in which an opponent has inferred more than was intended, may also be cited: "A writer disclaims attempting a certain task as above his powers, or doubts about deciding a proposition as beyond his knowledge. A self-sufficient opponent is very effective in assuring him that his diffidence is highly commendable, and fully justified in the circumstances."

IMPLICATIONS THAT ARE NOT DESIGNED

An excellent example of the *ignoratio elenchi* is contained in one of Hamlet's replies to Horatio and Marcellus. It was a favorite trick of the Prince of Denmark to return an answer that had nothing to do with the case, when he wished to conceal the true nature of his opinion. He says:

"There's ne'er a villain dwelling in all Denmark
But he's an arrant knave."

And to this seemingly irrelevant remark Horatio rejoins:

"There needs no ghost, my lord come from the grave,
To tell us this."

Now, one of the speediest means of bringing a discussion to an end is to accept an opponent's assertion in its literal meaning and by refusing to draw any inference at all. Accordingly, Hamlet remarks:

"Why, right; you are in the right;
And so, without more circumstance at all,
I hold it fit, that we shake hands, and part."

Again, a conviction or an act may be misrepresented, or an opinion may be retorted, by seeking an implication that was not designed. John Adams was once dining at the house of Judge Paine, who was a loyalist. When the host gave as a toast "the King," some of the guests would have refused to comply, had not Adams insisted. The latter then proposed a toast to "the Devil." The host was about to demur when his wife intervened: "My dear, as the gentleman has seen fit to drink to *our* friend, let us by no means refuse, in our turn, to drink to *his*."

The story is told of Whistler, that one day in a shop a customer, mistaking him for a salesman, rushed up to him and remarked: "I say, this 'at doesn't fit." "Neither does your coat," said Whistler, eying him critically.

A case in which the traits of a man's character are somewhat ironically inferred from his behavior, and which might be listed under this head, is contained in the following anecdote: The dramatist Piron and the poet Jean Baptiste Rousseau were walking one day in a solitary place. At the sound of the Angelus Rousseau fell on his knees. "It is unnecessary," Piron remarked, "God alone observes us."

POPULAR JUDGMENTS

Intolerance in the search for truth is often mistakenly charged by popular opinion to the expert or specialist in some department of knowledge, because he rejects without the trouble of examination the claims of some self-taught thinker. Cures extravagantly alleged to have been effected in independence of biological law, or the discovery of unknown facts through the revelations of returned spirits, suggest themselves as possible cases in point. William James urged what he called "fair play" in matters of this sort, but the nature of his liberality has been conveniently misunderstood and misrepresented by partizans on both sides. De Quincey's treatment of the case of those who pretend to have "squared the circle" might well serve as a model of refutation for claims thus mistakenly advanced:

"The general or unmathematical public are in a continual delusion about the nature of the barrier which separates us from the perfect solution of these problems. Every six months the newspapers announce that some self-taught mathematician or original genius has succeeded in squaring the circle. Upon this, the mathematician, without troubling himself to inquire into the particular form of this man's nonsense, contents himself with laughing. And to this laugh the non-mathematical observer replies by saying, or thinking, that *previous* to inquiry such a contemptuous dismissal of any man's pretensions is illiberal. . . . The man might fairly protest: Measure the value of my talent by the discovery I offer, and not the value of my discovery by my talent wantonly and invidiously assumed. Or, he might say (referring to Archimedes, Leibnitz, Euler, who had failed): Not as equal, still less as superior to these great men, but as standing on their shoulders, I pretend to have seen farther than they; or . . . simply insisting on the accidental *difference* of the station from which he had contemplated the question at issue; on any of these grounds, the candidate for the honors of discovery might roll back the burden of invidious feeling upon those that laughed at him *in limine,* were the barrier between us and the discovery of these truths merely subjective. But it is not so. The barrier is objective; it lies not in the person attempting but in the thing attempted. . . . The objection, therefore, to a pretended squarer of the circle is not: You, sir, by adding to our knowledge in a point impregnable to others, would compel us to believe you a greater than the greatest of those we honor; But this: You, sir, by propounding a discovery that would unsettle the foundations of our former knowledge, oblige us to disbelieve you on the faith of that very science to which you do and must appeal."

De Morgan has this to say on the same theme: "All the makers of systems who arrange the universe, square the circle, and so forth, not only comfort themselves by thinking of the neglect which Copernicus and other real discoverers met with for a time, but sometimes succeed in making followers. These last forget that for every true improvement, which has been for some time unregarded, a thousand absurdities have met that fate permanently. . . . Doctor Johnson tells a story of a lady who seriously meditated leaving out the classics in her son's education, because she had heard Shakespeare knew little of them. Telford is a standing proof (it is supposed by some) that special training is not essential for an engineer."

FALLACY OF AFFIRMING THE CONVERSE

There exists a very human tendency to affirm the converse of some proposition whose truth is granted, when the converse seems to support some fact that we should like to believe. Supposing it true that most artistic geniuses wear the hair long, some men try to create an impression of their talents by allowing the hair to grow or by wearing a wig. As already cited, it is supposed that the little Latin and less Greek that entered into the education of a Shakespeare has only to be adhered to in principle in order to produce his equal. We are constantly tempted to assume the converse of well-established truths. Consider the following passage from Emile Boutroux:

"The best men in a nation, says Renan, are those it crucifies. Martyrdom is the ransom of superiority. Death, then, is a witness to the effort made by the living being to rise above the environment in which he was born. Defeat is the mark of his greatness."

This passage is full of statements that are converted with some show of truth and analogous ones have become proverbial. The assumption of martyrdom in connection with the difficulties and distresses of daily experience is supposed to suggest vaguely a being morally superior to the petty circumstances in which he finds himself.

CHAPTER V

BEGGING THE QUESTION

A NOT uncommon fallacy is the one wherein we apply the axiom that "the whole is equal to the sum of its parts" to wholes that are not merely collections, but highly organized collections as well; wherein we assume that the properties of the whole are to be found in the parts, or that the properties of the parts added together will yield the properties of the whole, and conversely. Thus, the properties of water are not those of hydrogen and oxygen taken together. The characteristics of a jury are not to be found among the individuals that make it up.

In the second book of the Republic of Plato Socrates proposes to resolve the difficult problem of the nature of the just man by considering the same problem in relation to the larger whole of the state. He says:

"We speak of justice as residing in an individual mind and as residing also in an entire city do we not? . . . Perhaps, then, justice may exist in larger proportions in the greater subject, and thus be easier to discover; so, if you please, let us first investigate its character in cities; afterwards, let us apply the same inquiry to the individual, looking for the counterpart of the greater as it exists in the form of the less."

THE MATERIALISTIC FALLACY

While Socrates in this instance does not actually commit the fallacy in question, a breach of the proposed rule is certainly suggested. Emile Durkheim, the eminent sociologist, insists upon this point again and again. On his view a social fact is not a simple sum of facts that concern the individual. The properties of the whole are not here the sum of the properties of the parts that make it up. "Private sentiments do not become social save by combining under the action of forces *sui generis* which develop association; as the result of these combinations and mutual alterations, they become something else." Sociology deals with collective representations and therefore does not recognize the method of psychology or of anthropology as sufficient to deal with its subject-matter. "The causes of social facts must always be sought among other facts themselves social in character." When "a social fact is explained directly by psychical phenomena, one may rest assured the explanation is false." Thus when one ascribes the artistic character of Athenian civilization to the congenital powers of the race, "one proceeds as they did in the middle ages," by begging the question, "when they explained fire by phlogiston and the effects of opium by its dormative virtue." It is necessary "to explain the phenomena which are produced in the whole by the properties characteristic of the whole, the complex by the complex, social facts by society."

A FORMAL FALLACY IN EUCLID

If two or more propositions are true together, then any one may be taken to be true separately. This, like some others of the axioms of logic, may appear trivial at first blush. An important fallacy, however, originates in connection with it and in the following manner: Suppose that some proposition, whose truth we are seeking to establish, is only another form of some assertion, or set of assertions, whose truth is granted at the outset of our proof. It will then be easy to obtain the conclusion desired from the premises assumed, and if we should assume the right to assert the conclusion by itself, that is, in independence of the premises, we should commit the fallacy that goes by the name of **petitio principii.**

This breach of the rules of correct inference, which is commonly known as the fallacy of "begging the question," consists then in covertly assuming the truth of some principle equivalent to the one we are seeking to establish. Serious cases of this error might be drawn from the history of the sciences or from the philosophical disciplines. Thus, in the sixteenth proposition of the first book of his *Elements,* Euclid fails to state one of the axioms that is essential to the rigor of his proof, the axiom that two straight lines cannot intersect in more than one point. Now this omission entailed very serious consequences. Many eminent mathematicians of modern times, overlooking the error and assuming this proposition to be demonstrated, were able to establish certain truths which depend upon the axiom in question, such as "the fourth angle of a trirectangular quadrilateral cannot be obtuse" and "the angle sum of the triangle cannot be greater than two right angles," propositions inconsistent with a geometry that is now known to be possible and known as the geometry of Riemann. A somewhat similar fallacy is committed by the Italian geometer Saccheri in the thirty-third proposition of his book entitled *Euclid Freed of Every Blemish* (1733) by means of which he establishes the truth of Euclid's fifth postulate.

Most of the apparent demonstrations of Euclid's parallel axiom are cases of *petitio principii,* and such instances are very common indeed in the history of geometry. Always the demonstrator assumes tacitly some axiom that is only a disguised expression of the principle he seeks. For two thousand years the error has been committed again and again. In regard to his own efforts to establish a rigorous proof of this principle, the great geometer Gauss said in the year 1799: "Certainly I have come upon much that for the majority would pass as a proof, but in my eyes demonstrates nothing." The story is told of Lagrange that he on one occasion presented a memoir on the theory of parallels to the members of the French Academy, but withdrew the manuscript, when halfway through the reading, with the remark, "Il faut que j'y songe encore."

FURTHER EXAMPLES OF PETITIO PRINCIPII

Karl Pearson in his *Grammar of Science* is guilty of a fallacy of this same sort. He compares the mind to a clerk in a telephone exchange, who receives his messages at the brain terminals of the sensory nerves. "Messages in the form of sense impressions come flowing in. . . . But of the nature of things-in-themselves, of what may exist at the other end of our system of telephone wires, we know nothing at all."

It is evident, if we take this analogy seriously, that we have assumed not only the existence of an external world, but also that we know at once a great deal about it. We imagine we know, for example, that the nerves are *like* wires, however much we may be deceived by the messages which they convey. Of this case Professor Fullerton remarks: "It is interesting to see how a man of science, whose reflections compel him to deny the existence of the external world, that we all seem to perceive and that we somehow recognize as distinct from anything in our minds, is nevertheless compelled to admit the existence of this world at every turn."

Kant's second argument for the *a priori* character of space in the second edition of the *Kritik* seems to be a case in kind. He says: "One can never represent to himself that there is no space, although one can easily imagine that no objects are to be found in space." This means, it may be presumed: One can imagine objects moved out of a given space, but one cannot imagine space moved out of space. If this be a fair statement of the case, then we are asked to do that, which, by definition, we cannot do, and the odd compulsion of the argument contains no other kind of necessity than this.

Henri Poincaré mentions a case in which we should "beg the question" if we should try to settle a certain issue experimentally. We should always prefer to attribute the outcome of an experiment that would decide between the Euclidean and the Riemannian hypotheses in geometry, to the errors inherent in observation. But the very condition which is in question, would really be read into the conditions that surround the experiment and so would naturally be found to be verified there.

DEFINITION IN A CIRCLE

This case is of the highest importance in relation to the method of science, in particular in its relation to those delicate questions which center about the foundations of mathematics. The Italian geometer Veronese has been guilty of a fallacy of this sort. Thus, he defines the equality of numbers in the following way: "Numbers whose units correspond to one another uniquely and in the same order and of which the one is neither a part of the other nor equal to a part of the other are equal." Regarding this definition Georg Cantor observes:

"This definition of equality contains a circle and thus is meaningless. For what is the meaning of "not equal to a part of the other" in this addition? To answer this question we must first know when two numbers are equal or unequal. Thus, apart from the arbitrariness of his definition of equality, it presupposes a definition of equality, and this again presupposes a definition of equality, in which we must know again what equal and unequal are, and so on *ad infinitum*. After Veronese has, so to speak, given up of his own free will the indispensable foundation for the comparison of numbers, we ought not to be surprised at the lawlessness with which, later on, he operates with his pseudo-transfinite numbers, and ascribes properties to them which they cannot possess simply because they themselves, in the form imagined by him, have no existence except on paper. Thus, too, the striking similarity of his "numbers" to the very absurd "infinite numbers" in Fontenelle's *Géométrie de l'Infini* (Paris, 1727) becomes comprehensible. Recently W. Killing has given welcome expression to his doubts concerning the foundation of Veronese's book in the *Index lectionum* of the Münster Academy for 1895-1896."*

THE INCOMPLETE DISJUNCTION

The fallacy that results when we are asked to choose between two alternatives that do not exhaust the possibilities of any given case, is so common and so treacherous that it deserves to be well studied and understood. Thus Herbert Spencer in his Sociology attacking the "great man theory" of history, employs a disjunction that is incomplete. He says: "Whence comes the great man? . . . The question has two conceivable answers: his origin is supernatural or it is natural. Is his origin supernatural? Then he is a deputy god and we have theocracy once removed. . . . Is this an unacceptable solution? Then the origin of the great man is natural; and immediately this is recognized, he must be classed with all other phenomena in the

* Cantor's *Theory of Transfinite Numbers* trans. by Jourdain, Open Court, 1915.

society that gave him birth as a product of its antecedents."

A critic of this argument might suggest the following objections: If the term "natural origin" refers to an "environment" external to the man himself the argument is clearly false for the disjunction is not complete. It is not clear that what "original nature" one man brings with him into his environment is the same as what another *might* have brought. If on the other hand the "natural origin" refers to an environment so broadly defined as to include the man himself, then Spencer is open to the very charge of vagueness which he somewhat arrogantly urges against the opponents of his view. William James who quotes and criticizes this passage, exclaims:

"This outcry about the law of universal causation being undone . . . makes one impatient. These writers have no imagination of alternatives. With them there is no *tertium quid* between outward environment and miracle. *Aut Cæsar, aut nullus! Aut* Spencerism, *aut* catechism!"

THE FALLACY OF ACCIDENT

Many are the forms which the fallacy of accident may take on. I recall an argument, in which a man who was a good Grecian, though of poor logical pretentions, maintained that the term "classics" was only *properly* applied to the works of Greek and Roman writers. His general position in the matter amounted briefly to this: that he believed in calling things by their *right* names; that the common sense significance of terms is weakened or altogether lost by arbitrarily making them mean too much or too little. And yet this man must have known that the term "classic" was applied by the Romans to writers of the first rank, as to those who had "class." "Where Macgregor sits, there is the head of the table." It was the spirit of this maxim which this stubborn antiquarian denied.

When a man holds thus insistently to his own meaning of a word, it is wise to look about for hidden motives, and in his case these were easily found among his known opinions. Largely ignorant of the literature of modern Europe and, in any case, in no position to argue successfully that there are no modern works comparable to those of Greece or Rome, that is to say, no modern classics, he insisted upon the meaning of the word in its *accidental* sense; but with this end in view: The word once accepted in its accidental sense, he would have applied, either then, or at some convenient later date, in its *essential* meaning; he would then have enforced upon us all his favorite thesis, that only the *"classical"* writers have class, that is, possess preëminent rank. This *homo unius libri,* like others of his sort, was a stubborn disputant.

This case will illustrate one form of the fallacy of accident. As a further example consider a tale from Boccaccio which is cited by De Morgan:

"A servant who was roasting a stork for his master was prevailed upon by his sweetheart to cut off a leg for her to eat. When the bird came upon the table, the master desired to know what had become of the other leg. The man answered that storks had never more than one leg. The master very angry, but determined to strike his servant dumb before he punished him, took him next day into the fields where they saw storks, standing each on one leg, as storks do. The servant turned triumphantly to his master; on which the latter shouted and the birds put down their other legs and flew away. "Ah, sir," said the servant, "you did not shout to the stork at dinner yesterday; if you had done so, he would have shown his other leg too."

The servant had humorously assumed that what can be predicated in general of storks, can be predicated of roasted storks as well. Another example of De Morgan's deserves to be cited as well:

"The law in criminal cases demands a degree of accuracy in the statement of the *secundum quid,* which many people think is absurd. . . . Take two instances as follows: Some years ago, a man was tried for stealing a ham, and was acquitted on the ground that what was proven against him was that he had stolen a portion of a ham. Very recently a man was convicted of perjury, *in the year* 1846, and an objection (which the judge thought of importance enough to reserve) was taken, on the ground that it ought to have been in the *year of our Lord* 1846. . . . In the two instances, which by many will be held equally absurd, a great difference will be seen by anyone who will imagine the two descriptions, in each case, to be put before two different persons. One is told that a man has stolen a ham, another that he has stolen a part of a ham. The first will think he has robbed a provision warehouse, and is a deliberate thief; the second may suppose that he has pilfered from a cook shop, possibly from hunger. As things stand, the two descriptions may suggest different amounts of criminality, and different motives. But put the second pair of descriptions in the same way. One person is told that a man perjured himself in the year 1846; and another that he perjured himself in the year of our Lord 1846. As things stand, there is no imaginable difference; for there is only one era from which we reckon." Mr. Alfred Sidgwick says:

"One reason why clever young people are specially addicted to wordiness is that their confidence in the axioms of logic is not yet much tempered by experience. . . . As the old logicians used to put it, *A secundum quid* is not the same as *A simpliciter*. But modern logic leads us to doubt whether, in the world of fact, there is (strictly speaking) any such thing as *A simpliciter,* and thus to doubt whether the law of identity has any application at all except by consent of both parties to a dispute. . . . Where we are dealing with things that can be weighed and measured in the laboratory; where analysis and synthesis can be instrumentally checked and corrected at every step; this risk, though never entirely absent, is at its lowest, and therefore definition of the terms is seldom called for. Where, on the other hand, large and obviously complex matters, such as political and industrial phenomena, are dealt with, and much artificial simulation is needed to enable us to deal with them at all, the risk of verbalism is at its height. Every *A* in politics and economics is *A secundum quid,* and yet we are constantly tempted to treat it as *A simpliciter.*"

A SOPHIST IN SEARCH OF A DINNER

A case that properly falls under this head, that of the **fallacia accidentis,** is the tale told of a certain sophist, who was accustomed to approach his opponent bearing in his hands a covered dish. He would then inquire of the latter if he knew of a certainty that *all* chicken meat is nourishing. His opponent would at once subscribe to this as a medical fact of the most indubitable sort; for this principle might be termed an axiom of the Greek pharmacopœia. Asked if he knew that what was in the dish was nourishing and on receiving a negative reply, the knavish Athenian would remove the lid and display the chicken meat contained therein. It was vain for the victim of the argument who had thus contradicted himself, to plead a prior ignorance of the fact. What is known in general, or as a principle, is known in each one of its particular applications. What is known to be true of *all* chicken must be known to be true of this sample along with the rest.

CHAPTER VI

HUMOROUS SITUATIONS AND LITERAL STATEMENT

WHILE it is the intention of this essay to list and in part to classify some of those ambiguities that lead to erroneous opinions, we can hardly fail to note cases that present a humorous as well as a serious side. Such illustrations may be multiplied without limit. They are the stock in trade of the writer of comedies.

AMBIGUITIES THAT CAUSE HUMOROUS SITUATIONS

Horace Greeley once received an urgent invitation to lecture in a distant state. He replied:

"*Dear Sir,*—I am overworked and growing old. I shall be sixty next Feb. 3rd. On the whole, it seems that I must decline to lecture henceforth, except in this immediate vicinity, if I do at all. I cannot promise to visit Illinois on that errand—certainly not now."

While the defects of Greeley's chirography were notorious, and were of course through his relations with compositors, if for no other reason, well known to himself; nevertheless, the reply which he received by return mail, must have furnished some surprise. The substance was this:

"Dear Sir,—Your acceptance to lecture before our association came to hand this morning. We would say that the time you propose, Feb. 3rd, and terms, sixty dollars, are entirely satisfactory to us. As you suggest, we may be able to obtain for you other engagements in this immediate vicinity."

Another case of ambiguity, which is worth recording for its own sake, is the story told of Sir Walter Raleigh, who recently visited an American university in order to deliver a lecture. The president, preoccupied with the necessity of keeping a pressing engagement, had delegated one of his younger instructors to meet the visitor at the train. "You will have no difficulty in recognizing this gentleman," he said. "Go to him directly, introduce yourself, and present my apologies for not having come." Now as the matter turned out, the expected visitor had missed the train, and the person addressed, of distinguished appearance, was a well-known banker of New York. This person had been warned in advance of the pranks which undergraduates are known to practice and was determined to nip such undertakings in the bud. "Sir Walter Raleigh, I presume?" said the young instructor, addressing his distinguished stranger. "You are in error, sir," said the other. "I am Christopher Columbus. You will discover Sir Walter in the smoking car playing poker with Queen Elizabeth."

The subject of student pranks suggests another case which depends upon an ambiguity of an unusual kind. A woman who had been in the habit of letting rooms in the Latin Quarter to students, had become much attached to a tiny tortoise recently acquired. To her amazement the animal grew rapidly week by week until it had attained enormous size. Her roomers had replaced the pet each day by a larger duplicate. Then, when the maximum had been reached, this mushroom of the animal kingdom began to shrink. Gradually, becoming smaller and smaller for a number of weeks, the reptile slowly and unostentatiously resumed its original size; and in this state it remained to the time of its death some years later on.

AGREEMENT TO DISAGREE

While agreement on the part of disputants is commonly the end of discussion, rare are the cases where discussion might not be prolonged. Agreement in this sense is scarcely ever real. There are cases, indeed, where it is purely verbal; and there are cases where its verbal character is so patent that a continuation of the argument is actively provoked. The Frenchman Fontenelle detested controversy to the point that he would refuse to differ with any chronic organizer of disputes,

"Where nature's end of language is declined,
And men talk only to conceal the mind,"

declaring habitually that "all things are possible and that every man is in the right." He knew how to vary this formula, however, so that sometimes his answer had a sting. In his old age he was advised by his physician that the action of coffee on the human system is like that of a slow poison and should be abjured. "I quite agree with you," he replied, "and I have stoutly held to this opinion for more than eighty years."

But the refusal to meet an opponent on ground of his own choosing may take on other forms. When the Curé Freudenberger published a book entitled ***William Tell: a Danish Fable,*** the Swiss canton of Uri caused the work to be burned. To this case one might well apply the saying of Camille Desmoulins, one of the figures of the Revolution, and which has become proverbial:

"Brûler n'est pas répondre."

This reply he made to Robespierre, whose measures the Dantonists were then opposing.

WHISTLER VERSUS RUSKIN

The refusal to take a statement in its literal sense is always a tax on a man's intellect and imagination. It is commonly a simple affair, therefore, to get the average man to subscribe to the specious expression of any half truth. In a case at law both the prosecuting attorney and counsel for the defense may be actively at pains to convince the jury by giving a color of truth to statements of this kind. Such a general policy may prove to be dangerous, however, for once successfully retorted, such assertions argue only a certain poverty of intellect on the part of him who stands back of them. A part of the cross-examination in Whistler's famous libel suit against Ruskin will serve to illustrate this point. The reader will observe that the Attorney-General is consistently given over to the policy of the literal statement of half truths and that Whistler as consistently exposes his purposes:

"Now, Mr. Whistler, can you tell how long it took you to knock off that nocturne?"

