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ANALYTIC GEOMETRY

BY

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AND

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Exhibit Collection of Mathema

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PREFACE

This book is the result of several years of experience in teaching mathematics to students of engineering and science.

Since at the outset, analytic geometry opens to the student an entirely new method of approaching mathematical truth, much stress is placed on the first two chapters in which the student is building the concepts on which the future chapters rest. Emphasis has also been placed on those portions of analytic geometry in which experience has shown the student of calculus to be most frequently deficient. In this connection, in particular, polar coördinates have received more than usual attention and transcendental and parametric equations considerable space. The exercises are numerous and varied in character, and the teacher will thus be enabled to select from them those which best emphasize the points which he considers important.

The book has been used for two years in mimeographed form in the class room both by the authors and their colleagues, and many valuable suggestions arising from such use have been incorporated into the final form of the text.

The material is so arranged that the first ten chapters together with a portion of Chapter XIII include those subjects ordinarily offered to such freshman classes as cover in the first year the three subjects, college algebra, trigonometry and analytic geometry. The addition of Chapter XIV will round out a good course of five hours a week for a semester. The entire book should easily be covered in a three hour course throughout a year.

The authors take pleasure in expressing their thanks to their colleagues in the department of mathematics of the Iowa State College, for their assistance in reading proof and solving problems as well as for their many helpful suggestions.

MARIA M. ROBERTS,
JULIA T. COLPITTS.

AMES, IOWA, *January*, 1918.

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ANALYTIC GEOMETRY

CHAPTER I

CARTESIAN COÖRDINATES

1. Analytic geometry differs from other geometry mainly by the introduction of two new ideas: first, that a point in a plane is determined by its distances from two fixed intersecting lines in that plane, and second, that an equation in two variables completely represents a geometric locus.

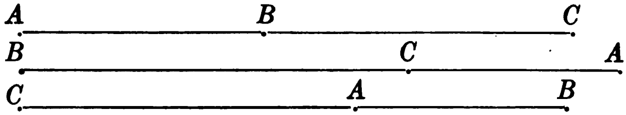
These inventions are due to René Descartes (1596–1650) who published his discoveries in 1637. In honor of his name, this branch of mathematics is often called Cartesian geometry, and the system of coördinates here used, Cartesian coördinates.

2. Directed lines. — On a fixed line $X'X$, let a fixed point O, called the *origin*, be chosen from which to measure distances.

It is customary to call distances measured to the right positive, and those measured to the left negative.

Let some unit of length be applied to OX, and suppose OB is 7 units long, then $+7$ is represented by OB. If the same measure is applied to OX', and OB' is 6 units long, then -6 is represented by OB'. Moreover, while OB measured to the right is $+7$ units, BO measured to the left is -7 units.

From the above definition, it is evident that if A, B, and C are three points on a line, then $AB + BC = AC$.



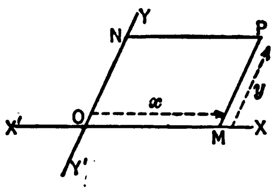
Construct three other figures also showing that $AB + BC = AC$.

Locate four points A, B, C, and D on a line and show in three different figures that $AB + BC + CD = AD$.

3. Position of a point in a plane. Cartesian Coördinates. — If it is known that a point is located on a given line, it is only necessary to know one number in order to locate the point, namely, that number which represents the distance and direction from the origin.

If the point is located anywhere in a plane, its position is fully determined by two numbers.

Let $X'X$ and $Y'Y$ be two straight lines intersecting at O, and let the point P be the given point. Draw NP and MP through P parallel to $X'X$ and $Y'Y$ respectively.



The position of the point P is fully determined if MP and NP are known.

The line $X'X$ (usually horizontal) is called the x-axis. The line $Y'Y$ is called the y-axis and their point of intersection is called the origin.

$x = OM = NP$ is the abscissa of the point.

The abscissa of a point is its distance from the y-axis measured parallel to the x-axis.

$y = ON = MP$ is the ordinate of the point.

The ordinate of a point is its distance from the x-axis measured parallel to the y-axis.

The two intersecting lines are called the **coördinate axes** and the two numbers which locate the position of the point, the **Cartesian coördinates** of the point.

Abscissas are taken as positive or negative according as they are measured to the right or left of the origin, and ordinates as positive or negative according as they are measured above or below the x-axis.

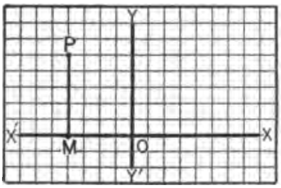
4. Rectangular coördinates. — The coördinate axes may intersect at any angle, but results are usually simpler if the axes are perpendicular, in which case, Cartesian coördinates are called **rectangular coördinates.** Cartesian coördinates when not rectangular are called **oblique coördinates.**

Unless otherwise specified, rectangular coördinates will always be used.

5. Notation. — The point whose coördinates are $x = a$, and $y = b$, is usually written $P = (a, b)$ or $P\ (a, b)$. This is read "P whose coördinates are a and b."

Variable points are in general represented by $P\ (x, y)$, and fixed points by $P_1\ (x_1, y_1)$, $P_2\ (x_2, y_2)$, etc.

To plot a point in Cartesian coördinates choose any convenient unit of measure, lay off from the origin on the x-axis a number equal to the abscissa, and from the extremity of this line, and on a parallel to the y-axis, a number equal to the ordinate.



Thus to plot the point $P\,(-4, 5)$, lay off $OM = -4$ on OX and draw $MP = 5$ parallel to OY.

The use of coördinate paper will be found to be of decided advantage in a rectangular system of coördinates. Such paper is constructed as in the figure in which the above point has been located.

EXERCISES

1. Plot accurately the points: (5, 6), (−2, −3), (0, 2), (−5, 0).

2. Let the axes OX and OY be inclined at an angle of 45°. Plot the points given in Ex. 1.

3. Draw the quadrilateral whose vertices are (3, 2), (−4, 2), (−4, −1), (3, −1). Prove the figure is a rectangle and find the lengths of its sides.

4. Where are the points whose ordinates are 0? Whose abscissas are 0? Whose abscissas are 2?

5. On what line will a point lie if its abscissa and ordinate are equal? If equal numerically but opposite in sign?

6. The origin is the middle point of a line one of whose extremities is (−2, −3). Find the other extremity.

7. What are the coördinates of a point half-way between the origin and the point (2, 4)? *Ans.* (1, 2).

8. In a rectangle whose sides are 4 and 3 one of the longer sides is chosen as the x-axis and a diagonal as the y-axis. What are the coördinates of the vertices and of the middle points of the sides?

9. An isosceles triangle has a base 6 and the equal sides each 5. The base is taken as the x-axis and the perpendicular from the vertex to the base as the y-axis. Find the coördinates of the vertices.

Find also the coördinates of the vertices if the base and one of the sides are chosen as axes.

10. What are the coördinates of the vertices of a square whose side is $2a$ if the origin is at the center of the square and the axes are parallel to the sides?

What are the coördinates of the vertices if the origin is at the center, one axis is parallel to a side, and the other is a diagonal?

11. What are the coördinates of the vertices of an equilateral triangle each side of which is a, the base being chosen as the x-axis and