. . . "I beg your pardon?" (*Laughter.*)

"Oh! I am afraid that I am using a term that applies rather perhaps to my own work. I should have said, "How long did it take you to paint that picture?"

"Oh no! permit me, I am too greatly flattered to think that you apply to work of mine, any term you are in the habit of using with reference to your own. Let us say then how long did it take to—'knock off,' I think that is it—to knock off that nocturne; well, as well as I remember, about a day."

"Only a day?"

"Well, I won't be quite positive; I may have still put a few more touches to it the next day if the painting were not dry. I had better say then, that I was two days at work on it."

"Oh, two days! The labor of two days, then, is that for which you ask two hundred guineas!"

"No;—I ask it for the knowledge of a lifetime." (*Applause.*)

"You have been told that your pictures exhibit some eccentricities?"

"Yes; often." (*Laughter.*)

"You send them to the galleries to incite the admiration of the public?"

"That would be such a vast absurdity on my part, that I don't think I could." (*Laughter.*)

"You know that many critics entirely disagree with your views as to these pictures?"

"It would be beyond me to agree with the critics."

"You don't approve of criticism then?"

"I should not disapprove in any way of technical criticism by a man whose whole life is passed in the practice of the science which he criticizes; but for the opinion of a man whose life is not so passed

I would have as little regard as you would, if he expressed an opinion on law."

"You expect to be criticized?"

"Yes; certainly. And I do not expect to be affected by it, until it becomes a case of this kind. It is not only when criticism is inimical that I object to it, but also when it is incompetent. I hold that none but an artist can be a competent critic."

LITERAL STATEMENT

The expression, "He calls a spade a spade," is employed to connote one whose habit of thought is literal, or, again, one whose frankness is coarse or rude. "Sire, I shall often displease you, but I shall never deceive you," said Dumouriez to Louis XVI. An illustration of the manner in which a discussion is effectually ended by taking an opponent at his word is contained in the following anecdote: Sydney Smith, the English divine, had won an argument, from an acquaintance. The latter said, "If I had a son who was idiot, I would make him a parson"; and Smith replied, "Your father was of a different opinion."

Matters which no one thinks of taking literally when so regarded may sometimes produce a surprising effect. The epigram of Nietzsche's: "It is a nice distinction, that God learned Greek, when he wished to turn author, and—that he learned it no better," is not free of the possibility of ambiguous interpretation.

Hamlet, in order to escape the questionings of a too inquisitive interlocutor, constantly employs the device of replying literally:

Pol.—Do you know me, my lord?
Ham.—Excellent well, you are a fishmonger.
Pol.—Not I, my lord.
Ham.—Then I would you were so honest a man.

* * * * * *

Pol.—. . . What do you read, my lord?
Ham.—Words, words, words.
Pol.—What is the matter, my lord?
Ham.—Between who?
Pol.—I mean, the matter that you read, my lord?
Ham.—Slanders, sir: for the satirical rogue says here that old men have grey beards. . . .

THE PRACTICAL MIND IS LITERAL

Romain Rolland in his unsurpassed arraignment of German acts and purposes at the time of the invasion of France exclaims in a dramatic passage: "Necessity knows no law. . . . Behold the eleventh commandment, the message you bring to the world to-day, Sons of Kant. . . . *To be or not to be say you?* Nay, 'tis not enough."

The faith of practical politicians in the creed of ***Realpolitik*** which is here denounced, as well as the worship of force generally and the accompanying contempt for ideas and ideals is, in effect, characteristic of the literal or practical mind. In violent contrast to such views is the superb optimism of Hegel as expressed in a passage from his *Philosophy of History:* "For, like the soul-conductor, Mercury, the idea is in truth the leader of peoples and of the world; and spirit, the rational and necessitated will of that conductor, is and has been the director of the events of the world's history."

AN EXAMPLE FROM THE TEACHING OF JESUS

The fundamental idea of the teaching of Jesus is the founding of the Kingdom of God. But in the mind of Jesus, or in the way in which he expressed the idea, the notion was ambiguous. Sometimes it means the reign of the poor and the disinherited; sometimes the literal accomplishment of the apocalyptic visions of Daniel; or again, the kingdom of souls and the deliverance of the spirit. Jesus himself was fully alive to the need of precision in a matter of such grave importance, and perhaps the most profound of all the reconciliations he attempted is contained in the saying:

"Neither shall they say, Lo here!
or lo there! for behold the kingdom
of God is within you."—*Luke XVII,* 20-21.

The kingdom of God is only to be found deep down in the heart of each one of us. The apocalyptic conception, that the actual order of humanity approaches its end, is on this interpretation a metaphor. The palingenesis, accompanied by agonies and calamities, is a rebirth of the spirit. The early Christian community, however, was preoccupied with the *literal sense* of these declarations of the Master. Only as time passed and the apocalyptic vision was not fulfilled, was the meaning of the prophecy given another sense.

AMBIGUITIES FROM QUESTIONS OF FACT

Sometimes the factors that conspire to produce an event are supposed to foreshadow an ambiguous result. This is the case when not all of the factors are known, so that the outcome is not precisely determined. It is notorious that a jury under such circumstances will condemn a man for a criminal act, whereas had more of the factors which controlled his behavior been known, the judgment might have been reversed. If the crime itself is an acknowledged fact and the criminal identified, the defense will be at pains to bring these unknown factors clearly to the light, preferring, when these are at hand, rational motives, like the defense of honor, family or life, or else compulsions in the man's environment that fairly lie beyond his own control.

Here the interest in precision, in so far as it concerns the fact in question, lies in demonstrating that the matter could not well have turned out otherwise, or, at the worst, that the *prima facie* responsibility of the accused is more than what is actually the case.

Sometimes the essential factors which determine an event are not only unknown but in the nature of things unknowable. Again, many so-called indubitable facts that we seem bound to accept, are only true because they are matters of definition. The philosopher may argue that all men seek only their own interest, that all acts altruistic in appearance are only concealed cases of self-seeking; but care must be taken to see if this is only forced on us because of the way in which he defines his terms; to see if the cogency of his argument is more or no more than the cogency of tantologous statement. "Swaggering paradoxes, when examined," said Burke, "often sink into pitiful logomachies."

There is a direct connection between our knowledge of fact and the appearance of ambiguity in the meaning of terms. In a metaphorical sense facts themselves become ambiguous as their nature becomes better and better known. For a long time in the history of chemistry everything that was not a solid or a liquid was called "air." This term became more and more ambiguous as different "airs" came to be distinguished. If a candle is burned in air under glass, the inert residuum possessing peculiar properties called for a special designation and was termed "nitrogen." At the very end of the century it was discovered that this term too was ambiguous, for it was proven that the inert residuum is not one chemical substance; that the new element argon is one of its constituents. Illustrations of this sort from the history of science might be greatly multiplied. The effort to hold consistently to the "molecular hypothesis" has led to subdivisions of the atom. The experiment of Michelson and Morley leading finally to the recent theory of relativity has introduced ambiguities into the meaning of mass, length and time.

LITERAL STATEMENT AND THE INTEREST OF TRUTH

The assumption of a blunt or literal attitude of mind is sometimes justified because it is taken to serve better the ends of truth. And, again, this tender sentiment for the maxim of the *magna est veritas* may only serve to cover up a playful mischief-making. The following (somewhat rearranged for our purpose) is taken from the life of Tristram Shandy:

"This contrariety of humors betwixt my father and my uncle, was the source of many a fraternal-squabble. The one could not bear to hear the tale of family disgrace recorded; and the other would scarce ever let a day pass to an end without some hint at it.

"My father, as I told you, was a philosopher in grain, speculative, systematical; and my aunt Dinah's affair was a matter of as much consequence to him, as the retrogradation of the planets to Copernicus: the backslidings of Venus in her orbit fortified the Copernican system, and the backslidings of my aunt Dinah in her orbit did the same service in establishing my father's system.

"In any other family-dishonor my father I believe had as nice a sense of shame as any man whatever. Nor would he, I dare say, have divulged the affair or taken the least notice of it to the world, but for the obligation he owed, as he thought to truth. *Amicus Plato,* my father would say, construing the words to my uncle Toby as he went along: *Amicus Plato,* that is, Dinah was my aunt; *sed magis amica veritas,* but Truth is my sister.

"For God's sake, my uncle Toby would cry, and for my sake, and for all our sakes, my dear brother, do let this story of our aunt's and her ashes sleep in peace. What is the character of a family to an hypothesis? my father would reply. The *life* of a family—my uncle Toby would say. Yes, the life, my father would say, maintaining his point. In my plain sense of things, my uncle Toby would answer, every such instance is downright murder, let who will commit it. There lies your mistake, my father would reply; for in *Foro Scientia* there is no such thing as murder; 'tis only death, brother."

And Sterne resumes: "As not one of our logical writers, or any of the commentators upon them, that I remember, have thought proper to give a name to this species of argument, I here take the liberty to do it myself, for two reasons: first, that to prevent all confusion in disputes, it may stand as much distinguished forever from every other species of argument, as the *Argumentum ad Vericundiam, ex Absurdo, ex Fortiori,* or any other argument, as the *Argumentum Fistulatorium,* and no other; and, secondly, that it may be said by my children's children that their learned grandfather had invented a name and generously thrown it into the treasury of the *Ars Logica,* for one of the most unanswerable arguments in the whole science; and if the end of argument is more to silence than convince, they may add, if they please, to one of the best arguments too."

CHAPTER VII

IN DEFENSE OF PREJUDICE

"For, of a truth, Love and strife were aforetime and shall be; nor ever, methinks, will boundless time be emptied of that pair.'

JOHANN HEINRICH LAMBERT in his *Theory of Parallels* published in 1786 after the death of the author, discovered the existence of a geometry many of whose characteristic propositions stand in contradiction to those of Euclid's own. Some of these paradoxical results possessed for his mind such a fascination in themselves that he would fain have held them to be true. In regard to one of these consequences he observed: "This result is so attractive that it easily creates the wish that the hypothesis on which it depends, might yet be true." But "these are *argumenta ab amore et invidia ducta,* which must be banished from geometry, as they must be banished from all other sciences." Had this eminent mathematician only had the courage to follow his instinct in this instance he would have been regarded as the discoverer of non-Euclidean geometry.

THE COLD COMFORT OF SCIENTIFIC INDIFFERENCE

This case illustrates pointedly the thesis of the present chapter, the thesis that the desire on the part of the scientist that something should be true, or, if you like, his *prejudice* in favor of a certain truth, facilitates its discovery. The great discoveries in science are very commonly the result of a certain bias on the part of the discoverer. Science has, through its technique, endeavored to impress upon the experimenter an indifference toward the outcome of his experiment. Regarding this so-called indifference of the objective observer, William James has this to say:

"For purposes of discovery such indifference is to be less highly recommended, and science would be far less advanced than she is if the passionate desires of individuals to get their own faiths confirmed had been kept out of the game. See, for example, the sagacity which Spencer and Weismann now display. On the other hand, if you want an absolute duffer in an investigation, you must, after all, take the man who has no interest whatever in its results: he is the warranted incapable, the positive fool. The most useful investigator, because the most sensitive observer, is always he whose eager interest in one side of the question is balanced by an equally keen nervousness lest he become deceived. Science has organized this nervousness into a regular *technique,* her so-called method of verification; and she has fallen so deeply in love with the method that one may even say she has ceased to care for truth by itself at all."

EMERSON'S PARADOX

Once during an address to one of Professor Child's classes at Harvard, Emerson made use of the occasion to voice a paradox. "Gentlemen," he said, "in twenty years the ranking list will be inverted." Many of those who are last will be first and those who rank high to-day will stand at the foot to-morrow. Emerson may well have been provoked to this remark by personal, though quite unconscious motives, for Professor Child had graduated with the highest honors, Emerson well down in the list of his classmates. The case of Emerson himself would most certainly bear out his generalization but a generalization based upon a single case would possess no inductive value at all.

Various are the ways that might be devised for trying out the truth of the Emersonian law. The obvious test would involve an appeal to statistical information. Psychological tests of how the mind grows might tend to show that those with greater potentialities develop at a slower rate, so that a college student who is mentally mature at twenty finds no difficulty in competition with a man whose full powers have only begun to develop a decade later. The point to observe, however, is the fact that the law must be discovered before it can be tried out and that Emerson, who is desirous from personal motives that it be true, is more likely to discover it than Professor Child, who would naturally be set against it from the beginning. The desire that a certain thing should be true makes the finding out of that truth an easier matter. The prejudice against prejudice may be of value in human affairs, but, unreflectively considered, it may impede our human progress.

THE HUMAN VALUE OF PREJUDICE

The human value of passion, of prejudice, of love and hate, as opposed to the cold indifference which science advocates, the author has defended elsewhere (*Open Court,* August, 1921). In that connection he remarked: "Had the Babylonians not believed that the stars of heaven controlled the high matters of human destiny they would never have found the patience, century on century, to record their observations, and Hellenism, one of the few sporadic attempts of man to surpass man, that renaissance of the oriental world, would have inherited no science upon which to build. Modern Chemistry owes its advancement in no small part to the persistent effort of the alchemist to transmute the baser metals into gold, and the misguided attempts of the geometer to square the circle by the use of rule and compass has left its mark on the science and furnished the clue to the discovery of unsuspected truths. If the world in which we find ourselves provokes our curiosity, it is because we build it up in large part out of those aspects of reality that interest us. 'Nothing has been accomplished in the world,' says Hegel, 'without interest, and, if interest be called passion, we may affirm that nothing great has been brought about in the world without passion on the part of the actors.' But it is important to remark that the truth which beckons is not always the one finally verified, just as the benefit sought is rarely the one accepted in the end. The law of the conservation of energy followed on the search for perpetual motion, and more gold has flowed from the application of chemistry than the alchemist could well have dreamed. . . . The world of Dante with the earth at the centre of the universe and the seven heavens encircling it, with Jerusalem at the top and the mountain of Purgatory, displaced by Satan as he plunged downward from the Empyrean, at the bottom, was of course a normal conception for him. The astronomical crank of his day would be the man who espoused, as against this geocentric conception, the eccentric opinion that the universe is heliocentric at bottom, the evidence of our senses to the contrary notwithstanding. A man who could express soberly such views might invent others equally absurd, and it was the custom of that day to put him quietly out of the way 'without the shedding of blood.' The majority of men has always insisted upon its inalienable right to deal as it sees fit with the 'abnormal' minority which strays too far from the norm."

THE HISTORIAN'S BIAS

Further in this same article "in defense of prejudice" the author had said: "The sphere, in which personal bias plays perhaps its most notably useful and important rôle, is the writing of history. The 'objective' historian, who opposes this view, we shall have with us always, like the rest of the poor in spirit; but his claims are readily exposed. According to this creature we must venture as little as may be beyond the 'documents' themselves. We must stand by the *ipsissima verba* at the risk of perverting the truth. If he sticks to his guns—he is par excellence the man who sticks to his dates—history is for him a colorless chronicle, whose only objective character is the 'facts' and their chronological order. His task would then be to establish this order 'without bias' and his history the documents set side by side. It is obvious from Euclid that his shelves, like the sentences of Kant, would have to be measured by a railroad engineer.

"What he does, then, in practice is to foreshorten the picture; not, indeed, by abstractions, the 'most trenchant of epitomizers,' for that would be his personal medium operating to pervert the truth; but rather by leaving out of account the unimportant facts, the ones that have no bearing upon the drama in its larger outlines. But see you not, Sir Historiographer, that by this admission the whole humbug about objectivity and the impersonal narrative is exposed? You *choose* the facts. Very well, Sir, and how do you choose them and why? Because they illustrate some general point of view, which is your own. Because they illuminate some personal insight suggested by your own personal bias and interesting in so far as your imagination is daring, colorful, shrewd and—objective. In this sense history is more than romance and only the poet can be safely entrusted to write it. Alexandre Dumas pointed this out long ago but such seeds fall on stony ground. It was the novelist's own habit, when writing of an event, to construct, as the phrase goes, *a priori,* all of its parts down to the minutest detail. He surpassed all other men in the range and in the accuracy of his topographical imagination; and whenever he took the trouble to visit the scene of his historical dramas, which he did upon occasion, when the historical accounts contradicted his own, he invariably discovered that he was right and that the historian was wrong. The search for objectivity, like the search for happiness, baffles all stupid folk, who know not how to forego the direct approach.

"If it be true that the historian selects those facts which illuminate his private point of view, it is no less true that the facts themselves are amenable to his interpretations. Facts to the unimaginative are hard and fast things; to the spiritually minded they are plastic. The mind of Plato is an historical fact. Who, then, was Plato? Was this mind best known to the author of the Dialogues? Beyond a doubt to Plato himself some aspects of it were pretty well revealed. But did he know it as it was really constituted? It is warranted that he possessed no such gift. I will wager that his illustrious pupil, Aristotle, knew its defects and its excellencies better than he knew them himself. Or was Plato the mind that was so well known to the scholars of the Renaissance? Each one of these points of view about the fact in question contains a measure of the truth, but none is absolute. Round about every historical fact there circles a halo of ambiguity and it is within the limits of this halo that the interpretation of the historian may have free range. The rim of fact is clear-cut only for him who has no magnifying lens at hand."

LOGIC AS THE ART OF SYMBOLS

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LOGIC AS THE ART OF SYMBOLS

I

"If I have a mind to, I can see Zeno," the tortoise said, "but I can not see a man in general." "That," Zeno replied, "is because you have an eye, but no mind." And this opinion was just, if the task of all science is to inform experience with the form of thought. If there is no experience entirely uninformed, there is none that is completely formed. Let us try to see through special instances just what this would mean.

If we appeal to our organized experience, two of the statements,

All Athenians are Greeks
All metals are elements
All squares are circles

will appear to be true and one will appear to be false. It is important to observe that each one resembles the others, that each one is different from the others. The part that remains unchanged in each one is called the *form* of the statement, the part that varies is called the *content*. The two words that stand for the content, Athenians-Greeks, metals-elements, squares-circles, are called respectively, in that order, the logical *subject* and the logical *predicate*. If we are not thinking of this distinction and are referring to them collectively, they are called the *terms* and the form into which they fall, is called the *relation* that connects the terms.

If one is now asked to abstract the content, that is, to leave it out altogether, and to express the form by itself, then we should have left a dismembered fragment, with the places for the terms left blank:

All —— are ——

It was exactly a fragment like this that Plato called a bodyless form, an idea.

Suppose that Peter and Paul have each a dictionary, which contains all the substantives in the language, and suppose that it is Peter's affair to insert substantives in the first blank, the subject, whereas Paul is charged to substitute in the second blank, which is the predicate. This list of possible substitutions is known as the *range of values* of the blanks, and it is clear, since Peter and Paul have each a copy of the same book, that each blank has the same range of values. If Peter puts in his first word and then allows Paul time to put in in succession each word in his list, and if Peter then puts in his second word and allows Paul time to repeat, and so on, and so on, it is clear that a great many assertions will be made, and that some of these will be true and some false.

It must be clear by now that the blank spaces have taken on the sense of symbols. They stand for substantives, but for no specific one, and it will not be stretching the point too much if we call them *variable terms.* The trouble with using the blanks themselves as symbols, lies partly in the fact that the one space is independent of the other, so that it becomes desirable to distinguish between them. The space, the symbol, the place to insert the specific content, is represented by letters of the alphabet, so that our general, or abstract proposition becomes:

All a's are b's

In what has gone before the content has been allowed to change, and we have fixed our attention on the one specific form or relation. Let us think of cases in which the form varies:

—— is the father of ——
—— is greater than ——
—— is parallel to ——

It is important to notice that in each of these new cases, the range of values of the blank spaces is delimited. We can not put any substantive whatever into the space and be sure to get an assertion that is meaningful.

> A cabbage is the father of a king
> Ugliness is greater than beauty
> Peter is parallel to Paul

Except as metaphor, such statements are neither true nor false. The terms of the relation have a range of values *appropriate* to it. The relation determines in a sense, and is determined by, its sphere of application. The form and the content fall in together. It is the way of common sense to distinguish relationships by comparing their spheres of application. Another way is to distinguish them by their *properties.* We shall presently see what this means.

We began a moment ago by speaking of the variation of form. This will be true of,

> a is prior to b
> a is synchronous with b
> a is the cause of b

wherein we think of the content as fixed, or as merely independent of the form. If we abstract, as before, from specific instances, we might write, a — b

or, if we replace the blank space by a letter like R, S, T,

a R b

In the case of some forms, R, S, T, the order in which the terms occur is indifferent, a R b and b R a, if the truth of each one implies the truth of the other, are logically equivalent. When this is the case, R is said to have the property of *symmetry*. Consider the instances,

If a is parallel to b, then b is parallel to a
If a is the spouse of b, then b is the spouse of a
If a is synchronous with b, b is synchronous with a

in order to assure ourselves that there exist many relations which possess this property. It is well to notice that as long as the terms have values that make, a is parallel to b, a is synchronous with b, meaningful statements, which are at the same time false, then the inference drawn is still just. John may not be the spouse of Jane but,

If John is the spouse of Jane, Jane is the spouse of John.

If there are three terms in question, a, b, and c, then **R** may or may not truly relate these in pairs of two. If **R** is such a relation that whenever a **R** b and b **R** c are assumed to hold, it follows that a **R** c holds, then **R** is said to have or to possess the property of *transitivity*. This property seems to hold of relations like: is prior to, is greater than, is synchronous with, implies, is included in, etc., but not to hold of relations like: is the father of, is the spouse of, is perpendicular to, is indistinguishable from, etc.

Curiously enough a relation may hold between a term and itself, that is, a **R** a may be true. This is even forced, as Vailati pointed out, unless we deny that a relation can be symmetrical and transitive at the same time,

If a **R** b and b **R** a, then a **R** a.

When this happens, when a relation holds between a term and itself, then the relation is said to be *reflexive*. Thus, implication is reflexive, that is, if a (is true) then a (is true), and so is inclusion. There is a geometry on a certain odd surface, in which perpendicularity is reflexive.

These are only a few of the properties, which a relation may or may not possess. One of their uses, together with their power to generate new properties through combination, is in enabling us to differentiate one relation from another. To illustrate this, suppose two terms, a and b, and four relations, all is, no is, some is, some is not. The following table shows how the properties which distinguish these relations, distinguish them from one another.

	reflexive	symmetrical	transitive
all is	yes	no	yes
no is	no	yes	no
some is	yes	yes	no
some is not	no	no	no

II

The principal notion we shall be dealing with in the sequel, is the concept of implication. This relation between propositions, which is translated by the word *implies,* or by the words *if then,* is very commonly represented by the symbol ($\supset$). Its use involves the idea of *deduction,* the process by which one proposition follows from another. The celebrated Schroeder conceived this as essentially that of assigning special meanings to the variables. That is to say, starting with general principles and specializing the different letters in the formulas in different ways, a series of new principles, not otherwise recognizable, will fall out. These new assertions are said to *follow* from the original ones. Or, otherwise, some of the blank spaces, the variables, as the result of this process, will swallow up some of the others, some may spread out into a network of new blanks. Let us assure ourselves that this is so by examining cases. If a, b, c, d, stand for propositions, if the symbol ($\supset$) reads *implies,* if the symbol ($'$) placed over a letter, *denies* the truth of what that letter asserts and reads *is untrue,* if the dot (.) means *and,* there ought to be no difficulty in giving a verbal rendering of the principles set down below:

i. $(a \supset b).(c \supset d) \supset (a.c \supset b.d)$
ii. $(a \supset b).(b \supset c) \supset (a \supset c)$
iii. $(a.b \supset c) \supset (a.c' \supset b')$
iv. $(a \supset a)$

A simple illustration of how a theorem follows from a principle, results if we substitute in (i) the value d for both b and c. We then have,

Theorem i. $(a \supset d).(d \supset d) \supset (a.d \supset d.d)$

We shall take this occasion to learn some of the simpler processes of the algebra of logic. One matter that is always taken for granted in this algebra is this: Whenever a proposition established as true, either as principle or theorem, is conjoined to another, that is, united with it by the word *and,* it can be suppressed, as if it added nothing to our information. Now the part in the second bracket of the theorem is the same as (iv). If we suppress this bracket, we get,

Theorem i. $(a \supset d) \supset (a.d \supset d.d)$

Another matter which the algebra of logic takes for granted is this: A proposition that repeats itself may be reduced to one, that is, d (is true) and d (is true) has exactly the sense, is logically equivalent to d (is true). If this change is made, the result will be in its simplest form, viz.

Theorem i. $(a \supset d) \supset (a.d \supset d)$

We should also discover, if we carried our study of the algebra further, that the second bracket, $(a.d \supset d)$, is true by itself, or in its own right, and this suggests a general truth, which would sooner or later be forced upon us, that a proposition unconditionally true is implied by any proposition whatever. The seeming paradox involved in this fact, must not be allowed to delay us here.

Another simple case results, if a be changed into $a \supset a$ in (iii) and the part $a \supset a$ be suppressed by (iv), viz.

Theorem ii. $(b \supset c) \supset (c' \supset b')$

which, by changing b into b′ and c into c′, and assuming that b″ is the same as b, c″ is the same as c, yields,

$$\text{Theorem iii.}\quad (b' \supset c') \supset (c \supset b)$$
$$(c' \supset b') \supset (b \supset c)$$

Two propositions are said to be logically equivalent if each one implies the other. By theorems ii, iii, $(b \supset c)$ and $(c' \supset b')$ are equivalent.

In deriving theorem iii, we assumed the right to substitute b for b″, c for c″. It is worth while to notice that this assumption is not necessary, if we assume their logical equivalence. Starting with,

$$(b \supset c) \supset (c' \supset b')$$

and changing b into b′, we get,

$$(b' \supset c) \supset (c' \supset b'')$$

In (i) let $a = (b' \supset c)$, $b = (c' \supset b'')$, $c = d = (b'' \supset b)$,

$$\{(b' \supset c) \supset (c' \supset b'')\} \supset \{(b' \supset c).(b'' \supset b) \supset (c' \supset b'').(b'' \supset b)\}$$

Therefore, $(b' \supset c) \supset (c' \supset b'').(b'' \supset b)$

In (ii) let $a = c'$, $b = b''$, $c = b$,

Therefore, $(c' \supset b'').(b'' \supset b) \supset (c' \supset b)$

In (ii) let $a = (b' \supset c)$, $b = (c' \supset b'').(b'' \supset b)$, $c = (c' \supset b)$,

$$\{(b' \supset c) \supset (c' \supset b'').(b'' \supset b)\}.\{(c' \supset b'').(b'' \supset b) \supset (c' \supset b)\}$$
$$\supset \{(b' \supset c) \supset (c' \supset b)\}$$

Therefore, $(b' \supset c) \supset (c' \supset b)$

Now changing c into c′, so that,

$$(b' \supset c') \supset (c'' \supset b)$$

we get, following the same steps as before,

$$(b' \supset c') \supset (c \supset b)$$

or, changing b into c, c into b,

$$(c' \supset b') \supset (b \supset c)$$

Let our fourth and final theorem be a case somewhat more complex. If we make c the same as d in (i) and suppress the second bracket in virtue of (iv), then,

$$(a \supset b) \supset (a.c \supset b.c)$$

The matter now becomes complicated. Replace,

$$\begin{aligned} &a \text{ by } (a \supset b) \\ &b \text{ by } (a.c \supset b.c) \\ &c \text{ by } (b.c \supset d) \end{aligned}$$

and there results,

$$\{(a \supset b) \supset (a.c \supset b.c)\} \\ \supset \{(a \supset b).(b.c \supset d) \supset (a.c \supset b.c).(b.c \supset d)\}$$

Observe that the first part of this long expression is true by itself, or in its own right, because of what we proved above. We now assume the right, already assumed above, to assert unconditionally anything that follows from something that is true. If we suppress the first part, then, and write the second part by itself, we get,

$$(a \supset b).(b.c \supset d) \supset (a.c \supset b.c).(b.c \supset d)$$

Now make the following substitutions in (ii), replace,

a by a.c, b by b.c, c by d,

then there results,

$$(a.c \supset b.c).(b.c \supset d) \supset (a.c \supset d)$$

Now make the following substitutions in (ii), replace,

a by $(a \supset b).(b.c \supset d)$
b by $(a.c \supset b.c).(b.c \supset d)$
c by $(a.c \supset d)$

Then, as before,

$$\begin{aligned} &\{(a \supset b).(b.c \supset d) \supset (a.c \supset b.c).(b.c \supset d)\}. \\ &\{(a.c \supset b.c).(b.c \supset d) \supset (a.c \supset d)\} \\ &\supset \{(a \supset b).(b.c \supset d) \supset (a.c \supset d)\} \end{aligned}$$

The first two lines in this long implication, the two brackets that come before the principal implication sign, have already been established as true. Consequently, we may suppress them and assert what is left by itself. If we do this we get,

$$\text{Theorem iv. } (a \supset b).(b.c \supset d) \supset (a.c \supset d)$$

a truth very important, because very useful, as we shall find later on.

The notion of a deductive science is one we have inherited from the Greeks. Such a science commonly begins with the consideration of certain terms and relationships, which at the outset are regarded as ultimate or as indefinable, and which are provisionally given a meaning in the context of common experience. Subsequently axioms and definitions are introduced, which exhibit the properties which these provisional indefinables are to possess, and these axioms and definitions, together with the theorems that follow from them, constitute a system in which they are now taken to be defined. The meaning or interpretation, which is finally attached to the indefinables, is called a *solution* of the axiom set. These solutions may be quite independent of one another, or they may differ from one another in point of generality. Ordinarily the specific relationships of a given science are constant, whereas the terms may take on a range of values. Commonly the terms or variables are represented by the letters of the alphabet, the relations which obtain among them, and other constants, by some symbol borrowed from traditional usage.

III

Let us suppose that a transcriber is confronted by an inscription, the one set down below, and that he proposes to himself the problem of finding an interpretation. The inscription is this:

$(a<a)'<o$	$(a<a')<o$
$(a<b').(b<a')'<o$	$(a<b).(a<b')<o$
$(a<b).(b<c).(a<c)'<o$	$(a<b).(a'<b)<o$

Already, in the provisional assumption that the marks set down above are not, say, the tracks of some prehistoric animal imprisoned in ancient rock, but that they are an inscription, and stand in need of interpretation, his first step is forced. The whole must stand, he says, for a proposition, or for a group of propositions, ordered in some way and related to one another, as otherwise the thing would be neither true nor false, and hence without meaning. We may suppose, then, that he has dropped his first supposition, that he is dealing with the fragment of a sentence, a substantive, the name of a king and his titles, or a date, with an implied *hic jacet,* and that what is written has been written, it may be, by a holy man, who is also a man of science.

We will suppose that our interpreter has tried out a vast number of promising solutions, and that the one that tempts him most, because it holds up best, is this: Each part of each line, as well as the signs a, b, c, o, are propositions. But the sign o, is different from a, b, c, for it is constant in meaning, and means an *impossibility,* or a statement that is always false, whereas the a, b, c, can stand for anything you like. The prime (′) interprets nicely as the *denial* of the statement it covers, the peculiar symbol (<), which made us think first of a bird track in the rock, means *implies.* Finally, the dot (.) works very well as *and,* whereas the brackets are simply punctuation marks.

Our interpreter has, of course, already begun to ask himself, what would happen, if a or b or c in any of the lines should be replaced by o, for he has been taking it pretty much for granted that the ancient scribe, who wrote the lines, intends them to be true over the whole range of values which a, b, c, may take on, and o looks very much like one of those values. What bothers him is how to make out the sense, particularly in the last three statements, whenever he replaces a, or a and b, or c, in different ways, by either o or o′.

Evidently there is a gap in the inscription. An essential matter has been rubbed out or otherwise lost, and this essential part is the definition of o. The only way to define o, which stands for an impossibility or an absurdity, that is, consistently with what is written, is to say: The proposition o implies and is implied by nothing but itself. Our interpreter may now be tempted, in the manner of transcribers, to interpolate something in the text. He may already have been pondering the addition of,

$$(a<b).(a.b'<o)'<o$$

which on making a and b the same, has suggested to him that a.a′ is a case of o. And so he may write in, in imitation of the original hand:

o is any a which satisfies

$$(a<a.a')'<o$$

from which the sense of o′ is at once deduced, that is, the proposition o′ implies and is implied by nothing but itself.

Our interpreter may reflect that this is not the sense that a Westerner attaches to the word *implies,* but that different logics, like different geometries, may arise in different cultures, and that what makes them differ is that they "speak in different tongues." However, the notion of implication as here defined, is not so foreign to his own culture as he may suppose. Just as Boolean implication, because it has the same properties, is the analogue of Boolean inclusion, so implication, as here defined, is the analogue of the *all is* relation of traditional Western logic. We shall presently show that either meaning can be deduced from, or translated into the other, by means of transformation formulas, which express the one relation in terms of the other.

In all this it is clear that the interpreter must bring to his interpretation an experience organized in the light of abstract ideas. This experience is as primitive as its science is fundamental. Take the notion of a class, that is, the notion of a group of objects which share a common characteristic, and let us try to imagine one of the many experiences out of which this concept must emerge.

One notion, which is widely diffused in all preliterate cultures, is this, namely, that one has a certain power over an object as soon as one knows its name. Thus if one knows John's name and calls John, John may appear, and it is in the light of this fact that one is careful not to pronounce lightly the name of god or devil, for then, not only might one of these beings appear, but even a whole swarm of deities, each one responding to the generic name. Suppose now that the king calls for John, and that in reply to the command a whole group of "Johnnies" appear before him. One of two things is likely to happen. He may refuse to interpret the experience, he may even become angry because his intention was not divined, or he may act intelligently. He may observe that John is generic, the name of a group, in effect, a common name. And so he says: I mean John the Red, or I mean John the Just, and then perhaps nobody will step forward, or at most, it may be, only one. Proper names invented to distinguish individuals, soon become *common*. That is one reason why Smith, Brown, Robinson, are not in themselves distinguished names. And yet one thinks of only one Adam Smith and the soul of John Brown goes marching on in song.

This idea, that a knowledge of the name rightly applied gives power, is the forerunner of a very late stage in man's reflection, that conception of seventeenth century rationalism that the definition is available, not only as an instrument of demonstration, but as an organon of discovery as well. A middle stage is represented by the thought of Confucius, *floruit* about 500 B. C., who conceives the whole problem of philosophy as that of the "rectification of names." "Confucius was asked by a disciple what he would first undertake were he to govern a state. The Master answered: It must needs be the rectifying of names. Indeed, said the bewildered disciple, that is far-fetched, sir! Why rectify them? Yü, said Confucius, addressing the disciple by name, thou art uncultivated. A gentleman should show a cautious reserve in regard to what he does not know. If names be incorrect, speech will not follow its natural sequence. If speech does not follow its natural sequence nothing can be established. If nothing can be established no rules of conduct or music will prevail. Where rules of conduct and music do not prevail law and punishments will not be just, the people will not know where to place their hands and feet. Therefore a superior man requires that names be capable of being spoken and that what is spoken must be capable of being put into practice. A superior man is never careless of words." *

The culture in which our supposed definition of inference originated, as well as the scribe who is supposed to have first set forth its properties, is purely hypothetical. There is no record that either has ever existed. But if either *had* existed, the task accomplished would have been legitimate in its own right. It is not, for example, the Greek conception of inference. Not all the conclusions of the Prior Analytics would follow because of it. They follow because of a theory of inference that is indistinguishable from the one we accept today, the so-called calculus of propositions.

Let us suppose that a manuscript of later date has rendered our inscription somewhat differently,

$(a<a)' \supset o$	$(a<a') \supset o$
$(a<b').(b<a') \supset o$	$(a<b).(a<b') \supset o$
$(a<b).(b<c).(a<c)' \supset o$	$(a<b).(a'<b) \supset o$

* *Development of the Logical Method in Ancient China,* by Hu Shih (Suh Hu), Shanghai, 1922.

It may be that the transcriber of this later text, who is a critic and commentator as well, has been unable to divine its original sense. He has found a solution of his own, but one that has a certain imperfection about it, for it will not allow the symbol (<) to stand in all the places that it stood before, and it is this that has prompted him to take liberties. He has edited his own text, and we may suppose that his commentary has descended to us as well.

Some of his surmises conform to the original intention, for the prime (′) stands for *denial* in some sense, and the dot means *and,* but the o which stands for an impossibility or an absurdity, he has contrived to define in quite a different way. What he says is this:

o is any a which satisfies

$$(a \supset a')' \supset o$$

and we may understand that this is so because he understands implication, which he represents by ($\supset$), in a novel and Western sense.

One begins to understand that Confucius could not have been far wrong when he said that the problem of the philosopher is to find the words that are sound, for our commentator has found an interpretation of the symbol (<) that is strange indeed. Let us set forth his meaning as best we can. Consider the area bounded by two radii of a circle and the subtended arc, and add to this area, in somewhat the way one adds a tail to a kite, the rest of the circumference of the circle. This area plus the loop hanging to it, is what a or b or c in the formulas mean.

Then $a < b$ is said to hold, or be true, if the area of b contains the area of a, and the loop of a contains the loop of b.

One feels that the word *contains* has a certain dubious sound, as if one were asked to gather its meaning by an appeal to geometrical intuition. As a matter of fact it will be necessary to limit this intuition in an arbitrary way. Suppose the two radii approach one another so that the area tends to disappear, and vanishes altogether in the limiting case. Then all that is left is the loop, which makes the whole circumference of the circle. The tail has swallowed up the kite. This limiting value which a or b or c may take on, our commentator represents by o, probably because he feels that there is some vague analogy with the other o of propositions, which appears in the formulas. Perhaps he feels there will be no confusion since this latter o does not lie within the range of values of his variables, and perhaps because he thinks it is more elegant to point out the analogy by using the same symbol.

Suppose, on the other hand, that the two radii recede from one another, generating a larger and larger area, and finally approach one another in the opposite sense. Then all that is finally left is the area of the circle, for the loop has entirely disappeared. The kite has swallowed up the tail. This limiting value which a or b or c may take on, our commentator represents by i. We shall presently see that it means the same thing as o′.

Shall we say, or shall we not say, that a $<$ b holds, when a or b becomes one of these limiting cases? We shall say this, and this is entirely arbitrary, that the circumference contains no other loop but itself, and that a zero-loop, no loop at all, is contained in no other loop but itself. Without this understanding our formulas would not hold for every possible value of the variables, a, b, and c.

What, now, does a′ mean, and how is it to be interpreted? This is rather suggested by the idea of denial, what is left of the whole when the part standing for a is excluded. That is, a′ is all of the area of the circle which does not belong to a, plus that part of the circumference which does not belong to a, the part that a's radii subtend.

While our formulas contain nothing that looks like a.b, a.c, b.c, our critic insists that expressions of this sort lie within the range of values of the variables, and that they must be interpreted. He calls a.b what is common to a and b. If a and b overlap they have a common area, and this common area has an arc of its own. This is what a.b means. If a is all inside of b, then a.b is the same as a. If b is all inside of a, then a.b is the same as b. If the areas of a and b do not overlap at all, then a.b will have no area and will be represented by the circumference of the circle and will be the same thing as o. Evidently a.b and b.a are equivalent. Here our commentator uses a hard word. He says that the relation expressed by the dot, the *and,* is commutative or symmetrical.

This o, this tail which has swallowed up the kite, and its negative, o′ or i, the kite which has swallowed up the tail, are thus defined:

o is any a which satisfies

$$(a < a.a')' \supset o$$

i is any a which satisfies

$$(a' < a.a')' \supset o$$

Finally, our critic insists that all this proves that the original set of six propositions are *consistent* with one another, that never by combining them with one another, that never by assigning special meanings to the variables, could we ever be led to a contradiction. The reason he gives for this has a metaphysical ring and may, therefore, be tainted from the beginning, though the great Lambert said the same thing. What he says is this: What has once been shown to be possible, in virtue of having been shown to be actual, can never lead to an impossibility, that is to say, can never lead to contradiction.

Consider now another system, defined by another set of properties, viz.

$$(a \supset a)' \supset o$$
$$(a \supset b').(b \supset a')' \supset o$$
$$(a.b \supset b)' \supset o$$
$$(a \supset b).(a.b' \supset o)' \supset o$$
$$(a \supset b)'.(a.b' \supset o) \supset o$$
$$(a.b \supset c).(a.c' \supset b')' \supset o$$
$$(a \supset b).(b \supset c).(a \supset c)' \supset o$$
$$(a \supset b).(c \supset d).(a.c \supset b.d)' \supset o$$
$$(d \supset b).(a.b \supset c).(a.d \supset c)' \supset o$$

This is a system whose variables can be interpreted to mean either propositions or classes, and, therefore, the symbol ($\supset$) to mean either implication or inclusion. If the letters are to mean classes, the last symbol ($\supset$), just before the zero, reads *implies*.

We may, however, interpret the variables exactly as we did in the previous system, provided we emend slightly the meaning of inclusion. This emendation consists simply in leaving out the *arbitrary* proviso, viz.

The proposition $a \supset b$ is said to hold, or be true, if the area of b contains the area of a, and the arc of a contains the arc of b.

IV

The first system we considered above, interpreted for classes, is the same as traditional logic and may be termed Aristotelean. The last one, which we just set down, is called Boolean, because it was developed by George Boole in the middle of the last century. It is all but universally taken for granted today that these two logics can not be reconciled, but we shall now show that this is a misunderstanding, by deriving each system from the other. This derivation will depend on our finding transformation formulas, which will express Aristotelean inclusion in terms of Boolean, and vice versa. The first of these has the form:

$$(a<b) = (a\supset b).\{(b\supset a) + (a\supset b')'.(b'\supset a)'\}$$

wherein the new symbol (+) is to be read *either, or.* These proofs will involve certain principles of the algebra not yet met with. We shall learn these as we go along.

If a and b be identified in the formula, we have,

$$(a<a) = (a\supset a).\{(a\supset a) + (a\supset a')'.(a'\supset a)'\}$$

In this algebra a proposition unconditionally true, like $a\supset a$, is represented by i, the negation of o, and anything whatever added to it leaves it unchanged, that is,

$$i = i + a = i + b \text{ etc.}$$

so that the long expression in the bracket reduces to $a \supset a$.

Therefore, $(a < a) = (a \supset a).(a \supset a) = (a \supset a)$
Therefore, $(a < a)' \supset o = (a \supset a)' \supset o$

and, consequently,

$$(a < a)' \supset o$$

is true unconditionally.

Again, if b be replaced by a′ in the formula,

$$\begin{aligned}(a < a') &= (a \supset a').\{(a' \supset a) + (a \supset a)'.(a \supset a)'\} \\ &= (a \supset a').(a' \supset a)\end{aligned}$$

and it will be proper at this point to recall the definition of the null-class and its negative, the universe.

These are defined as any a which satisfies, respectively,

$$(1)\ a \supset a' \qquad (2)\ a' \supset a$$

Substitute a′ for b, and then a′ for a, a for b, in

$$(a \supset b).(a.b' \supset o)' \supset o$$

and we get,

$$(a \supset a').(a \supset o)' \supset o$$
$$(a' \supset a).(a' \supset o)' \supset o$$

Substitute $a = (a \supset a'),\ b = (a \supset o)',\ c = o$, in

$$(a.b \supset c).(a.c' \supset b')' \supset o$$

so that $\{(a \supset a').(a \supset o)' \supset o\} . \{(a \supset a') \supset (a \supset o)\}' \supset o$

and suppressing the first bracket because it is true by itself,

$$\{(a \supset a') \supset (a \supset o)\}' \supset o$$

Then $$(a \supset a') \supset (a \supset o)$$

must be true by itself, as well as its variant,

$$(a' \supset a) \supset (a' \supset o)$$

If we assume, now, another law of the algebra,

$$(a \supset b).(c \supset d) \supset (a + c \supset b + d)$$

as well as $$i \supset a + a'$$

then, $$\begin{aligned}(a \supset a').(a' \supset a) &\supset (a \supset o).(a' \supset o)\\ &\supset (a + a' \supset o)\\ &\supset (i \supset o)\\ &\supset o\end{aligned}$$

for this is a fundamental form of invalidity.

Consequently, $$(a < a') \supset o$$

To derive another property of the Aristotelean system,

$$(a < b').(b < a')' \supset o$$

we must find $(b < a')'$, and to do this we must introduce a formula that involves *either, or, and* and *denial.* This is expressed in this way:

$$(a\ .\ b)' = (a' + b')$$
$$(a + b)' = (a'\ .\ b')$$

Applying these in succession,

$$(b<a') = (b\supset a').\{(a'\supset b) + (a\supset b)'.(b\supset a)'\}$$
$$(b<a')' = (b\supset a')' + \{(a'\supset b) + (a\supset b)'.(b\supset a)'\}'$$
$$(b\supset a')' + (a'\supset b)'.\{(a\supset b)'.(b\supset a)'\}'$$
$$(b\supset a')' + (a'\supset b)'.\{(a\supset b) + (b\supset a)\}$$

and since by the formula,

$$(a<b') = (a\supset b').\{(b'\supset a) + (a\supset b)'.(b\supset a)'\}$$

the two sides of these equations must be multiplied together to get

$$(a<b').(b<a')'$$

The general rule for doing this is the same as in ordinary algebra, for the same laws govern "plus" and "times" here. Let us multiply out the right hand side of each equation and place the one "sum" underneath the other, thus,

$$(b\supset a')' + (a'\supset b)'.(a\supset b) + (a'\supset b)'.(b\supset a)$$
$$(a\supset b').(b'\supset a) + (a\supset b').(a\supset b)'.(b\supset a)'$$

Then, as in algebra, the product of these is,

$$
\begin{aligned}
&(b'\supset a).(a\supset b').(b\supset a')' \\
+\ &(a\supset b').(a\supset b).(b'\supset a).(a'\supset b)' \\
+\ &(a\supset b').(b\supset a).(b'\supset a).(a'\supset b)' \\
+\ &(a\supset b)'.(b\supset a)'.(a\supset b').(b\supset a')' \\
+\ &(a'\supset b)'.(a\supset b').(b\supset a)'.(a\supset b).(a\supset b)' \\
+\ &(a'\supset b)'.(a\supset b').(a\supset b)'.(b\supset a).(b\supset a)'
\end{aligned}
$$

The factors in each term have been rearranged, but the product of the last two is zero in each case. Accordingly,

$$(a<b').(b<a')'\supset o$$

if zero times anything is zero and the sum of any number of zeros is zero.

Finally, since,

$$
\begin{aligned}
(a<b) &= (a\supset b).\{(b\supset a) + (a\supset b')'.(b'\supset a)'\} \\
(a<b') &= (a\supset b').\{(b'\supset a) + (a\supset b)'.(b\supset a)'\} \\
(a<b).(a<b') &= (a\supset b).\{(b'\supset a) + (a\supset b)'.(b\supset a)'\}. \\
&\quad (a\supset b').\{(b\supset a) + (a\supset b')'.(b'\supset a)\} \\
&\supset (a\supset b).(a\supset b').(b\supset a).(b'\supset a) \\
&\supset (a.a\supset b.b').(b+b'\supset a+a) \\
&\supset (a\supset o).(i\supset a) \\
&\supset (i\supset a).(a\supset o) \\
&\supset (i\supset o) \\
&\supset o
\end{aligned}
$$

Therefore, $(a<b).(a<b')\supset o$

which for $a=a'$, $b=b'$, yields,

$$(b<a).(b'<a)\supset o$$

In order to prove,

$$(a<b).(b<c).(a<c)'\supset o$$

we should write,

$$(a<b) = (a\supset b).\{(b\supset a) + (a\supset b')'.(b'\supset a)'\}$$
$$(b<c) = (b\supset c).\{(c\supset b) + (b\supset c')'.(c'\supset b)'\}$$
$$(a<c)' = (a\supset c)' + (c\supset a)'.\{(a\supset c') + (c'\supset a)\}$$

and multiply these expressions together. We should discover on inspection a zero factor in each term of the product, and from this our result would follow.

The characteristic features of Aristotle's logic have now been established, and since we have laid down no restriction whatever as to what the terms shall mean, it must hold true for every meaning which the terms can have, provided the Boolean system holds true. Before we derive Boole's system from that of Aristotle, we shall add a number of theorems to what has gone before, because these theorems will make this derivation very brief.

1.1 $(a<b).(a'<a.a').(b'<b.b')'\supset o$

for, $(a'<o) = (a'\supset o).(o\supset a') + (a'\supset o).(a'\supset o')'.(o'\supset a')'$
$$= (a'\supset o)$$

since in this algebra,

$$(o\supset a) \text{ and } (a\supset o')$$

are unconditionally true.

$\therefore\quad (a<b).(a'<o).(b'<o)' =$
$$(o'\supset a).(a\supset b).(o'\supset b)'.(b\supset a) +$$
$$(o'\supset a).(a\supset b).(o'\supset b)'.(a\supset b')'.(b'\supset a)'$$

and the product of the first three factors in each term vanishes.

1.2 $(a<b).(a<a.a')'.(b<b.b')\supset o$

changing a into a′, b into b′.

2.1 $(a<b).(a.b'<o)'\supset o$

2.2 $(a<a.a').(a.b'<o)'\supset o$

We have, $(a<o).(a.b'<o)' = (a\supset o).(a\supset b)'$

$$\{(a\supset o)\supset(a\supset b)\}$$

$$\supset\{(a\supset o).(a\supset b)'\supset(a\supset b).(a\supset b)'\}$$

But the part before the main implication sign is true, and therefore,

$(a\supset o).(a\supset b)'\supset o$

2.3 $(b'<b.b').(ab'<o)'\supset o$

3.1 $(a<b)'.(a.b'<o).(a<o)'.(b'<o)'\supset o$

The product vanishes, since,

$(a\supset b)'.(a.b'\supset o)\supset o$

$(a\supset b).(a\supset b').(a\supset o)'\supset o$

4.1 $(a.b<b)'.(a.b<o)'.(b'<o)'\supset o$

5.1 $(a<b).(c<d).(a.c<b.d)'.$
$(a.c<o)'.(b'+d'<o)'\supset o$

The product vanishes in virtue of

$(a\supset b).(c\supset d).(a.c\supset b'+d').(a.c\supset o)'\supset o$

for $(a\supset b).(c\supset d).(a.c\supset b'+d')$

$$\supset (a.c \supset b' + d').(a.c \supset b.d)$$
$$\supset (a.c \supset o)$$

and $(a \supset b).(c \supset d).(b' + d' \supset o)'.(b' + d' \supset a.c) \supset o$

for $(a \supset b).(c \supset d).(b' + d' \supset a.c)$

$$\supset (b' + d' \supset a.c).(b' + d' \supset a' + c')$$
$$\supset (b' + d' \supset o)$$

5.2 $(c < d).(b' < o).(b' + d' < o)'.(a.c < o)'.(a.c < b.d)' \supset o$

for since, $(c \supset d).(b' \supset o) \supset (a \supset b).(c \supset d)$

$(c \supset d).(b' \supset o).(a.c \supset b.d)' \supset$

$(a \supset b).(c \supset d).(a.c \supset b.d)' \supset o$

5.3 $(a < b).(d' < o).(b' + d' < o)'.(a.c < o)'.(a.c < b.d)' \supset o$

5.4 $(b' < o).(d' < o).(b' + d' < o)' \supset o$

6.1 $(a.b < c).(a.c' < b')'.(a.c' < o)'.(b < o)' \supset o$

The terms vanish because,

$$(a.b \supset c).(a.c' \supset b).(a.c' \supset o)' \supset o$$

since $(a.b \supset c).(a.c' \supset b) \supset (a.c' \supset b').(a.c' \supset b)$

$$\supset (a.c' \supset o)$$

and $(a.b \supset c).(b \supset a.c').(b \supset o)' \supset o$

since $(a.b \supset c) \supset (a.c' \supset b')$

$$\supset (b \supset a' + c)$$

$(b \supset a.c').(b \supset a' + c) \supset (b \supset o)$

6.2 $(a.b < o).(b < o)'.(a.c' < o)'.(a.c' < b')' \supset o$

for $(a.c' \supset b).(a.b \supset o).(a.c' \supset o)' \supset o$

since $(a.c' \supset b).(a.b \supset o) \supset (a.c' \supset a.b).(a.b \supset o)$

$$\supset (a.c' \supset o)$$

and $(b \supset a.c').(a.b \supset o).(b \supset o)' \supset o$

since $(a.b \supset o) \supset (a \supset b')$

$$\supset (a.c' \supset b')$$

$(b \supset a.c').(a.b \supset o) \supset (b \supset a.c').(a.c' \supset b')$

$$\supset (b \supset b')$$
$$\supset (b \supset o)$$

7.1 $(a.b < c).(d < b).(a.d < c)'.(a.d < o)'.(c' < o)' \supset o$

The terms in the product vanish because,

$$(a.b \supset c).(d \supset b).(a.d \supset c').(a.d \supset o)' \supset o$$

$$\text{for } (a.b \supset c).(d \supset b).(a.d \supset c') \supset (a.d \supset c).(a.d \supset c')$$
$$\supset (a.d \supset o)$$

$$\text{and } (a.b \supset c).(d \supset b).(c' \supset a.d).(c' \supset o)' \supset o$$

$$\text{for } (a.b \supset c).(d \supset b).(c' \supset a.d) \supset (a.d \supset c).(c' \supset a.d)$$
$$\supset (c' \supset a' + d').(c' \supset a.d)$$
$$\supset (c' \supset o)$$

7.2 $(a.b < c).(b' < o).(a.d < c)'.(a.d < o)'.(c' < o)' \supset o$

The terms in the product vanish,

$$\text{because } (a.b \supset c).(b' \supset o).(a.d \supset c)'$$
$$\supset (a.c' \supset b').(b' \supset o).(a.d \supset c)'$$
$$\supset (a.c' \supset o).(a.d \supset c)'$$
$$\supset (a \supset c).(a.d \supset c)'$$
$$\supset (a.d \supset c).(a.d \supset c)'$$

$$\text{and } (b' \supset o).(a.b \supset c).(a.d \supset c').(a.d \supset o)'$$
$$\supset (a.c' \supset b').(b' \supset o).(a.d \supset c').(a.d \supset o)'$$
$$\supset (a.c' \supset o).(a.d \supset c').(a.d \supset o)'$$
$$\supset (a \supset c).(a.d \supset c').(a.d \supset o)'$$
$$\supset (a.d \supset o).(a.d \supset o)'$$

$$\text{and } (a.b \supset c).(c' \supset a.d).(b' \supset o).(c' \supset o)'$$
$$\supset (a.c' \supset b').(b' \supset o).(c' \supset a.d).(c' \supset o)'$$
$$\supset (a.c' \supset o).(a' + d' \supset c).(c' \supset o)'$$
$$\supset (a \supset c).(a' \supset c).(c' \supset o)'$$
$$\supset (c' \supset o).(c' \supset o)'$$

7.3 $(a.b < o).(d < b).(a.d < c)'.(a.d < o)'.(c' < o)' \supset o$
in virtue of $(a.b \supset o).(d \supset b).(a.d \supset o)' \supset o$

7.4 $(a.b < o).(b' < o).(a.d < c)'.(a.d < o)'.(c' < o)' \supset o$
in virtue of $(a.b \supset o).(b' \supset o).(a.d \supset o)' \supset o$

We are now prepared to deduce Boole's system from that of Aristotle, or Aristotle's system as we have extended it, for the theorems we deduced at the end do not appear in the manuals of traditional logic. The formula which enables us to carry out this proof, expresses Boolean inclusion in terms of the "all, is" relation of Aristotle, viz.

$$(a \supset b) = (a < b) + (a < a.a') + (b' < b.b')$$

Thus we have very simply,

$$(a \supset a)' = (a < a)'.(a < o)'.(a' < o)' \supset o$$
$$(a \supset b') = (a < b') + (a < o) + (b < o)$$
$$(b \supset a') = (b < a') + (b < o) + (a < o)$$

Therefore, $(a \supset b') = (b \supset a')$

$(a \supset b').(b \supset a')' \supset o$

1.0 $(a \supset b).(b \supset c).(a \supset c)' \supset o$
by 1.1, 1.2

2.0 $(a \supset b).(a.b' \supset o)' \supset o$
by 2.1, 2.2, 2.3

3.0 $(a \supset b)'.(a.b \supset o) \supset o$
by 3.1

4.0 $(a.b \supset b)' \supset o$
by 4.1

5.0 $(a \supset b).(c \supset d).(a.c \supset b.d)' \supset o$
by 5.1, 5.2, 5.3, 5.4

6.0 $(a.b \supset c).(a.c' \supset b')' \supset o$
by 6.1, 6.2

7.0 $(a.b \supset c).(d \supset b).(a.d \supset c)' \supset o$
by 7.1, 7.2, 7.3, 7.4

The logic of Aristotle having been deduced from that of Boole, and conversely, we conclude that we are confronted, not by two irreconcilable systems, but by two systems that "speak in different tongues." The seeming contradiction, which algebrists of logic have always taken to be final, turns out to be a purely verbal one.

ABSTRACT LOGIC

PROBLEM, METHOD, AND DEVELOPMENT

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Abstract Logic

I

IN the pages which follow we shall be dealing with terms, which are the same as substantives or classes, a group of objects conceived by the aid of a common property, and with propositional forms. We shall understand by a proposition a sentence that is either true or false, whether it is true for all meanings of the terms that enter into it, or true for no meanings of the terms that enter into it, or its truth is, in general, contingent on the meaning of its terms. Terms and relations may be combined to form propositions, and propositions and relations may be combined to form propositions, postulates, principles and theorems. Arbitrarily we shall understand a postulate to be an assumption, provisional or final, whose truth is independent of any meaning that may be assigned to its terms, and a principle to be an assumption whose truth is independent of any propositional form which its variables may take on. Thus,

$$A(ab) \angle I(ab)$$

"If all a is b then some a is b,"

$$A(ab) \angle V(ab)$$

"If all a is b then some non-a is non-b,"

if they appeared in our system as fundamental and not derived, would be termed postulates,

$$(p \angle q) \cdot (q \angle r) \angle (p \angle r)$$

"If p implies q, and q implies r, then p implies r,"

$$(p \angle q) \angle (q' \angle p')$$

"If p (is true) implies q (is true) then q (is untrue) implies p (is untrue),"

if they were assumed and not proven, would be called principles. Clearly, since the truth of a principle is independent of the propositional forms that enter into it, its truth will also be independent of the meaning of the terms that enter into these propositional forms. We may now define our problem and with it our science:

> "Logic is the science whose problem it is to construct all propositions whose truth is independent of the meaning of terms." (E. A. Singer)

We shall recognize the following propositional forms as necessary and sufficient for the expression of any truth:

The Categorical forms:

$A(ab)$ = All a is b
$E(ab)$ = No a is b
$I(ab)$ = Some a is b

$O(ab)$ = Some a is not b
$U(ab)$ = All non-a is b
$V(ab)$ = Some non-a is non-b

The form of Negation:

p' = p is untrue

The Hypothetical form:

$(p \angle q)$ = p implies q = If p (is true) then q (is true)

$(p \angle q)'$ = p does not imply q

The Conjunctive form:

$p \times q = p \cdot q$ = p (is true) and q (is true)

The Disjunctive form:

$p + q$ = p (is true) or q (is true)

It is sometimes erroneously assumed that, if one form can be expressed in terms of another, it is thereby shown to be unessential to the algebra and is only allowed to remain as a matter of convenience. Either conjunction or disjunction can be represented in terms of the other,

$$(p \times q)' = p' + q' \qquad (p + q)' = p' \times q'$$

but we shall find out later that they are differentiated by the properties which certain expansion formulas express.

The categorical forms contain, besides the traditional ones, two that were discovered by De Morgan and which we represent by U and V. One may satisfy himself provisionally that these six forms are necessary and sufficient, by negating the terms in all possible ways and noticing that always one of the six forms recurs. The truth of the illustrations below will be established later on:

$$A(a'b') = A(ba) \qquad A(ab') = E(ab) \qquad E(a'b') = A(a'b) = U(ab)$$

$$I(a'b) = I(ba') = O(ba) \qquad O(a'b') = I(a'b) = I(ba') = O(ba) \text{ etc.}$$

using the prime (′) by analogy to stand for the denial of a class as well as for the denial of a proposition, that is, $a' =$ non-a.

Whenever we wish to express the fact that the term-order in a categorical form is unsettled, we shall place a comma between the terms in the bracket. Thus, $P(a, b)$ will represent ambiguously $P(ab)$ or $P(ba)$. The set of all possible variants of a given form of proposition will be called the *array* of propositions of that form. We shall begin by constructing all the true and all the false propositions of the form $P(a, b) \angle Q(a, b)$.

Definition: $(p \angle q)(q \angle p) \angle (p = q)$

$(p = q) \angle (p \angle q)(q \angle p)$

Theorem: $(p \angle q)(q \angle p) = (p = q)$

In general for the sake of brevity it will be assumed that, when the equality of two propositions has been established, we may substitute one for the other. Illustrations will be given, however, of the fact that it is not necessary to assume this property of equality, because the same result can always be effected by the longer process of elimination.

If P and Q are two propositions which satisfy the conditions,

$$(1)\ P \angle Q' \qquad (2)\ Q' \angle P$$

then P is said to be the *contradictory* of Q.

Theorem: If P is the contradictory of Q, then Q is the contradictory of P. This follows at once if we write down two variants of the principle listed below,

$$(P \angle Q') \angle (Q \angle P') \qquad (Q' \angle P) \angle (P' \angle Q)$$

Postulate:	$A(ab) \angle O'(ab)$	$O'(ab) \angle A(ab)$
Theorem:	$O(ab) \angle A'(ab)$	$A'(ab) \angle O(ab)$
Postulate:	$A(a'b) = U(ab)$	$O(a'b) = V(ab)$
Theorem:	$A(ab) = U(a'b)$	$O(ab) = V(a'b)$
Theorem:	$U(ab) \angle V'(ab)$	$V'(ab) \angle U(ab)$
	$V(ab) \angle U'(ab)$	$U'(ab) \angle V(ab)$
Postulate:	$A(ab) = E(ab')$	$I(ab) = O(ab')$
Theorem:	$E(ab) = A(ab')$	$O(ab) = I(ab')$
Theorem:	$E(ab) \angle I'(ab)$	$I'(ab) \angle E(ab)$
	$I(ab) \angle E'(ab)$	$E'(ab) \angle I(ab)$

Theorem: A, O and E, I and U, V are contradictory pairs.

Postulate: $E(ab) \angle E(ba)$

Theorem: $E(ab) = E(ba)$

for, changing a into b and b into a in the postulate, $E(ba) \angle E(ab)$.

Theorem: $I(ab) = I(ba)$

for, $E(ab) \angle E(ba) \cdot \angle \cdot E'(ba) \angle E'(ab)$ by the principle listed below and substituting I for E'.

Let us, however, prove the same thing without assuming the right to substitute.

Principle: $(p \angle q) \angle (q' \angle p')$

Principle:* $(p \angle q)(q \angle r) \angle (p \angle r)$

$$E(ab) \angle E(ba) \cdot \angle \cdot E'(ba) \angle E'(ab)$$

by the first principle.

$$\therefore \quad E'(ba) \angle E'(ab)$$

for we shall assume the right to suppress the antecedent, the part to the left of the main implication symbol, and to assert the consequent, the part to the right of the main implication symbol, by itself.

$$I(ba) \angle E'(ba) \cdot E'(ba) \angle E'(ab) \cdot \angle \cdot I(ba) \angle E'(ab)$$

by the second principle.

$$\therefore \quad I(ba) \angle E'(ab)$$

$$I(ba) \angle E'(ab) \cdot E'(ab) \angle I(ab) \cdot \angle \cdot I(ba) \angle I(ab)$$

by the second principle.

$$\therefore \quad I(ba) \angle I(ab)$$

and since $I(ab) \angle I(ba)$ results on changing a into b and b into a, we have by definition,

* It can be shown that some of the principles here introduced are true in their applications but not when the variables are unrestricted.

$$I(ab) \angle I(ba) \cdot I(ba) \angle I(ab) \cdot = \cdot I(ab) = I(ba)$$

Theorem: $A(ab) = A(b'a')$

for $A(ab) = E(ab') = E(b'a) = A(b'a')$

Theorem: $O(ab) = O(b'a')$

for $O(ab) = I(ab') = I(b'a) = O(b'a')$

Theorem: $U(ab) = U(ba)$

for $U(ab) = A(a'b) = A(b'a) = U(ba)$

Theorem: $V(ab) = V(ba)$

Postulate: $A(ab) \angle I(ab)$

Theorem: $A(ab) \angle V(ab)$

$$A(ab) \angle A(b'a') \cdot A(b'a') \angle I(b'a') \cdot \angle \cdot A(ab) \angle I(b'a')$$

$$A(ab) \angle I(b'a') \cdot I(b'a') \angle V(ba) \cdot \angle \cdot A(ab) \angle V(ba)$$

$$A(ab) \angle V(ba) \cdot V(ba) \angle V(ab) \cdot \angle \cdot A(ab) \angle V(ab)$$

Theorem: $A(ab) \angle I(ba)$

Theorem: $A(ab) \angle V(ba)$

Theorem: $E(ab) \angle O(ab)$

$$A(ab) \angle I(ab) \cdot \angle \cdot I'(ab) \angle A'(ab)$$

by the first principle and

$$E(ab) \angle I'(ab) \cdot I'(ab) \angle A'(ab) \cdot \angle \cdot E(ab) \angle A'(ab)$$

$$E(ab) \angle A'(ab) \cdot A'(ab) \angle O(ab) \cdot \angle \cdot E(ab) \angle O(ab)$$

by the second principle.

Theorem: $E(ab) \angle O(ba)$

$$E(ab) \angle E(ba) \cdot E(ba) \angle O(ba) \cdot \angle \cdot E(ab) \angle O(ba)$$

Theorem: $U(ab) \angle O(ab)$

$$A(ab) \angle V(ab) \cdot \angle \cdot V'(ab) \angle A'(ab)$$

by the first principle, etc.

Theorem: $U(ab) \angle O(ba)$

In order to prove our remaining theorems we introduce a third

Principle: $(p \angle q) \angle (p \angle p)$

Theorem:

$A(ab) \angle A(ab)$ $\quad$ $E(ab) \angle E(ab)$ $\quad$ $U(ab) \angle U(ab)$

$V(ab) \angle V(ab)$ $\quad$ $I(ab) \angle I(ab)$ $\quad$ $O(ab) \angle O(ab)$

$A(ab) \angle I(ab) \cdot \angle \cdot A(ab) \angle A(ab)$

$O(ab) \angle A'(ab) \cdot \angle \cdot O(ab) \angle O(ab)$ $\quad$ etc.

We have now constructed all true propositions, or as we shall call them, *valid moods*, following tradition, of the form $P(a, b) \angle Q(a, b)$. The array of all propositions of this form is known as the array of *immediate inference*. There will evidently be only two *figures* or term orders, $P(ab) \angle Q(ab)$ being the same as $P(ba) \angle Q(ba)$, and $P(ab) \angle Q(ba)$ being the same as $P(ba) \angle Q(ab)$. If we allow P and Q to take on in every possible way the values A, E, I, O, U, V, we should have thirty-six *moods*, the permutations of the six letters taken two at a time. Leaving out for brevity the parts (ab) and (ba) and the sign of implication $(\angle)$ but understanding them to be present, we should have:

The array $P(ab) \angle Q(ab)$

AA	*EA*	*IA*	*OA*	*UA*	*VA*
AE	***EE***	*IE*	*OE*	*UE*	*VE*
AI	*EI*	***II***	*OI*	*UI*	*VI*
AO	***EO***	*IO*	***OO***	***UO***	*VO*
AU	*EU*	*IU*	*OU*	***UU***	*VU*
AV	*EV*	*IV*	*OV*	*UV*	***VV***

The array $P(ab) \angle Q(ba)$

AA	*EA*	*IA*	*OA*	*UA*	*VA*
AE	***EE***	*IE*	*OE*	*UE*	*VE*
AI	*EI*	***II***	*OI*	*UI*	*VI*
AO	***EO***	*IO*	*OO*	***UO***	*VO*
AU	*EU*	*IU*	*OU*	***UU***	*VU*
AV	*EV*	*IV*	*OV*	*UV*	***VV***

the valid moods being printed in boldface.

It remains to establish the invalidity of those moods that are not valid. Our task will be simplified if we solve first a special case of the present array, viz. the set of all propositions of the form,

$$P(aa) \angle Q(aa)$$

If we leave out for brevity the part (*aa*) and the symbol ($\angle$) the array of all propositions of this form will be:

AA	***EA***	***IA***	***OA***	***UA***	***VA***
AE	***EE***	*IE*	***OE***	***UE***	*VE*
AI	***EI***	***II***	***OI***	***UI***	***VI***
AO	***EO***	*IO*	***OO***	***UO***	*VO*
AU	***EU***	*IU*	***OU***	***UU***	*VU*
AV	***EV***	***IV***	***OV***	***UV***	***VV***

the valid moods being printed in boldface.

Principle: $(p' \angle p) \angle (q \angle p)$

Postulate: $A'(aa) \angle A(aa)$

Theorem: $I'(aa) \angle I(aa)$

for $A(aa) \angle I(aa) \cdot \angle \cdot I'(aa) \angle A'(aa)$

$I'(aa) \angle A'(aa) \cdot A'(aa) \angle A(aa) \cdot \angle \cdot I'(aa) \angle A(aa)$

$I'(aa) \angle A(aa) \cdot A(aa) \angle I(aa) \cdot \angle \cdot I'(aa) \angle I(aa)$

Theorem: $V'(aa) \angle V(aa)$

for $A(aa) \angle V(aa) \cdot \angle \cdot V'(aa) \angle A'(aa)$

$V'(aa) \angle A'(aa) \cdot A'(aa) \angle A(aa) \cdot \angle \cdot V'(aa) \angle A(aa)$

$V'(aa) \angle A(aa) \cdot A(aa) \angle V(aa) \cdot \angle \cdot V'(aa) \angle V(aa)$

Theorem: All the moods ending in A, I or V are valid.

Thus, $V'(aa) \angle V(aa) \cdot \angle \cdot E(aa) \angle V(aa)$ etc.

Theorem: All the moods valid in the array $P(ab) \angle Q(ab)$ will be valid in the array $P(aa) \angle Q(aa)$, by equating a and b.

Theorem: $O(aa) \angle E(aa)$

$I(aa) \angle A(aa) \cdot \angle \cdot A'(aa) \angle I'(aa)$

$O(aa) \angle A'(aa) \cdot A'(aa) \angle I'(aa) \cdot \angle \cdot O(aa) \angle I'(aa)$

$O(aa) \angle I'(aa) \cdot I'(aa) \angle E(aa) \cdot \angle \cdot O(aa) \angle E(aa)$

Theorem: $U(aa) \angle E(aa)$

for $I(aa) \angle V(aa) \cdot \angle \cdot V'(aa) \angle I'(aa)$

$U(aa) \angle V'(aa) \cdot V'(aa) \angle I'(aa) \cdot \angle \cdot U(aa) \angle I'(aa)$

$U(aa) \angle I'(aa) \cdot I'(aa) \angle E(aa) \cdot \angle \cdot U(aa) \angle E(aa)$

Theorem: $E(aa) \angle U(aa)$

for $V(aa) \angle I(aa) \cdot \angle \cdot I'(aa) \angle V'(aa)$

$E(aa) \angle I'(aa) \cdot I'(aa) \angle V'(aa) \cdot \angle \cdot E(aa) \angle V'(aa)$

$E(aa) \angle V'(aa) \cdot V'(aa) \angle U(aa) \cdot \angle \cdot E(aa) \angle U(aa)$

Theorem: $O(aa) \angle U(aa)$

for $V(aa) \angle A(aa) \cdot \angle \cdot A'(aa) \angle V'(aa)$

$O(aa) \angle A'(aa) \cdot A'(aa) \angle V'(aa) \cdot \angle \cdot O(aa) \angle V'(aa)$

$O(aa) \angle V'(aa) \cdot V'(aa) \angle U(aa) \cdot \angle \cdot O(aa) \angle U(aa)$

Principle: $(p' \angle p) \angle (p \angle p')'$

Principle: $(pq \angle r) \angle (pr' \angle q')$

Principle: $pq \angle qp$

Theorem: $(p \angle q) \cdot (p \angle r)' \angle (q \angle r)'$

$(p \angle r)' \cdot (q \angle r) \angle (p \angle q)'$

Theorem: $A(aa) \angle O(aa)$ is untrue.

for $[A'(aa) \angle A(aa)] \angle [A(aa) \angle A'(aa)]'$

$[A(aa) \angle A'(aa)]' \cdot [O(aa) \angle A'(aa)] \angle [A(aa) \angle O(aa)]'$

Theorem: $A(aa) \angle E(aa)$ is untrue.

$E(aa) \angle O(aa) \cdot O(aa) \angle A'(aa) \cdot \angle \cdot E(aa) \angle A'(aa)$

$[A(aa) \angle A'(aa)]' \cdot [E(aa) \angle A'(aa)] \angle [A(aa) \angle E(aa)]'$

Theorem: $A(aa) \angle U(aa)$ is untrue.

$U(aa) \angle O(aa) \cdot O(aa) \angle A'(aa) \cdot \angle \cdot U(aa) \angle A'(aa)$

$[A(aa) \angle A'(aa)]' \cdot [U(aa) \angle A'(aa)] \angle [A(aa) \angle U(aa)]'$

Theorem: $I(aa) \angle E(aa)$ is untrue.

$[I'(aa) \angle I(aa)] \angle [I(aa) \angle I'(aa)]'$

$[I(aa) \angle I'(aa)]' \cdot [E(aa) \angle I(aa)] \angle [I(aa) \angle E(aa)]'$

Theorem: $I(aa) \angle O(aa)$ is untrue.

$O(aa) \angle A(aa) \cdot A(aa) \angle O'(aa) \cdot \angle \cdot O(aa) \angle O'(aa)$

$[A(aa) \angle I(aa)] \cdot [A(aa) \angle O(aa)]' \angle [I(aa) \angle O(aa)]'$

Theorem: $I(aa) \angle U(aa)$ is untrue.

$[A(aa) \angle I(aa)] \cdot [A(aa) \angle U(aa)]' \angle [I(aa) \angle U(aa)]'$

Theorem: $V(aa) \angle E(aa)$ is untrue.

$[A(aa) \angle V(aa)] \cdot [A(aa) \angle E(aa)]' \angle [V(aa) \angle E(aa)]'$

Theorem: $V(aa) \angle U(aa)$ is untrue.

$[A(aa) \angle V(aa)] \cdot [A(aa) \angle U(aa)]' \angle [V(aa) \angle U(aa)]'$

Theorem: $V(aa) \angle O(aa)$ is untrue.

$[V(aa) \angle E(aa)]' \cdot [O(aa) \angle E(aa)] \angle [V(aa) \angle O(aa)]'$

Theorem: $E(aa) \angle E'(aa)$

$E(aa) \angle I(aa) \cdot I(aa) \angle E'(aa) \cdot \angle \cdot E(aa) \angle E'(aa)$

Theorem: $U(aa) \angle U'(aa)$

$U(aa) \angle V(aa) \cdot V(aa) \angle U'(aa) \cdot \angle \cdot U(aa) \angle U'(aa)$

Theorem: $O(aa) \angle O'(aa)$

$O(aa) \angle A(aa) \cdot A(aa) \angle O'(aa) \cdot \angle \cdot O(aa) \angle O'(aa)$

Theorem: $(p \angle p') \angle (p' \angle p)'$

Theorem: $E'(aa) \angle E(aa)$ is untrue.

Theorem: $U'(aa) \angle U(aa)$ is untrue.

Theorem: $O'(aa) \angle O(aa)$ is untrue.

If $P(ab)$ and $Q(ab)$ satisfy the first of the conditions,

$$(1)\ P \angle Q' \qquad (2)\ Q' \angle P$$

but not the second, P is said to be the *contrary* of Q.

If $P(ab)$ and $Q(ab)$ satisfy the second of these conditions but not the first, P is said to be the *subcontrary* of Q.

If $P(ab)$ and $Q(ab)$ satisfy neither condition, P is said to be the *subaltern* of Q.

Theorem: If P is respectively the contrary, subcontrary, subaltern of Q, then Q is respectively the contrary, subcontrary, subaltern of P.

From the results just obtained we have the

Theorem: $E(aa)$, $U(aa)$ and $O(aa)$ are the contraries of themselves.

Theorem: $A(aa)$, $I(aa)$ and $V(aa)$ are the subcontraries of themselves.

Definition: A form that is the contrary of itself is called a *null-form*.

It is represented by the symbol (0).

Definition: A form that is the subcontrary of itself is called a *one-form*.

It is represented by the symbol (1).

The solution of the array $P(aa') \angle Q(aa')$ involves no novelty of method and no new assumption. We shall content ourselves with a few illustrations and then set down the results:

Since $A(ab) = E(ab'),\ A(aa) = E(aa')$

$$E'(aa') \angle A'(aa) \cdot A'(aa) \angle A(aa) \cdot \angle \cdot E'(aa') \angle A(aa)$$

$$E'(aa') \angle A(aa) \cdot A(aa) \angle E(aa') \cdot \angle \cdot E'(aa') \angle E(aa')$$

Since $V(ab) = O(a'b),\ V(aa) = O(a'a)$

$$O'(a'a) \angle V'(aa) \cdot V'(aa) \angle V(aa) \cdot \angle \cdot O'(a'a) \angle V(aa)$$

$$O'(a'a) \angle V(aa) \cdot V(aa) \angle O(a'a) \cdot \angle \cdot O'(a'a) \angle O(a'a)$$

Since $A(ab) = U(a'b),\ A(a'a) = U(aa)$

$$A(a'a) \angle U(aa) \cdot U(aa) \angle U'(aa) \cdot \angle \cdot A(a'a) \angle U'(aa)$$

$$A(a'a) \angle U'(aa) \cdot U'(aa) \angle A'(a'a) \cdot \angle \cdot A(a'a) \angle A'(a'a)$$

Since $O(ab) = I(ab'),\ O(aa) = I(aa')$

$$I(aa') \angle O(aa) \cdot O(aa) \angle O'(aa) \cdot \angle \cdot I(aa') \angle O'(aa)$$

$$I(aa') \angle O'(aa) \cdot O'(aa) \angle I'(aa') \cdot \angle \cdot I(aa') \angle I'(aa')$$

These results summarized would be:

$$A(aa) = I(aa) = V(aa) = E(aa') = U(aa') = O(aa') = 1 \text{ (one)}$$

$$A(aa') = I(aa') = V(aa') = E(aa) = U(aa) = O(aa) = 0 \text{ (zero)}$$

Let us now return to the array $P(a, b) \angle Q(a, b)$ in order to establish the untruth of those moods not already proved to be valid. We remark in the first place that any mood that has been proven invalid in either of the special arrays, $P(aa) \angle Q(aa)$ or $P(aa') \angle Q(aa')$, will also be invalid in the more general array $P(a, b) \angle Q(a, b)$, because it will have been shown to fail in a special case. Thus, $E(aa') \angle A(aa')$ is a special case in which $E(ab)$

$\angle A(ab)$ fails, so that this latter mood is untrue. $A(aa) \angle E(aa)$ is a special case in which $A(ab) \angle E(ba)$ fails, so that $A(ab) \angle E(ba)$ is not true in general, that is, is untrue. We may, therefore, assume that we have established the invalidity of the following moods of the array $P(a, b) \angle Q(a, b)$: *AE, AO, AU, IE, IO, IU, VE, VO, VU, EA, EI, EV, OA, OI, OV, UA, UI, UV.*

In order to prove the invalidity of those that remain it will be necessary to introduce an existential postulate, which will constantly reappear in the work which follows:

Postulate: There exists an $a, b, c,$ such that,

$$O(ab) \angle O'(ab), \qquad O(cb) \angle O'(cb), \qquad I(ca) \angle I'(ca)$$

Theorem: There exists an $a, b,$ such that,

$$A(ba) \angle A'(ba), \qquad O(ab) \angle O'(ab)$$

We will abridge the proof of this theorem. From a result established later on we have,

$$A(ba) \cdot A(cb) \angle I(ca)$$

$$\therefore \quad O'(cb) \cdot I'(ca) \angle A'(ba)$$

$$\text{But } O'(cb) = 1, \qquad I'(ca) = 1, \qquad A'(ba) = 1$$

$$\therefore \quad A(ba) = 0, \qquad O(ab) = 0$$

Theorem: $A(ab) \angle A(ba)$ is untrue.

$$[A'(ab) \angle A(ab)] \angle [A(ab) \angle A'(ab)]' \, p' \angle p \cdot \angle \cdot (p \angle p')'$$

$$[A(ba) \angle A'(ba)] \angle [A(ba) \angle A'(ab)] \, p \angle p' \cdot \angle \cdot p \angle q$$

$[A(ab) \angle A'(ab)]' \cdot [A(ba) \angle A'(ab)] \angle [A(ab) \angle A(ba)]'$

$[A(ab) \angle A(ba)]' \cdot [E(ba') \angle A(ba)] \angle [A(ab) \angle E(ba')]'$

$[A(ab) \angle U(a'b)] \cdot [A(ab) \angle E(ba')]' \angle [U(a'b) \angle E(ba')]'$

$[A(ab) \angle A(ba)]' \cdot [U(b'a) \angle A(ba)] \angle [A(ab) \angle U(b'a)]'$

$[A(ab) \angle E(ab')] \cdot [A(ab) \angle U(b'a)]' \angle [E(ab') \angle U(b'a)]'$

Theorem: $E(a, b) \angle U(a, b)$ is untrue.

$U(a, b) \angle E(a, b)$ is untrue.

Theorem: $V(a, b) \angle I(a, b)$ is untrue.

$I(a, b) \angle V(a, b)$ is untrue.

Theorem: $I(a, b) \angle A(a, b)$ is untrue.

$[A(ab) \angle I(ab)] \cdot [A(ab) \angle A(ba)]' \angle [I(ab) \angle A(ba)]'$

$[A(ab) \angle I(ba)] \cdot [A(ab) \angle A(ba)]' \angle [I(ba) \angle A(ba)]'$

Theorem: $O(a, b) \angle E(a, b)$ is untrue.

Theorem: $V(a, b) \angle A(a, b)$ is untrue.

Theorem: $O(a, b) \angle U(a, b)$ is untrue.

Since the time of Leibnitz it has been commonly accepted that the traditional logic is not true in all of its parts, that subalternation ($A \angle I$, $E \angle O$) and some of the moods of the syllogism, recognized as valid, fail when the terms are allowed to take on the limiting values nothing (zero) and universe (one). The motive which impels us to introduce into logic the notion of an empty class, a class which has no members, is one that concerns the generality of the science. Thus, if we had not defined (so as to be able to control) the notion of a null-class and the notion of a null-proposition, certain cases would arise which would not be interpretable and which our science would then have to declare to be meaningless. We have already met with such cases in connection with the *zero* and *one* of propositions. Let us mention another. The syllogism in *Celarent*,

$$E(ba) \cdot A(cb) \angle E(ca)$$

becomes, when we identify terms in the major premise,

$$E(aa) \cdot A(ca) \angle E(ca)$$

and this result is only interpretable when we have recognized $E(aa)$ as a null-proposition. If the major premise "vanishes" then so too does the antecedent, the conclusion follows and *Celarent* remains valid. The older logician might take account of this situation by saying, "the terms are by presumption distinct." But by so doing he set a restriction on the meaning of his variables and thus reduced the generality of his science. Once we have understood that $E(aa)$ is a null-proposition, we "lift out" this limitation and may assert the validity of *Celarent* in a perfectly general sense.

Similar considerations lead us to introduce into logic the notion of an empty class. There may, for example, appear in our formulas not only the letters

a, b, c, but "sums" and "products" made up of these,

$$a \times b = \text{(what is) } a \text{ and } b$$

$$b + c = \text{(what is) } b \text{ or } c$$

and we should like to interpret these as classes as well. We should like to say, whenever (what is) a is a class and (what is) b is a class, then (what is) a and b is also a class. But if we wish to keep on saying this we must recognize the existence of empty classes, for sometimes a and b have no common membership.

Again, there may appear in our formulas besides the letters a, b, c, the negatives of these letters, a', b', c', and we should want to know how to interpret a' when a has taken on the value "everything" (universe). That is, if we insist that every class has a negative within the universe of discourse, we shall have to ask: what is the negative of the universe? The introduction of the null-class depends, then, upon considerations of generality.

Once we have recognized this conception, however, some of the implications already taken to be valid seem to turn out to be untrue. Besides the relation of subalternation, $A \angle I$, criticized by Leibnitz and his successors, some of the moods of the syllogism, traditionally recognized as valid, seem to fail. It has seemed very natural to identify the "all is" relation of Aristotle with the Boolean relation of "inclusion." In that case, if we represent Boolean "inclusion" by the symbol ($\angle$), then

$$A(ab) = a \angle b = a \text{ is included in } b$$

$$E(ab) = a \angle b' = a \text{ is included in non-}b$$

in order to preserve the so-called relation of "obversion," and then, if $I(ab)$ is to be the contradictory of $E(ab)$,

$$I(ab) = (a \angle b')' = a \text{ is not included in non-}b$$

Subalternation then becomes, for the case in which a is a null-class,

$$0 \angle b \cdot \angle \cdot (0 \angle b')'$$

and in the Boolean algebra the antecedent of this expression is true, while the consequent is false. Subalternation, then, is said to fail unless we refuse to allow zero (0) and one (1) as possible meanings of the terms. The traditional logic would lack generality, since restrictions have been placed on the possible values of its terms. In the case of the syllogism, the mood *Darapti*, for example, would fail whenever the conclusion is false and the middle term has become an empty class.

All of this criticism would be unanswerable if it were not for one of the assumptions which it makes, an assumption simple and natural in itself, but one that is not forced upon us. It is assumed that Aristotelian inclusion and Boolean inclusion must mean the same thing, that is, that they enter into the same kind of algebra and have the same properties. The fact is that these two algebras are essentially distinct. Either one may be deduced from the other by a certain transformation. They are, accordingly, of equal generality. Each one is true without any restriction whatever on the meaning of the terms. The transformation formula gives the correct relation between the two kinds of "inclusion." Let us give first the one which enables us to deduce Aristotelian logic in its complete generality from the logic of Boole. It is:

$$A(ab) = (a \angle b) \cdot [(b \angle a) + (a \angle b')' \cdot (b' \angle a)']$$

In order to preserve "obversion," $A(ab) = E(ab')$ and its analogue $A(ab) = U(a'b)$, we assume,

$$E(ab) = (a \angle b') \cdot [(b' \angle a) + (a \angle b)' \cdot (b \angle a)']$$

$$U(ab) = (a' \angle b) \cdot [(b \angle a') + (b \angle a)' \cdot (a \angle b)']$$

The definition of $O(ab)$, $I(ab)$ and $V(ab)$ will be the respective contradictories of these, that is, by the well-known rules of Boolean algebra,

$$O(ab) = (a \angle b)' + (b \angle a)' \cdot [(a \angle b') + (b' \angle a)]$$

$$I(ab) = (a \angle b')' + (b' \angle a)' \cdot [(a \angle b) + (b \angle a)]$$

$$V(ab) = (a' \angle b)' + (b \angle a')' \cdot [(b \angle a) + (a \angle b)]$$

It will be evident at once to anyone familiar with the classical Boole-Schroeder algebra that these definitions lead at once to the following results: (1) *A, O* and *E, I* and *U, V* are contradictory pairs, (2) the ordinary process of privative conception is valid, (3) conversion by contraposition holds of *A* and *O* but not of the other forms, (4) all of the forms except *A* and *O* are simply convertible, (5) *A, I* and *V* become true when the terms are identified and the other forms then become false, (6) *E, O* and *U* become true when the terms become contradictory and the other forms then become false. In order to complete everything that remains of traditional doctrine, it is only necessary to prove subalternation and the syllogistic mood in *Barbara*. In the case of subalternation, since,

$$(A \cdot E \angle \text{zero}) \angle (A \angle I)$$

from $$(p \cdot q \angle 0) \angle (p \angle q')$$

it will only be necessary to prove that the product of *A* and *E* vanishes. We have:

$$A(ab) = (a \angle b) \cdot [(b \angle a) + (a \angle b')' \cdot (b' \angle a)']$$

$$E(ab) = (a \angle b') \cdot [(b' \angle a) + (a \angle b)' \cdot (b \angle a)']$$

$$\therefore \quad A(ab) \cdot E(ab) = (a \angle b) \cdot (a \angle b') \cdot (b \angle a) \cdot (b' \angle a)$$

$$\cdot \angle \cdot (a \cdot a \angle b \cdot b') \cdot (b + b' \angle a + a)$$

$$\cdot \angle \cdot (a \angle 0) \cdot (1 \angle a)$$

$$\cdot \angle \cdot (1 \angle 0)$$

$$\cdot \angle \cdot 0$$

In the case of the mood in *Barbara*, since,

$$\mathrm{A}(ba) \cdot A(cb) \angle A(ca) \cdot \angle \cdot A(ba) \cdot A(cb) \cdot O(ca) \angle \text{zero}$$

and conversely, it will only be necessary to prove that the product of $A(ba)$, $A(cb)$ and $O(ca)$ vanishes. We have as before,

$$A(ba) = (b \angle a) \cdot [(a \angle b) + (b \angle a')' \cdot (a' \angle b)']$$

$$A(cb) = (c \angle b) \cdot [(b \angle c) + (c \angle b')' \cdot (b' \angle c)']$$

$$O(ca) = (c \angle a)' + (a \angle c)' \cdot [(c \angle a') + (a' \angle c)]$$

If we multiply these out we shall find that each term contains a zero factor, thus:

$$[(c\angle b)(b\angle a)(c\angle a)'](a\angle b)(b\angle c)+$$

$$[(c\angle b)(b\angle a)(c\angle a)'](a\angle b)(c\angle b')'(b'\angle c)'+$$

$$[(c\angle b)(b\angle a)(c\angle a)'](b\angle c)(a\angle b')'(b'\angle a)'+$$

$$[(c\angle b)(b\angle a)(c\angle a)'](a\angle b')'(b'\angle a)'(c\angle b')'(b'\angle c)'+$$

$$[(a\angle b)(b\angle c)(a\angle c)'](b\angle a)(c\angle b)(c\angle a')+$$

$$[(b\angle a)(a\angle c')(b\angle c')'](a\angle b)(c\angle b)(b'\angle c)'(a\angle c)'+$$

$$[(b\angle c)(c\angle a')(b\angle a')'](c\angle b)(b\angle a)(b'\angle a)'(a\angle c)'+$$

$$[(b\angle a)(a\angle c')(b\angle c')'](c\angle b)(a\angle b')'(b'\angle a)'(b'\angle c)'(a\angle c)'+$$

$$[(a\angle b)(b\angle c)(a\angle c)'](a'\angle c)(b\angle a)(c\angle b)+$$

$$[(c'\angle a)(a\angle b)(c'\angle b)'](b\angle a)(c\angle b)(c\angle b')'(a\angle c)'+$$

$$[(a'\angle c)(c\angle b)(a'\angle b)'](b\angle c)(b\angle a)(a\angle b')'(a\angle c)'+$$

$$[(a'\angle c)(c\angle b)(a'\angle b)'](b\angle a)(a\angle b')'(c\angle b')'(b'\angle c)'(a\angle c)'$$

We have rearranged the factors in each term so that the "zero" part, placed in square brackets, shall appear at the beginning, and we have made some changes corresponding to $(c \angle a') = (a \angle c')$, $(b' \angle c)' = (c' \angle b)'$ etc. The truth of the mood in *Barbara* being established, the remaining twenty-three valid moods of the syllogism follow at once from the results already set down.

With the introduction of the U- and V-forms there is another critical case, viz. $A(ab) \angle V(ab)$, which is proved in the same way as subalternation, this time by showing that the product of $A(ab)$ and $U(ab)$ vanishes. The array of the new syllogism will contain eight hundred and sixty-four moods, instead of the traditional two hundred and fifty-six, and there will be twenty valid moods in each figure except the fourth, which will have twenty-one.

The underlying properties of Aristotle's logic having now been established, and no restriction whatever having been laid down as to what the terms shall mean, this logic will be true in all of its parts and for every meaning which the terms may take on, provided the logic of Boole holds true. Before deriving Boole's system from that of Aristotle, and so showing that the two systems, so far from standing in contradiction to one another, are equivalent and have the same generality, we shall add a number of lemmas to what has gone before, because these lemmas will add brevity to our derivation. Let us for convenience represent the "all, is" relation of traditional logic thus,

$$A(ab) = a \supset b = \text{all } a \text{ is } b$$

1.1 $$[a \supset b] \cdot [a' \supset a \cdot a'] \cdot [b' \supset b \cdot b']' \angle 0$$

for, $[a' \supset 0] = [a' \angle 0][0 \angle a'] + [a' \angle 0][a' \angle 0']'[0' \angle a']'$

$$= [a' \angle 0]$$

since in this algebra,

$$[0 \angle a] \text{ and } [a \angle 0']$$

are unconditionally true.

$$\therefore \quad [a \supset b][a' \supset 0][b' \supset 0]' =$$
$$[0' \angle a][a \angle b][0' \angle b]'[b \angle a] +$$
$$[0' \angle a][a \angle b][0' \angle b]'[a \angle b']'[b' \angle a]'$$

and the product of the first three factors in each term vanishes.

1.2 $$[a \supset b] \cdot [a \supset a \cdot a']' \cdot [b \supset b \cdot b'] \angle 0$$

by changing a into a' and b into b'.

2.1 $$[a \supset b] \cdot [a \cdot b' \supset 0]' \angle 0$$

2.2 $$[a \supset a \cdot a'] \cdot [ab' \supset 0]' \angle 0$$

We have,

$$[a \supset 0][a \cdot b' \supset 0]' = [a \angle 0][a \angle b]'$$
$$[a \angle 0 \cdot \angle \cdot a \angle b] \angle$$
$$[(a \angle 0)(a \angle b)' \angle (a \angle b)(a \angle b)']$$

But since the antecedent of this long expression is true,

$$a \angle 0 \cdot (a \angle b)' \angle 0$$

2.3 $[b' \supset b \cdot b'] \cdot [a \cdot b' \supset 0]' \angle 0$

3.1 $[a \supset b]' \cdot [a \cdot b' \supset 0] \cdot [a \supset 0]' \cdot [b' \supset 0]' \angle 0$

The product vanishes since,

$$[a \angle b]'[a \cdot b' \angle 0] \angle 0$$

$$[a \angle b][a \angle b'][a \angle 0]' \angle 0$$

4.1 $[a \cdot b \supset b]' \cdot [a \cdot b \supset 0]' \cdot [b' \supset 0]' \angle 0$

5.1 $[a \supset b] \cdot [c \supset d] \cdot [a \cdot c \supset b \cdot d]' \cdot$
$[a \cdot c \supset 0]' \cdot [b' + d' \supset 0]' \angle 0$

The product vanishes in virtue of

$$[a \angle b][c \angle d][a \cdot c \angle b' + d'][a \cdot c \angle 0]' \angle 0$$

for $[a \angle b][c \angle d][a \cdot c \angle b' + d'] \angle$
$[a \cdot c \angle b' + d'][a \cdot c \angle b \cdot d] \angle$
$[a \cdot c \angle 0]$

The remaining results we shall set down without proof.*

5.2 $[c \supset d] \cdot [b' \supset 0] \cdot [b' + d' \supset 0]' \cdot [a \cdot c \supset 0]' \cdot$
$[a \cdot c \supset b \cdot d]' \angle 0$

5.3 $[a \supset b] \cdot [d' \supset 0] \cdot [b' + d' \supset 0]' \cdot [a \cdot c \supset 0]' \cdot$
$[a \cdot c \supset b \cdot d]' \angle 0$

* For these proofs see *Logic as the Art of Symbols* at the end of the second edition of the author's *First Book in Logic* (New York, F. S. Crofts & Co., 1933).

5.4 $[b' \supset 0] \cdot [d' \supset 0] \cdot [b' + d' \supset 0]' \angle 0$

6.1 $[a \cdot b \supset c] \cdot [a \cdot c' \supset b']' \cdot [a \cdot c' \supset 0]' \cdot [b \supset 0]' \angle 0$

6.2 $[a \cdot b \supset 0] \cdot [b \supset 0]' \cdot [a \cdot c' \supset 0]' \cdot [a \cdot c' \supset b']' \angle 0$

7.1 $[a \cdot b \supset c] \cdot [d \supset b] \cdot [a \cdot d \supset c]' \cdot [a \cdot d \supset 0]' \cdot [c' \supset 0]' \angle 0$

7.2 $[a \cdot b \supset c] \cdot [b' \supset 0] \cdot [a \cdot d \supset c]' \cdot [a \cdot d \supset 0]' \cdot [c' \supset 0]' \angle 0$

7.3 $[a \cdot b \supset 0] \cdot [d \supset b] \cdot [a \cdot d \supset c]' \cdot [a \cdot d \supset 0]' \cdot [c' \supset 0]' \angle 0$

7.4 $[a \cdot b \supset 0] \cdot [b' \supset 0] \cdot [a \cdot d \supset c]' \cdot [a \cdot d \supset 0]' \cdot [c' \supset 0]' \angle 0$

We are now prepared, assuming Aristotle's system to be true, to derive the fundamental properties of Boolean algebra. Some of the multiplications will be long and involved but the only terms in our products which do not obviously disappear will be those whose vanishing we have established above. The formula which enables us to carry out this proof will evidently be one which expresses Boolean inclusion in terms of the "all, is" relation of Aristotle. It is to be written as follows:

$$[a \angle b] = [a \supset b] + [a \supset a \cdot a'] + [b' \supset b \cdot b']$$

From this formula we have very simply,

$$[a \angle a]' = [a \supset a]'[a \supset 0]'[a' \supset 0]' \angle 0$$

$$[a \angle b'] = [a \supset b'] + [a \supset 0] + [b \supset 0]$$

$$[b \angle a'] = [b \supset a'] + [b \supset 0] + [a \supset 0]$$

$$\therefore \quad [a \angle b'] = [b \angle a']$$

1.0 $[a \angle b][b \angle c][a \angle c]' \angle 0$
by 1.1 and 1.2

2.0 $[a \angle b][a \cdot b' \angle 0]' \angle 0$
by 2.1, 2.2, 2.3

3.0 $[a \angle b]'[a \cdot b' \angle 0] \angle 0$
by 3.1

4.0 $[a \cdot b \angle b]' \angle 0$
by 4.1

5.0 $[a \angle b][c \angle d][a \cdot c \angle b \cdot d]' \angle 0$
by 5.1, 5.2, 5.3, 5.4

6.0 $[a \cdot b \angle c][a \cdot c' \angle b']' \angle 0$
by 6.1, 6.2

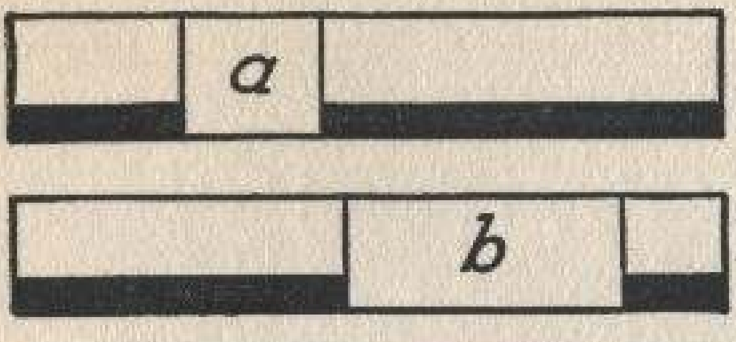
7.0 $[a \cdot b \angle c][d \angle b][a \cdot d \angle c]' \angle 0$
by 7.1, 7.2, 7.3, 7.4

The logic of Aristotle and the logic of Boole having been shown to be intertranslatable, we conclude that we are confronted, not by two irreconcilable doctrines, but by two systems that "speak in different tongues." Aristotle's system is no more and no less than a non-Boolean algebra of logic, and Boole's system is, if you like, non-Aristotelian. The apparent contradiction, which algebrists of logic have always taken to be final, turns out to be purely verbal.

Let us now, for purposes of comparison, set down side by side some of the characteristics of the two systems, it not being relevant in this context to inquire which of these properties are fundamental and which derived:

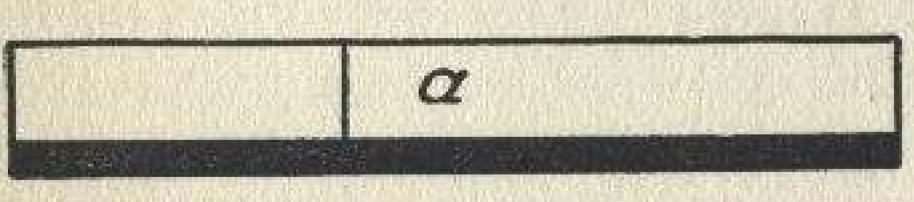
$[a \supset b][c \supset d][a \cdot c \supset b \cdot d]' \neq 0$	$[a \angle b][c \angle d][a \cdot c \angle b \cdot d]' = 0$
$[a \supset a]' = 0$	$[a \angle a]' = 0$
$[a \supset b'] = [b \supset a']$	$[a \angle b'] = [b \angle a']$
$[a \supset b][b \supset c][a \supset c]' = 0$	$[a \angle b][b \angle c][a \angle c]' = 0$
$[a \supset b][a \cdot b' \supset 0]' = 0$	$[a \angle b][a \cdot b' \angle 0]' = 0$
$[a \supset b]'[a \cdot b' \supset 0] \neq 0$	$[a \angle b]'[a \cdot b' \angle 0] = 0$
$[a \supset b][a \supset b'] = 0$	$[a \angle b][a \angle b'] \neq 0$
$[a \supset b][a' \supset b] = 0$	$[a \angle b][a' \angle b] \neq 0$
$[a \supset a'] = 0$	$[a \angle a'] \neq 0$
$[a \cdot b \supset b]' \neq 0$	$[a \cdot b \angle b]' = 0$
$[a \cdot b \supset c][a \cdot c' \supset b']' \neq 0$	$[a \cdot b \angle c][a \cdot c' \angle b']' = 0$
$[d \supset b][a \cdot b \supset c][a \cdot d \supset c]' \neq 0$	$[d \angle b][a \cdot b \angle c][a \cdot d \angle c]' = 0$

Since each one of these systems is derivable from the other by a transformation, they must either both be true (consistent) or both be false (inconsistent). Let us invent a matrix (in this case a geometrical representation) of each system which will exhibit its inner consistency. Plus (+) as well as times (× or ·) and the symbol of negation (′) have the same meaning in the two systems and hence will have the same matrices in each. But inclusion [$\supset$ or $\angle$] have different meanings and will be represented by different matrices. Let the letters *a*, *b*, *c*, . . . be represented as in the diagrams below:



the whole area being the universe, the part between the uprights in the middle being known as the *area* of *a*, *b*, and the black line at the base extending from these uprights to the ends of the universe being known as the *supplement* of *a*, *b*. Any class *a*, *b*, etc., will be understood to be represented by its area conjoined to its supplement.

Imagine, now, the uprights to approach one another, so that the area diminishes, and to coincide with one another in the limiting case. Then *a*, *b* will be all supplement and no area, as in the figure below:



Such a class will be called a null-class.

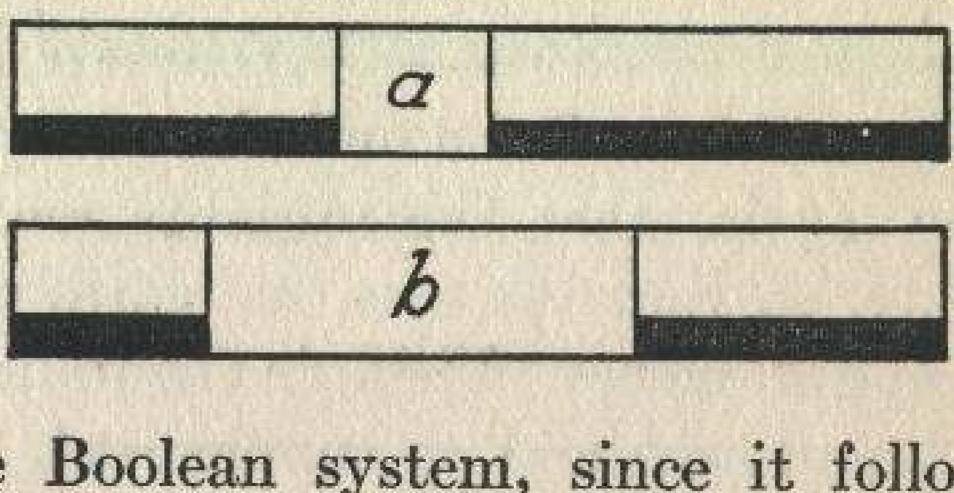
Imagine, again, that the uprights recede from one another, so that the area increases while the supplement diminishes. Then, in the limiting case, *a*, *b*, will be all area and no supplement, as in the figure below:

Such a class will be called a one-class.

Definition: A class a is said to be included in the class b, in the Boolean sense of inclusion, [$a \angle b$], if the area of a is contained in the area of b and the supplement of b is contained in the supplement of a.

Definition: A class a is said to be included in the class b, in the Aristotelian sense of inclusion, [$a \supset b$], if the area of a is contained in the area of b and the supplement of b is contained in the supplement of a; with the proviso that the supplement of the null-class contains no other supplement but itself, and that the supplement of the one-class is contained in no other supplement but its own.

When a and b are distinct from nothing and universe, the two meanings of inclusion are the same. Thus, if the two figures below be superimposed, it will be seen that they verify either of the conditions, [$a \supset b$] or [$a \angle b$].



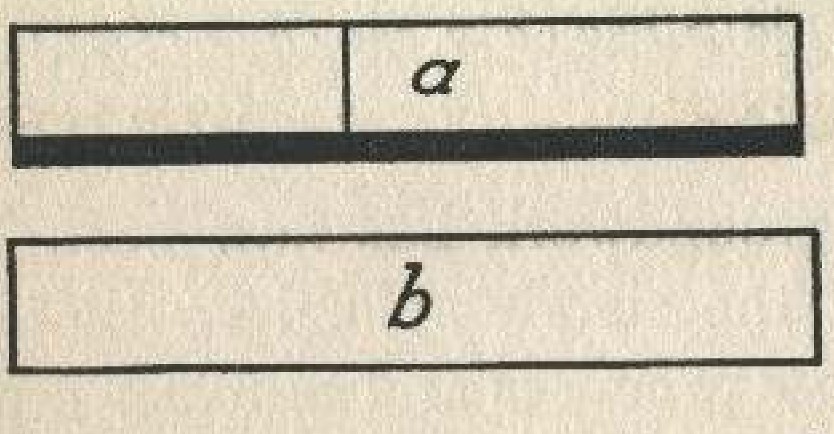
In the Boolean system, since it follows from $a' \cdot a' \angle a'$ that $a' \cdot a \angle a$, or $0 \angle a$, the null-class is contained in any class whatever. In the Aristotelian system (that is, Aristotle's system as we have extended it) zero is contained in nothing but itself and the universe contains nothing but itself, for while $a \cdot a \supset a$ holds here as there, the law of contradiction and interchange fails, and $0 \supset a$ can not be derived as a theorem. We add now the matrices that are common to the two systems.

Definition: The product of two classes is the common area conjoined to the sum of their supplements.

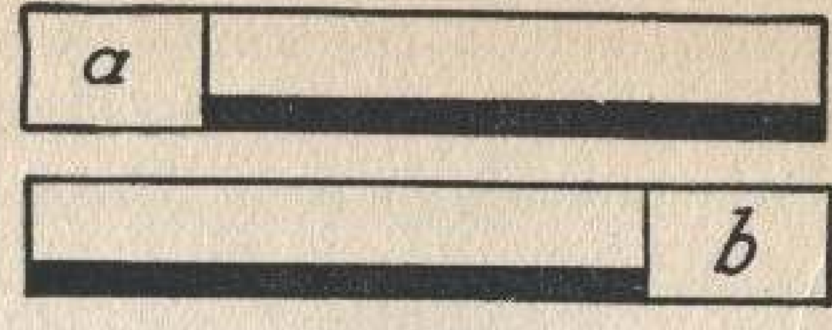
Definition: The sum of two classes is the sum of their areas conjoined to the supplement that is common to the two.

Definition: The negative of a class a, viz. a', is the area of a subtracted from the area of the universe conjoined to the supplement of a subtracted from the supplement of nothing.

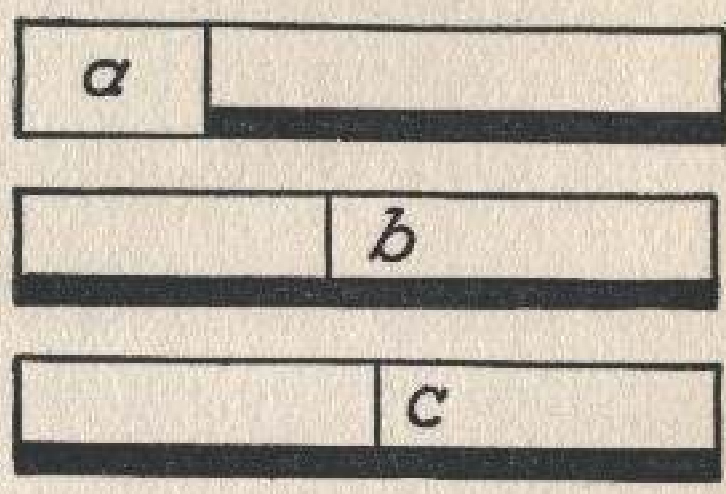
These matrices will verify the postulates of the two systems and will also show how some property not characteristic of the system fails. In the illustrations below the figures for each case are supposed to be superimposed. They then show a case in which the indicated proposition fails.



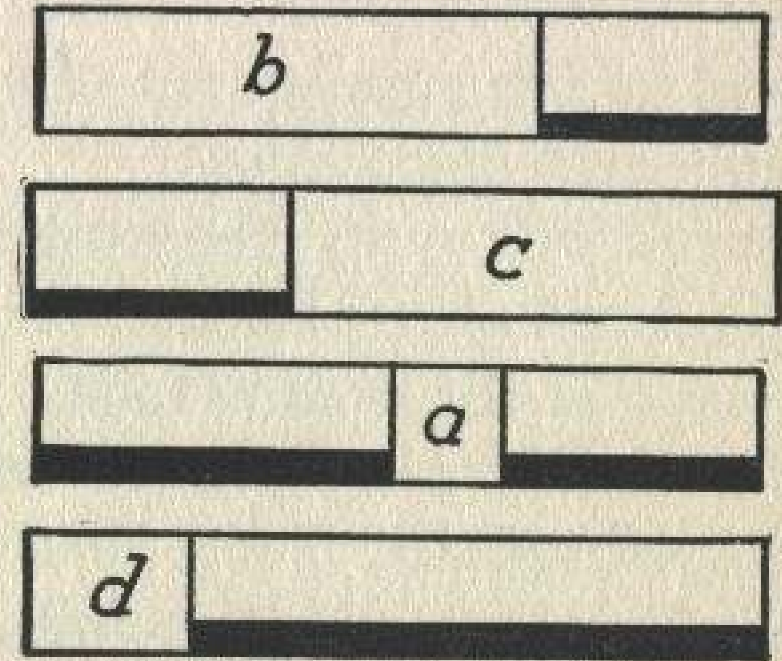
$[a \supset b]'[a \cdot b' \supset 0] \neq 0$ (figure above)



$$[a \cdot b \supset b]' \neq 0$$



$$[a \cdot b \supset c][a \cdot c' \supset b']' \neq 0$$



$$[d \supset b][a \cdot b \supset c][a \cdot d \supset c]' \neq 0$$

$$[d \supset b][a \supset c][a \cdot d \supset b \cdot c]' \neq 0$$

All the groundwork has now been laid for the solution of the simple syllogism, that is, the syllogism whose premises and conclusion are not conditioned by a modal operator. This more general case will be considered in the sequel, when we attack the most general problem of the class calculus.

The array of the syllogism is the array of all propositions of the form,

$$P(a, b) \cdot Q(b, c) \angle R(c, a)$$

an implication which shows two categorical forms conjoined in the antecedent, a single categorical form as the consequent and with the terms arranged in cyclical order. In the categorical form $R(ca)$ the terms are the subject c and the predicate a, and the term-order is the order subject-predicate. The two forms conjoined in the antecedent are called the premises and the consequent is called the conclusion. The predicate of the conclusion is called the major term and points out the major premise. Since logical multiplication is commutative, the order of the premises is indifferent, but we agree as a matter of convention always to write the major premise first in the antecedent. The subject of the conclusion is called the minor term and points out the minor premise. The term common to the premises, which does not appear in the conclusion, is called the middle term. There will evidently be four and only four term-orders, obtained by permuting the three letters in every possible way. These have been established by convention as:

1. $P(ba) \cdot Q(cb) \angle R(ca)$, the first figure

2. $P(ab) \cdot Q(cb) \angle R(ca)$, the second figure

3. $P(ba) \cdot Q(bc) \angle R(ca)$, the third figure

4. $P(ab) \cdot Q(bc) \angle R(ca)$, the fourth figure

It is obvious that, if we were to change $R(ca)$ into $R(ac)$, we should, after we had restored the conventional order of the premises, obtain the same four figures over again but in a different order. It is clear too that any one of the three letters a, b, c, may appear as major, minor or middle term. Thus,

$$P(ab) \cdot Q(bc) \angle R(ca)$$

$$P(ac) \cdot Q(cb) \angle R(ba)$$

$$P(ba) \cdot Q(ac) \angle R(cb)$$

are different equivalent ways of writing the fourth figure, for figure means nothing more than the ordered relation of major, minor and middle term to one another.

A little further back we proved that the "all, is" relation of traditional logic is transitive, if we assume an equation connecting it with the Boolean relation of "inclusion" and the properties of the Boolean algebra of classes. That is, we proved that

$$A(ba) \cdot A(cb) \angle A(ca)$$

the mood in *Barbara.*

If in this syllogism we were to change a into a', b into b', c into c', in succession, there would result,

$$A(ba') \cdot A(cb) \angle A(ca')$$

which is the same as $E(ba) \cdot A(cb) \angle E(ca)$

$$A(b'a) \cdot A(cb') \angle A(ca)$$

which is the same as $U(ba) \cdot E(cb) \angle A(ca)$

$$A(ba) \cdot A(c'b) \angle A(c'a)$$

which is the same as $A(ba) \cdot U(cb) \angle U(ca)$

From these four valid moods,

Postulate: $A(ba) \cdot A(cb) \angle A(ca)$

Theorem: $E(ba) \cdot A(cb) \angle E(ca)$

Theorem: $U(ba) \cdot E(cb) \angle A(ca)$

Theorem: $A(ba) \cdot U(cb) \angle U(ca)$

and principles already set down, viz.

$$(pq \angle r) \angle (qp \angle r)$$

$$(pq \angle r) \angle (pr' \angle q')$$

$$(pq \angle r) \angle (r'q \angle p')$$

$$(s \angle p) \cdot (pq \angle r) \angle (sq \angle r)$$

$$(s \angle q) \cdot (pq \angle r) \angle (ps \angle r)$$

$$(pq \angle r) \cdot (r \angle s) \angle (pq \angle s)\ *$$

follow the remaining seventy-seven valid moods of the syllogism, twenty in each one of the first three figures and twenty-one in the fourth. The array of the syllogism, each mood represented by one of the

* We have already noted that some of these principles employed are not true if the variables are unrestricted, but only on the condition that the antecedent is verified. Our results will be valid here because they are only employed in this sense and under this condition. These reservations will become clear when we come to deal with the algebra of propositions.

permutations of the six letters taken three at a time, is given on a separate page. A number after any mood indicates that it is valid in that figure.

The array of all propositions of the form, $P(a, b) \cdot Q(b, c) \angle R(c, a)$.

AAA 1	EAA	IAA	OAA	UAA	VAA
E	E 12	E	E	E	E
I 134	I	I 34	I	I	I
O	O 1234	O	O 3	O 34	O
U	U	U	U	U 34	U
V 124	V	V	V	V	V 12
AEA	EEA	IEA	OEA	UEA 1234	VEA
E 24	E	E	E	E	E
I	I	I	I	I 1234	I
O 24	O	O	O	O	O
U	U	U	U	U	U
V	V 1234	V	V 13	V 1234	V
AIA	EIA	IIA	OIA	UIA	VIA
E	E	E	E	E	E
I 13	I	I	I	I	I
O	O 1234	O	O	O	O
U	U	U	U	U	U
V	V	V	V	V	V
AOA	EOA	IOA	OOA	UOA	VOA
E	E	E	E	E	E
I	I	I	I	I 12	I
O 2	O	O	O	O	O
U	U	U	U	U	U
V	V 34	V	V	V	V
AUA	EUA	IUA	OUA	UUA	VUA
E	E	E	E	E	E
I	I 1234	I	I 24	I 1234	I
O 1234	O	O	O	O	O 1234
U 13	U	U	U	U	U
V	V 1234	V	V	V	V
AVA	EVA	IVA	OVA	UVA	VVA
E	E	E	E	E	E
I	I	I	I	I	I
O	O	O	O	O	O
U	U	U	U	U	U
V 24	V	V	V	V	V

We shall not take the time and space here to deduce all of the valid moods of the syllogism, but we shall give ample illustrations, so that the reader will be in no doubt about any of the processes:

$$[A(ba) \cdot A(cb) \angle A(ca)] \cdot [A(ca) \angle I(ac)] \angle [A(ba) \cdot A(cb) \angle I(ac)]$$

by $(pq \angle r) \cdot (r \angle s) \angle (pq \angle s)$

$$[A(ba) \cdot A(cb) \angle I(ac)] \angle [A(cb) \cdot A(ba) \angle I(ac)]$$

by $(pq \angle r) \angle (qp \angle r)$

$$\therefore \quad A(ba) \cdot A(cb) \angle A(ca)] \cdot [A(ca) \angle I(ac)] \angle [A(cb) \cdot A(ba) \angle I(ac)]$$

by $(p \angle q) \cdot (q \angle r) \angle (p \angle r)$

and suppressing the two factors before the main implication sign, because they are true, that is, suppressing the antecedent, we have:

Theorem: $A(cb) \cdot A(ba) \angle I(ac)$ [Bramantip]

$$[A(ab) \cdot A(bc) \angle I(ca)] \angle [I'(ca) \cdot A(bc) \angle A'(ab)]$$

$$[I'(ca) \cdot A(bc) \angle A'(ab)] \angle [A(bc) \cdot I'(ca) \angle A'(ab)]$$

by $(pq \angle r) \angle (r'q \angle p')$ and $(pq \angle r) \angle (qp \angle r)$

$$[E(ca) \angle I'(ca)] \cdot [A(bc) \cdot I'(ca) \angle A'(ab)] \angle [A(bc) \cdot E(ca) \angle A'(ab)]$$

by $(s \angle q) \cdot (pq \angle r) \angle (ps \angle r)$

$$[A(bc) \cdot E(ca) \angle A'(ab)] \cdot [A'(ab) \angle O(ab)] \angle [A(bc) \cdot E(ca) \angle O(ab)]$$

by $(pq \angle r) \cdot (r \angle s) \angle (pq \angle s)$

and suppressing the antecedent, because it is true, there results:

Theorem: $A(bc) \cdot E(ca) \angle O(ab)$ [a weakened Camenes]

$[E(ba) \cdot A(cb) \angle E(ca)] \angle [E'(ca) \cdot A(cb) \angle E'(ba)]$

$(pq \angle r) \angle (r'q \angle p')$

$[I(ac) \angle E'(ca)] \cdot [E'(ca) \cdot A(cb) \angle E'(ba)] \angle [I(ac) \cdot A(cb) \angle E'(ba)]$

by $(s \angle p) \cdot (pq \angle r) \angle (sq \angle r)$

$[I(ac) \cdot A(cb) \angle E'(ba)] \cdot [E'(ba) \angle I(ab)] \angle [I(ac) \cdot A(cb) \angle I(ab)]$

by $(pq \angle r) \cdot (r \angle s) \angle (pq \angle s)$

$[I(ac) \cdot A(cb) \angle I(ab)] \angle [A(cb) \cdot I(ac) \angle I(ab)]$

by $(pq \angle r) \angle (qp \angle r)$

Consequently, there results as before:

Theorem: $A(cb) \cdot I(ac) \angle I(ab)$ [Darii]

$[A(ca) \angle I(ac)] \cdot [A(cb) \cdot I(ac) \angle I(ab)] \angle [A(cb) \cdot A(ca) \angle I(ab)]$

by $(s \angle q) \cdot (pq \angle r) \angle (ps \angle r)$

So that we have, suppressing the antecedent:

Theorem: $A(cb) \cdot A(ca) \angle I(ab)$ [Darapti]

$[A(ba) \cdot A(bc) \angle I(ca)] \angle [A(ba) \cdot I'(ca) \angle A'(bc)]$

by $(pq \angle r) \angle (pr' \angle q')$

$[E(ca) \angle I'(ca)] \cdot [A(ba) \cdot I'(ca) \angle A'(bc) \angle [A(ba) \cdot E(ca) \angle A'(bc)]$

by $(s \angle q) \cdot (pq \angle r) \angle (ps \angle r)$

$$[A(ba) \cdot E(ca) \angle A'(bc)] \cdot [A'(bc) \angle O(bc)] \angle [A(ba) \cdot E(ca) \angle O(bc)]$$

by $(pq \angle r) \cdot (r \angle s) \angle (pq \angle s)$

$$[A(ba) \cdot E(ca) \angle O(bc)] \angle [E(ca) \cdot A(ba) \angle O(bc)]$$

by $(pq \angle r) \angle (qp \angle r)$

Consequently, as before:

Theorem: $E(ca) \cdot A(ba) \angle O(bc)$ [a weakened Cesare]

$$[A(ba) \cdot I(cb) \angle I(ca)] \cdot [I(ca) \angle I(ac)] \angle [I(cb) \cdot A(ba) \angle I(ac)]$$

by $(pq \angle r) \cdot (r \angle s) \angle (qp \angle s)$

Accordingly, suppressing the two factors in the antecedent:

Theorem: $I(ab) \cdot A(bc) \angle I(ca)$ [Dimaris]

The examples that have been given illustrate all of the processes that would have to be employed in order to deduce from *AAA* (in the first figure) the eighty valid moods that remain. Our task, however, would not be complete until it had been proved that all of the other moods are invalid. In order to establish this fact, let us recall an existential postulate, which was introduced in connection with the invalid moods of immediate inference, that is, in solving the array of all propositions of the form $P(a, b) \angle Q(a, b)$. This was stated as follows:

Postulate: There exists an *a, b, c,* such that,

$$O(ab) \angle O'(ab), \qquad O(cb) \angle O'(cb), \qquad I(ca) \angle I'(ca)$$

These last three implications are equivalent to,

$$O(ab) \angle 0 \text{ (zero)}, \qquad O(cb) \angle 0 \text{ (zero)}, \qquad I(ca) \angle 0 \text{ (zero)}$$

by the principle $(p \angle p') \angle (p \angle q)$,
or, again, they may be written,

$$1 \text{ (one)} \angle A(ab), \qquad 1 \text{ (one)} \angle A(cb), \qquad 1 \text{ (one)} \angle E(ca)$$

From these results we have:

$$[1 \angle A(ab)] \cdot [A(ab) \cdot A(cb) \angle I(ca)] \angle [1 \cdot A(cb) \angle I(ca)]$$

by $(s \angle p) \cdot (pq \angle r) \angle (sq \angle r)$

$$[1 \angle A(cb)] \cdot [1 \cdot A(cb) \angle I(ca)] \angle [1 \cdot 1 \angle I(ca)]$$

by $(s \angle q) \cdot (pq \angle r) \angle (ps \angle r)$

$$[1 \cdot 1 \angle I(ca)] \cdot [I(ca) \angle 0] \angle [1 \cdot 1 \angle 0]$$

Assuming, now, that $[1 \cdot 1 \angle 0]$ is the same as $[1 \angle 0]$, and since $[1 \angle 0] \angle 0$, we have, suppressing the parts, $[1 \angle A(ab)]$, $[1 \angle A(cb)]$, $[I(ca) \angle 0]$,

$$[A(ab) \cdot A(cb) \angle I(ca)] \angle 0$$

by $(p \angle q) \cdot (q \angle r) \angle (p \angle r)$

and from this there results the

Theorem: $A(ab) \cdot A(cb) \angle I(ca)$ is untrue,

since this syllogism fails in one case, viz. for $a = a$, $b = b$, $c = c$.

By means of further principles, which follow as theorems from assumptions already set down, many other cases can be deduced. Let us enumerate some of these:

Theorems $(pq \angle r)' \angle (pr' \angle q')'$

$(p \angle s) \cdot (pq \angle r)' \angle (sq \angle r)'$

$(pq \angle r)' \cdot (s \angle r) \angle (pq \angle s)'$

$$[A(ab) \angle I(ba)] \cdot [A(ab) \cdot A(cb) \angle I(ca)]' \angle [I(ba) \cdot A(cb) \angle I(ca)]'$$

Theorem: $I(ba) \cdot A(cb) \angle I(ca)$ is untrue,

since there is one case in which it fails.

$$[I(ba) \cdot A(cb) \angle I(ca)]' \cdot [A(cb) \angle I(cb)] \angle [I(ba) \cdot I(cb) \angle I(ca)]'$$

Theorem: $I(ba) \cdot I(cb) \angle I(ca)$ is untrue.

$$[I(ba) \cdot I(cb) \angle I(ca)]' \cdot [A(ca) \angle I(ca)] \angle [I(ba) \cdot I(cb) \angle A(ca)]'$$

Theorem: $I(ba) \cdot I(cb) \angle A(ca)$ is untrue.

$$[A(ab) \cdot A(cb) \angle I(ca)]' \angle [A(ab) \cdot I'(ca) \angle A'(cb)]'$$

$$[A(ab) \cdot I'(ca) \angle A'(cb)]' \cdot [I'(ca) \angle E(ca)] \angle [A(ab) \cdot E(ca) \angle A'(cb)]'$$

$$[A(ab) \cdot E(ca) \angle A'(cb)]' \cdot [O(cb) \angle A'(cb)] \angle [A(ab) \cdot E(ca) \angle O(cb)]'$$

Theorem: $A(ab) \cdot E(ca) \angle O(cb)$ is untrue.

$$[A(ab) \cdot E(ca) \angle O(cb)' \cdot [E(bc) \angle O(cb)] \angle [A(ab) \cdot E(ca) \angle E(bc)]'$$

$$[A(ab) \cdot E(ca) \angle E(bc)]' \angle [E(ca) \cdot A(ab) \angle E(bc)]'$$

Theorem: $E(ca) \cdot A(ab) \angle E(bc)$ is untrue.

$$[A(ab) \cdot A(cb) \angle I(ca)]' \cdot [A(ca) \angle I(ca)] \angle [A(ab) \cdot A(cb) \angle A(ca)]'$$

Theorem: $A(ab) \cdot A(cb) \angle A(ca)$ is untrue.

$$[A(ab) \angle U(a'b)] \cdot [A(ab) \cdot A(cb) \angle I(ca)]' \angle [U(a'b) \cdot A(cb) \angle I(ca)]'$$

$$[U(a'b) \cdot A(cb) \angle I(ca)]' \cdot [O(ca') \angle I(ca)] \angle [U(a'b) \cdot A(cb) \angle O(ca')]'$$

Theorem: $U(ab) \cdot A(cb) \angle O(ca)$ is untrue,

since it fails for $a = a'$, $b = b$, $c = c$.

The examples given are sufficient to illustrate the method. The moods whose invalidity can not be established by these means may be shown to be untrue by reduction to invalid moods of immediate inference which are already established. Thus we have:

Suppose $O(ba) \cdot O(cb) \angle I(ca)$ were valid, and substitute $b = c'$. Then, as a result of the fundamental assumption that a true proposition must remain true for all meanings of the variables, it follows that,

$$[O(ba) \cdot O(cb) \angle I(ca)] \angle [O(c'a) \cdot O(cc') \angle I(ca)]$$

$$[O(c'a) \cdot O(cc') \angle I(ca)] \angle [1 \angle O(cc')] \cdot [O(c'a) \cdot O(cc') \angle I(ca)]$$

for since $1 \angle O(cc')$ is true, it can be regarded as a unit multiplier, and applying the principle $p \angle 1 \cdot p$.

$[1 \angle O(cc')] \cdot [O(c'a) \cdot O(cc') \angle I(ca)] \angle [O(c'a) \angle I(ca)]$

$[V(ca) \angle O(c'a)] \cdot [O(c'a) \angle I(ca)] \angle [V(ca) \angle I(ca)]$

by $(p \angle q) \cdot (q \angle r) \angle (p \angle r)$.

$\therefore \quad [O(ba) \cdot O(cb) \angle I(ca)] \angle [V(ca) \angle I(ca)]$

by $(p \angle q) \cdot (q \angle r) \angle (p \angle r)$.

$\therefore \quad [V(ca) \angle I(ca)]' \angle [O(ba) \cdot O(cb) \angle I(ca)]'$

by $(p \angle q) \angle (q' \angle p')$.

But it has been shown that there is a special meaning of the terms that verifies the antecedent. Accordingly, for this special meaning the consequent will follow, and we have:

Theorem: $O(ba) \cdot O(cb) \angle I(ca)$ is untrue.

Suppose, as a further example, that $E(ba) \cdot U(cb) \angle A(ca)$ were valid, then:

$[E(ab) \angle E(ba)] \cdot [E(ba) \cdot U(cb) \angle A(ca)] \angle [E(ab) \cdot U(cb) \angle A(ca)]$

by $(s \angle p) \cdot (pq \angle r) \angle (sq \angle r)$

$[U(bc) \angle U(cb)] \cdot [E(ab) \cdot U(cb) \angle A(ca)] \angle [E(ab) \cdot U(bc) \angle A(ca)]$

by $(s \angle q) \cdot (pq \angle r) \angle (ps \angle r)$

and if this expression is true, it will be true for $b = b'$, so that

$[E(ab) \cdot U(bc) \angle A(ca)] \angle [E(ab') \cdot U(b'c) \angle A(ca)]$

$[A(ab) \angle E(ab')] \cdot [E(ab') \cdot U(b'c) \angle A(ca)] \angle [A(ab) \cdot U(b'c) \angle A(ca)]$

$$[A(bc) \angle U(b'c)] \cdot [A(ab) \cdot U(b'c) \angle A(ca)] \angle [A(ab) \cdot A(bc) \angle A(ca)]$$

and if this last were true, it would be true for $b = c$, so that,

$$[A(ab) \cdot A(bc) \angle A(ca)] \angle [A(ac) \cdot A(cc) \angle A(ca)]$$

$$[A(ac) \cdot A(cc) \angle A(ca)] \angle [1 \angle A(cc)] \cdot [A(ac) \cdot A(cc) \angle A(ca)]$$

$$[1 \angle A(cc)] \cdot [A(ac) \cdot A(cc) \angle A(ca)] \angle [A(ac) \angle A(ca)]$$

from which it follows, by successive applications of the law of transitivity, that,

$$[E(ba) \cdot U(cb) \angle A(ca)] \angle [A(ac) \angle A(ca)]$$

and consequently that

$$[A(ac) \angle A(ca)]' \angle [E(ba) \cdot U(cb) \angle A(ca)]'$$

But it has been shown that there are special meanings of the terms which verify the antecedent. Accordingly, the consequent is true for these meanings and we have:

Theorem: $E(ba) \cdot U(cb) \angle A(ca)$ is untrue.

All the processes involved in the deduction of the valid and the invalid moods of the syllogism having been amply illustrated, we shall now assume that this deduction has been made. This is only the beginning of the solution of our general problem, which is to construct all propositions whose truth is independent of the meaning of the variables. We can not profitably continue, however, without developing more completely the algebra of propositions. It is to this task that we now turn.

II

In this section we shall develop a method by which the most general problem of the algebra of propositions may be attacked. By the term proposition we shall mean a sentence that is either true or false. It will designate either a proposition or a propositional function as currently understood, that is, it will mean a variable which may take on constant values. If we were to say, for example, that our variables must be restricted to zero or one values, we would be imposing on them a restriction that would reduce the generality of our science.

Our most general problem is to construct all propositions whose truth is independent of the form of the variables. Its solution may be understood in two senses. We may develop a method by which the truth or falsehood of any proposition taken at random may be determined, or we may actually isolate and exhibit the form of all true propositions. The second solution would presuppose the first. In what follows it will be sufficiently indicated just how both tasks may be carried out. For the most part our solutions will be carried only as far as modal functions of the second order, but such an induction will make it clear how these results may be extended indefinitely.

The following propositional forms, together with the categorical forms already enumerated, are recognized as necessary and sufficient for the expression of any truth, but it is important to notice that a propositional form is not shown to be unessential to the algebra because it can be expressed in terms of some other form. We say important, because current opinion is quite dogmatic in holding the opposite view. Thus, conjunction may be expressed in terms of disjunction and conversely and for many properties there is a symmetrical reciprocity between the two relations, but they are differentiated by the properties which certain expansion formulas express.

The null-proposition will be defined as a p that satisfies,

$$p \angle p'$$

and will be represented by the symbol for zero, 0.

The one-proposition, its negative, will be defined as a p that satisfies,

$$p' \angle p$$

and will be represented by the symbol for one, 1.

The propositional forms necessary and sufficient in our algebra are:

The form of negation:

$p' = p$ is untrue.

The hypothetical form:

$$p \angle q = p \text{ implies } q,$$
$$(p \angle q)' = p \text{ does not imply } q.$$

The conjunctive form:

$$p \cdot q = p \times q = p \text{ (is true) and } q \text{ (is true)}.$$

The disjunctive form:

$$p + q = \text{either } p \text{ (is true) or } q \text{ (is true)}.$$

Propositions will be represented by the letters $p, q, r \ldots$ and the expression $|p, q, r, \ldots|$ will be defined as,

$$\begin{aligned} |p, q, r, \ldots|' &= p \cdot q \angle r' + \ldots \\ &= p \angle q' + r' + \ldots \\ &= 1 \angle p' + q' + r' + \ldots \end{aligned}$$

The expression $|p, q, r, \ldots|$ is called a *modal* function of the n elements $p, q, r, \ldots$ while the straight bracket which expresses this function is called a modal *operator*. The modal function, if it have n elements, is said to be of the nth *degree*, for the operator acts on the elements as if they constituted a product and were permutable among themselves, and of the first *order* if none of the elements is governed by a modal operator.

We shall find that the relation of any variable p to any modal function of p of the first degree and of the nth order follows at once from the assumptions set down below:

Definition: $|p| = (p \angle 0)'$

Principle: $p \angle (p \angle 0)'$

Principle: $p \angle \|\,|p|'\,|'_n \cdot \angle \cdot p \angle \|\,|p|'\,|'_{n+1}$

The notation employed in the last principle requires explanation. The n placed as a subscript below the straight bracket is intended to mean that this bracket is repeated n times. That is, when we wish to indicate that the modal operator is repeated, we write,

$$|p|_2 = ||p||, \quad |p|_3 = |||p|||, \quad ||p|'|'_0 = |p|'' = |p|$$

On making $n = 0$, then, in the second principle, there results:

$$p \angle |p| \cdot \angle \cdot p \angle ||p|'|'$$

and since the antecedent is true by the first principle, we have,

$$p \angle ||p|'|'$$

Continuing this process, then, for successive values of n, we have generally,

Theorem: $p \angle ||p|'|'_n$

If it is desired to express in the form of a table the interconnection of all modal functions of a single variable up to modals of the fourth order, it will be necessary to employ this theorem for the four cases, $n = 0, 1, 2, 3$. Accordingly we have:

(1) $p \angle |p|$ (2) $p \angle ||p|'|'$

(3) $p \angle |||p|'||'$ (4) $p \angle ||||p|'|||'$

As we go on it will often be convenient to omit the first half of the modal operator in order to save space. With this understanding the four results just written down would appear thus,

$$p \angle p|, \qquad p \angle p|'|', \qquad p \angle p|'||', \qquad p \angle p|'|||'$$

Keeping in mind the formula,

$$(p \angle q) \angle (q' \angle p') \tag{5}$$

which follows as a matter of definition if the elements in the modal function are commutative, we have:

$p \angle p|'|||'$

$p|'|||' \angle p|'||'$ by making $p = p|'||$ in (1) and applying (5)

$p|'||' \angle p|'||'|$ $p = p|'||'$ in (1)

$p|'||'| \angle p|'|'$ $p = p|$ in (2) and applying (5) and then making $p = p|'$

$p|'|' \angle p|'|'|'|'$ $p = p|'$ in (2) and applying (5), then making $p = p|'$ and applying (5)

$p|'|'|'|' \angle p|'|'|$ $p = p|'$ in (1), then (5) and $p = p|'|'$

$p|'|'| \angle p|'|'||$ $p = p|'|'|$ in (1)

$p|'|'|| \angle p|$ $p = p|'$ in (3), then (5)

$p| \angle p||'||'$ $p = p|$ in (3)

$p||'||' \angle p||'|'$ by making $p = p|$ in (1), and applying (5) and then making $p = p||'$

$p||'|' \angle p||'|'|$ $p = p||'|'$ in (1)

$p||'|'| \angle p||$ $p = p||'$ in (2), then (5)

$p|| \angle p|||'|'$ $p = p||$ in (2)

$p|||'|' \angle p|||$ $p = p|'$ in (1), then (5) and $p = p||$

$p||| \angle p||||$ $p = p|$ in (1)

The other half of the table is obtained by changing p into p' throughout.

<table>
<tr><td>P||||</td><td>P||||'</td></tr>
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In order to establish the fact that the converse of each one of these implications is untrue, we proceed as follows:

Definition: A modal function, whose operator contains an even number of symbols of negation, is termed an affirmative mode; otherwise, a negative mode, or a mode that denies. Let Lp, Mp, Np, stand for modal functions of p.

We shall discover later that the law of transitivity holds only when the antecedent is verified. Consequently, under these conditions,

$$(Lp \angle Mp)(Mp \angle Np) \angle (Lp \angle Np)$$

Definition: If $Lp \angle Mp$ is a true implication and L and M are not identical, then M is said to be higher in the modal scale than L.

Theorem: If M is higher than L in the modal scale, then L' is higher in the modal scale than M'.

The principle set down below is a theorem from results that have gone before:

Principle: If L and M are, the one affirmative and the other negative, and if L is higher in the modal scale than M' (so that M is higher than L'), then the equation $Lp \cdot Mp = 0$ has two solutions, zero and one.

Postulate: If L and M are, the one affirmative and the other negative, and if P stands for the function,

$$(a \angle b) \cdot [(b \angle a) + (a \angle b')' \cdot (b' \angle a)'],$$

and if L is higher in the modal scale than M', then P is not a solution of the equation, $Lp \cdot Mp = 0$.

Theorem: $(a' \angle b) \cdot [(b \angle a') + (a \angle b)' \cdot (b \angle a)']$

$$(a \angle b') \cdot [(b' \angle a) + (a \angle b)' \cdot (b \angle a)']$$

are not solutions of the equation, $Lp \cdot Mp = 0$.

Theorem: $(a \angle b)' + (b \angle a)' \cdot [(a \angle b') + (b' \angle a)]$

$$(a' \angle b)' + (b \angle a')' \cdot [(a \angle b) + (b \angle a)]$$

$$(a \angle b')' + (b' \angle a)' \cdot [(a \angle b) + (b \angle a)]$$

are not solutions of the equation, $Lp \cdot Mp = 0$, since if P is not a solution of this equation, P' is evidently not a solution.

The particular form of the P selected here anticipates developments that follow later on in part III.

From these facts it follows that the equalities set down below are untrue, since there is one value of p at least, viz. $p = P$, which does not satisfy them. These equalities are:

$$p|'|'||' \cdot p| = 0$$
$$p|'|'|' \cdot p|'|'|| = 0$$
$$p|'|'|'| \cdot p|'|'| = 0$$
$$p|'| \cdot p|'|'|'|' = 0$$
$$p|'||'|' \cdot p|'|' = 0$$
$$p|'|| \cdot p|'||'| = 0$$
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$$p||'| \cdot p||'|'| = 0$$
$$p||'|'|' \cdot p|| = 0$$
$$p||' \cdot p|||'|' = 0$$
$$p|||'| \cdot p||| = 0$$
$$p' \cdot p|'|||' = 0$$
$$p|||' \cdot p|||| = 0$$

and since these are untrue, it follows by $(pq \angle 0)' \angle (p \angle q')'$, that

$$p| \angle p|'|'|| \text{ is untrue.} \quad p|' \angle p||'|| \text{ is untrue.}$$
$$p|'|'||' \angle p|' \text{ is untrue.} \quad p||'||' \angle p| \text{ is untrue.}$$
etc., etc.

III

Keeping always in mind the general problem, which was stated at the outset, that of constructing all propositions whose truth is independent of the meaning of terms, we proceed inductively toward its solution, beginning with the next simplest case. The array of immediate inference is the set of all propositions of the form,

$$Mp(ab) \angle Nq(ab)$$

$$Mp(ab) \angle Nq(ba)$$

wherein Mp and Nq stand for modal functions of p and q, and wherein p and q are conceived as capable of taking on any one of the constant values, A, E, I, O, U, V. There will thus be thirty-six moods to consider in each figure, obtained by taking the permutations of the six letters two at a time and by taking each letter once with itself. Difference of figure being conceived as a difference of term-order, the set $Mp(ab) \angle Nq(ab)$ will be referred to as belonging to the first figure of immediate inference, the members of the set $Mp(ab)$

$\angle Nq(ba)$ as belonging to the second figure of immediate inference.

If p were to take on the value A and q the value E, yielding a mood in each set,

$$MA(ab) \angle NE(ab), \qquad MA(ab) \angle NE(ba)$$

then we might substitute first $a = a'$, then $b = b'$, and finally $a = a'$ and $b = b'$. In that way there would result six new moods of the array,

$$MU(ab) \angle NA(ab) \qquad MU(ab) \angle NA(ba)$$

$$ME(ab) \angle NA(ab) \qquad ME(ab) \angle NA(ba)$$

$$MA(ab) \angle NU(ab) \qquad MA(ab) \angle NU(ba)$$

so that if we had once determined all the true and all the untrue variants of the first type, it would only be a matter of bookkeeping to record the true and untrue variants for the other six. Now it is a fact that if in addition to the substitutions $p = A$ and $q = E$, we had gone on to substitute the pairs, A,A, and then A,O, and then A,I, and finally, O,A, I,A, O,O, O,I, and had put in the negative terms as before, all the other possible moods would have resulted. Consequently, we may consider only these pairs and ignore all the others.

Moreover, our problem can be still further simplified, for since the prime over any letter can be regarded as a part of the modal operator, and since the pairs set down above can be rewritten, A,A,

A,I', A,A', A,I, A',A, I,A, A',A', A',I, it will only be necessary to determine the true and untrue propositions in the four arrays,

$$MA(ab) \angle NA(ab) \qquad MA(ab) \angle NA(ba)$$
$$MA(ab) \angle NI(ab) \qquad MI(ab) \angle NA(ab)$$

for since $I(ab)$ is simply convertible, the first and second figure will have the same valid cases for the last two types set down. The first of these four arrays was solved in the last chapter, if we consider P to have everywhere the sense of A. In order to solve $MA(ab) \angle NA(ba)$ it will be necessary to introduce again the fundamental existential postulate.

Postulate: There exists an $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, such that,

$$O(\mathbf{ab}) \angle 0, \qquad O(\mathbf{cb}) \angle 0, \qquad I(\mathbf{ca}) \angle 0$$

Theorem: There exists an $\mathbf{a}$, $\mathbf{b}$, such that,

$$A(\mathbf{ba}) \angle 0, \qquad O(\mathbf{ab}) \angle 0$$

We are now in a position to show that there are no valid moods in the array,

$$MA(ab) \angle NA(ba)$$

Suppose in the first instance that M and N are both affirmative modal operators and that we make $a = \mathbf{a}$ and $b = \mathbf{b}$. Then the resulting mood,

$$MA(\mathbf{ab}) \angle NA(\mathbf{ba})$$

will be clearly invalid.

If M and N are both negative modal operators, then the substitution $a = b$, $b = a$, shows that there are no valid moods of this type.

If M is affirmative and N negative, we let $b = a$ and the resulting type,

$$MA(aa) \angle NA(aa)$$

is clearly invalid.

If M is negative and N affirmative, the substitution $b = a'$ yields,

$$MA(aa') \angle NA(a'a)$$

which, like the other, is untrue by the results of chapter I.

That there are no valid moods in the arrays $MA(ab) \angle I(ab)$ or $I(ab) \angle A(ab)$, wherein M is affirmative and N negative, will appear on making $a = b$, and that there is none when M is negative and N is affirmative, will appear on making $b = a'$. We now introduce a theorem connected with our task of establishing invalidity.

Theorem: $|A(ab)|_n' \cdot |E(ab)|_n' \angle 0$ is untrue.

for since $E(ab) \angle O(ba) \cdot$ and since $O(ba) \angle 0$, it follows that $E(ab) \angle 0$. Therefore, if the above expression were true in general, then we should have,

$$|A(ba)|_n' \cdot |E(ab)|_n' \angle 0$$

But this is untrue, so that the theorem follows. Let us write down two variations of this theorem,

$$(1) \quad |A|_n' \angle |I'|_n \text{ is untrue}$$

$$(2) \quad |I'|_n' \angle |A|_n \text{ is untrue}$$

If M and N are both negative, it may be shown by means of (1) that any mood of the array $MA(ab) \angle NI(ab)$ is invalid by the principles,

$$(p \angle q)(p \angle r)' \angle (q \angle r)'$$

$$(p \angle r)'(q \angle r) \angle (p \angle q)'$$

Thus in particular we should have,

$$(A|||' \angle A'|||'|') \cdot (A|||' \angle I'|||)' \angle (A'|||'|' \angle I'|||)'$$

$$(A|||' \angle I'|||)' \cdot (I'|'|' \angle I'|||) \angle (A|||' \angle I'|'|')'$$

In the same way, when M and N are both affirmative, we can show by (2) that any mood of the array $MI(ab) \angle NA(ab)$ is invalid.

All the remaining types are valid. To show this we introduce provisionally an induction formula, which is a theorem from another one introduced later on.

$$A(ba)|_n \cdot |A(cb)|_n \angle |A(ca)|_n \cdot \angle \cdot |A(ba)|_{n+1} \cdot |A(cb)|_{n+1} \angle |A(ca)|_{n+1}$$

and the antecedent can be suppressed, since we have proved in the first chapter that it is true for $n = 0$, so that we have in general,

$$|A(ba)|_n \cdot |A(cb)|_n \angle |A(ca)|_n$$

From this there results on making $a = c'$, and since $A(cc') \angle 0$, the theorem,

$$|A(ab)|_n \cdot |E(ab)|_n \angle 0$$

If we write down two variations of this theorem, viz.

$$|A(ab)|_n \angle |I'(ab)|'_n$$

$$|I'(ab)|_n \angle |A(ab)|'_n$$

all the remaining valid moods follow at once by the principle,

$$(p \angle q)(q \angle r) \angle (p \angle r)$$

In the sequel it will become clear that this principle holds true whenever the factors in the antecedent are verified.

Proceeding now inductively toward the solution of the general problem, the next type of inference we should encounter would be the syllogism. It will have one of the four forms, figures or term-orders, as follows:

$$Lp(ba) \cdot Mq(cb) \angle Nr(ca) \qquad \text{1st figure}$$

$$Lp(ab) \cdot Mq(cb) \angle Nr(ca) \qquad \text{2nd figure}$$

$$Lp(ba) \cdot Mq(bc) \angle Nr(ca) \qquad \text{3rd figure}$$

$$Lp(ab) \cdot Mq(bc) \angle Nr(ca) \qquad \text{4th figure}$$

wherein, as before, the variables p, q and r are conceived as capable of taking on any one of the six constant values, A, E, I, O, U, V. Thus, in constructing the array of the syllogism, we should have to form the permutations of the six letters taken three at a time. This will yield two hundred and sixteen moods in each one of the four figures, and this number will be multiplied again as we allow L, M and N to vary.

Let us begin with an illustration of how all this complexity of form may be reduced to a manageable simplicity. Suppose initially that p, q and r have all the value A, and that we substitute in all four figures first $a = a'$, then $b = b'$, then $c = c'$. And, beginning again, suppose we substitute again $a = a'$, $b = b'$, then $b = b'$, $c = c'$, and then $a = a'$, $c = c'$. Finally, beginning all over again as before, let us substitute $a = a'$, $b = b'$, $c = c'$. It is evident that a great many other moods will result, so that once we have solved the array,

$$LA(a, b) \cdot MA(b, c) \angle NA(c, a)$$

it will only be a matter of bookkeeping to record the true and the untrue moods in the other equivalent instances. As a matter of fact, if we were to start with the permutations *AAA*, *AAE*, *AAO*, *AAI*, *AOA*, *AOU*, *AOO*, *AOI*, *OOA*, *OOE*, *OOI*, *OOO*, and make these same substitutions, all the moods that remain would be accounted for. It follows, then, that if we solve these cases, we will have all the others.

However, as in the case of immediate inference, the matter may be still further simplified, for these twelve permutations may be written, AAA, AAI', AAA', AAI, $AA'A$, $AA'V'$, $AA'A'$, $AA'I$, $A'A'A$, $A'A'I'$, $A'A'I$, $A'A'A'$, and, since the prime or symbol of negation may be regarded as part of the modal operator, the complete solution of the syllogism is reduced to the solution of,

$$LA(a, b) \cdot MA(c, b) \angle NA(c, a)$$

$$LA(a, b) \cdot MA(c, b) \angle NI(c, a)$$

for $AA'V'$ turns into the last of these on negating all the terms.

Let us begin with the first type,

$$LA(a, b) \cdot MA(c, b) \angle NA(c, a)$$

and examine in succession the possible alternatives:

(1) L affirmative, M affirmative, N negative.
Seen to be invalid on making $a = b$.
(2) L affirmative, M negative, N affirmative.
Seen to be invalid on making $a = b$.
(3) L negative, M affirmative, N affirmative.
Seen to be invalid on making $c = b$.
(4) L negative, M negative, N affirmative.
If we make $b = a'$, this reduces to one of the two,

$$MI(ca) \angle NA(ca)$$

$$MV(ca) \angle NA(ca)$$

in which M and N are affirmative, and this is invalid.

(5) L negative, M negative, N negative.

This last case reduces to (3) by contradicting and interchanging the minor premise and the conclusion.

(6) L affirmative, M affirmative, N affirmative.

All the valid moods occur in the first figure, for the invalidity of those in the second figure is established on making $c = b$, those in the third and fourth figure on making $a = b$. The valid moods are all a consequence of an induction principle,

$$[MA(ba) \cdot MA(cb) \angle MA(ca)] \angle [|MA(ba)| \cdot |MA(cb)| \angle |MA(ca)|]$$

$$[MA(ca) \angle MA(ba) + MA(cb)] \angle [|MA(ca)| \angle |MA(ba)| + |MA(cb)|]$$

Thus:

$$[A(ba) \cdot A(cb) \angle A(ca)] \angle [|A(ba)| \cdot |A(cb)| \angle |A(ca)|]$$

$$[|A(ca)|' \angle |A(ba)|' + |A(cb)|'] \angle [||A(ca)|'| \angle ||A(ba)|'| + ||A(cb)|'|]$$

$$[||A(ba)|'|' \cdot ||A(cb)|'|' \angle ||A(ca)|'|'] \angle [|||A(ba)|'|'| \cdot |||A(cb)|'|'| \angle |||A(ca)|'|'|]$$

All moods not derived in this way can be shown to be invalid by identifying terms in one of the premises.

(7) L affirmative, M negative, N negative.

All the valid moods are to be found in the second figure and follow from those under the last case by,

$$(pq \angle r) \angle (pr' \angle q')$$

There are no more, for any such mood would be invalid under (6) by the same principle.

(8) L negative, M affirmative, N negative.

The valid moods are all in the third figure and follow from those under (6) by,

$$(pq \angle r) \angle (r'q \angle p')$$

and there are no more by the same reasoning given under (7).

We proceed now to consider the other type of syllogism, viz.

$$LA(a, b) \cdot MA(c, b) \angle NI(c, a)$$

which may be treated as the last case except for the alternatives under (4), (6), (7) and (8).

(4) L negative, M negative, N affirmative.

If we substitute $c = a'$, this will reduce to one of the special cases,

$$MO(ab) \angle NE(ab)$$

$$MO(ab) \angle NU(ab)$$

wherein M and N are new modals and both affirmative, and these are invalid.

(6) L affirmative, M affirmative, N affirmative.

The moods in the second figure are invalid by the existential postulate, which was assumed above, but all the moods in the first, third and fourth figures are valid. As a special case we have:

$$|A(ca)| \cdot |E(ca)| \angle \mathbf{0} \cdot \angle \cdot |A(ca)| \angle |I'(ca)|'$$

(a) $|A(ba)| \cdot |A(cb)| \angle |I'(ca)|'$

(b) $|A(cb)| \cdot |A(ba)| \angle |I'(ac)|'$

and the analogous mood in the third figure is derived as follows:

$$|A(ba')| \cdot |A(cb)| \angle |A(ca')|$$

$$\therefore \quad |E(ca)|' \cdot |A(cb)| \angle |E(ba)|'$$

$$|I'(ca)|' \cdot |A(cb)| \angle |I'(ba)|'$$

(c) $|A(ca)| \cdot |A(cb)| \angle |I'(ba)|'$

and other variants follow from (a), (b) and (c) by

$$(s \angle p)(pq \angle r) \angle (sq \angle r)$$

$$(pq \angle r)(r \angle s) \angle (pq \angle s)$$

These principles of the classical logic hold, as we shall discover later on, whenever the antecedent is verified. The cases (7) and (8) are treated as before.

A zero cycle is a product of modal functions of single categorical forms, the terms of which are arranged in cyclical order, the whole product being represented as implying zero. A product of cycles, which is represented as implying zero, is termed a zero conjunction of cycles. A chain of modal functions of single categorical forms is a product of such functions, wherein the terms are arranged in sequence but are not necessarily closed at the ends to form a cycle. An intersection is a term which two or more cycles, or more generally chains, of categorical forms have in common. A cross-product is any cycle formed at the intersections in a product of chains or cycles. In the discussion which follows we shall take for granted the following:

Principle: If none of the cycles of a valid mood of the zero conjunction of cycles vanishes, then one of its cross-products vanishes.

Since the sign of negation over any letter can always be regarded as a part of the modal operator, and since any letter can be replaced by its contradictory negated, we may habitually think of an affirmative form as *A*, an *I* or a *V* modified by an operator containing an even number of primes, a negative form as an *A*, an *I* or a *V* modified by an operator containing an odd number of primes. The solution of the sorites is equivalent to the solution of the zero cycle.

The general form of the sorites is represented thus:

$$p(1, 2) \cdot q(2, 3) \ . \ . \ . \ r(n, n-1) \angle s(n, 1)$$

wherein we think of the $pq \ . \ . \ . \ rs$ as governed by an arbitrary modal operator, whether this operator is expressed or not, and in which the comma between the terms means that the term-order is not settled.

Definition: If $p(ab) \angle q(ab)$ is a true implication, but the converse does not hold, then $p(ab)$ is said to be universal and $q(ab)$ is said to be particular.

We shall at the outset state certain theorems, which will shorten the work of establishing the general solution that follows.

Theorem 1: If the conclusion of the sorites is affirmative, then all the premises are affirmative.

If one or more premises are negative, we may identify terms in each one of affirmative premises, supposing such to be present, and suppress each one of these as if it were a unit multiplier. The resulting sorites, which now contains only negative premises, may then be reduced to a chain, or a chain with a three term cycle at the end, by identifying in succession terms that are next but one. The general method of procedure is easily abstracted from a single illustration. Suppose that, as a special case, all the affirmative premises have been removed and that there remains,

$$E(12) \cdot U(23) \cdot O(34) \cdot E(45) \cdot U(56) \cdot O(67) \angle A(71)$$

Changing first 3 into 1 and then 5 into 1, the sorites breaks up into a zero conjunction of cycles, whose separate parts are,

$$E(12) \cdot U(21) \qquad O(14) \cdot E(41) \qquad U(16) \cdot O(67) \cdot O(71)$$

and since none of these parts vanish separately, the original sorites with which we started, whatever it may have been, is invalid by our principle.

Theorem 2: If the conclusion is negative, then one and only one premise is negative.

For, if all the premises were affirmative, we may reduce the sorites to an invalid syllogism by identifying in succession adjacent terms, and if more than one is negative, the method described above is applicable.

Theorem 3: If the conclusion is a universal modified (governed) by an affirmative modal operator, then all the premises are universals governed by affirmative modal operators.

The only possibility we should have to consider here is the presence of particular premises governed by affirmative modals, for a universal governed by a negative modal is the same as this, and a negative modal over a particular is the same as a universal governed by an affirmative modal. If the conclusion is an *A*-form and one or more *I*- or *V*-premises be present, we have only to identify terms in succession, being careful to preserve one of the particular premises. If the conclusion is an *E*- or *U*-form, one and only one premise is negative. If this premise prove to be an *O*-form, we identify terms in all the affirmative premises that occur. If an *I*- or *V*-premise occur and the negative premise should be *E* or *U*, we identify terms in all the premises except one particular and the negative.

Theorem 4: If the conclusion is a particular governed by an affirmative modal operator, then not more than one premise is a particular governed by an affirmative modal operator.

Definition: If $Mp \angle Np$ holds true, but the converse is untrue, the modal operator N is said to be weaker than M, and M is said to be stronger than N.

CONCLUSION IN THE *A*-FORM

If the conclusion is an A-form governed by an affirmative modal operator, then each premise is of the same form (theorems 1, 3) and it is not hard to show that the modal operator governing any premise is no weaker than the modal governing the conclusion.

Once the premises have been arranged in normal order, the larger ordinal number in each one must appear first as subject. Otherwise, by the methods already illustrated, it would be possible to reduce the sorites to an invalid mood of the syllogism. Consequently, to be valid, the sorites must have the form,

$$A(21) \cdot A(32) \; . \; . \; . \; A(n\, n-1) \angle A(n\, 1)$$

each premise being conceived as governed by an affirmative modal operator no weaker than the conclusion. This mood of the sorites is evidently to be constructed or generated from the following chain of valid syllogisms:

$$A(21) \cdot A(32) \angle A(31)$$

$$A(31) \cdot A(43) \angle A(41)$$

$$\cdots\cdots\cdots\cdots$$

$$A(n-1\, 1) \cdot A(n\, n-1) \angle A(n\, 1)$$

CONCLUSION IN THE *I*-FORM

Since the conclusion is affirmative, it follows that all the premises are affirmative (theorem 1), and since it is particular it follows that not more than one premise is particular (theorem 4). Moreover, it is easy to show that the particular premise can not be in the *V*-form, if the mood is to be valid.

If the premises all have the *A*-form and are governed by affirmative modal operators, the first premise which presents its terms in the order $(s - 1\ s)$, that is, with the larger ordinal number coming first, establishes that term-order in each premise which follows. For suppose a premise following the premise in question should exhibit the term-order $(s\ s - 1)$. Then, by identifying terms in successive premises, the sorites would be reducible to an invalid syllogism of the form:

$$A(s - 2\ s - 1) \cdot A(s\ s - 1) \angle I(s\ s - 2)$$

If we suppose that the term-order $(s\ s - 1)$ is preserved as far as the rth premise and is then reversed, then the term-order $(s - 1\ s)$ is established from the rth premise to the end, and the sorites becomes,

$$A(21) \cdot A(32) \ldots A(r\ r - 1) \cdot A(r\ r + 1) \ldots A(n - 1\ n) \angle I(n\ 1)$$

the position of the governing operators in the modal scale being easily determined as the same as for the case of the analogous syllogism. This valid mood of the sorites is constructible from the following chain of valid syllogisms:

$$A(21) \cdot A(32) \angle A(31)$$

$$\cdots\cdots\cdots\cdots$$

$$A(r-1\;1) \cdot A(r\;r-1) \angle A(r\;1)$$

$$A(r\;1) \cdot A(r\;r+1) \angle I(r+1\;1)$$

$$I(r+1\;1) \cdot A(r+1\;r+2) \angle I(r+2\;1)$$

$$\cdots\cdots\cdots\cdots$$

$$I(n-1\;1) \cdot A(n-1\;n) \angle I(n\;1)$$

Should the tth premise be in the I-form, then the term-order in the first $t-1$ premises can be shown to be $(s\;s-1)$, that is, the larger ordinal number will appear first as subject. Otherwise, the sorites would be reducible to an invalid syllogism of the form,

$$A(s-1\;s) \cdot I(s\;s+1) \angle I(s+1\;s+2)$$

In the same way, the order of terms in the last $n-t-1$ premises is established as $(s-1\;s)$. For should any premise coming after the I-premise present the term-order $(s\;s-1)$, the sorites could be reduced to an invalid syllogism of the form,

$$I(s-2\;s-1) \cdot A(s\;s-1) \angle I(s\;s-2)$$

The form of the sorites is, therefore, determined as,

$$A(21)\;\ldots\;A(t\;t-1) \cdot I(t\;t+1) \cdot A(t+1\;t+2)\;\ldots\;A(n-1\;n) \angle I(n\;1)$$

and it is easy to determine the nature of the governing modal operators from the case of the analogous syllogism. In this case the only restriction on the affirmative modal operators that govern the premises is that the affirmation of $I(t\,t+1)$ is no weaker than the affirmation of $I(n\,1)$.

The validity of this form of the sorites is evidently implied by the following chain of valid syllogisms:

$$A(21) \cdot A(32) \angle A(31)$$

$$\ldots\ldots\ldots\ldots$$

$$A(t-1\,1) \cdot A(t\,t-1) \angle A(t\,1)$$

$$A(t\,1) \cdot I(t\,t+1) \angle I(t+1\,1)$$

$$I(t+1\,1) \cdot A(t+1\,t+2) \angle I(t+2\,1)$$

$$\ldots\ldots\ldots\ldots$$

$$I(n-1\,1) \cdot A(n-1\,n) \angle I(n\,1)$$

All the valid moods with a *V*-conclusion can be had by negating all the terms in the sorites with an *I*-conclusion, and it is easy to show that there are no others. The moods with an *E*-, *U*-, or *O*-conclusion evidently arise by contradicting and interchanging a premise and conclusion in one of the cases already determined.

In solving the sorites, and the solution of the sorites contains the solution of the zero cycle, we have assumed an existential postulate to this effect: No zero conjunction of cycles can vanish unless one of the cycles or one of the cross-products vanishes.

This principle yields the solution of the zero conjunction of cycles, for if C be a cycle or a cross-product in $C_1 \cdot C_2 \ldots C_n$, then

$$(C \angle 0) \angle (C_1 \cdot C_2 \ldots C_n \angle 0)$$

A chain or a conjunction of chains that are not cycles and which contain no cross-product can evidently be reduced to unity by identifying terms in the affirmative forms and making them contradictory in the negatives, and it is evident that a conjunction of chains can be reduced to the conjunction of cycles it contains in the same way without introducing a zero factor.

The most general case that could occur would be a sum of products of chains represented as implying zero, since for a sum to vanish each part must vanish separately, and because a product of sums can be reduced to a sum of products by direct multiplication. This case is now solved and with it the general problem we have proposed.

One qualification. Certain expansion formulas in the algebra of propositions have not been stated and must be added.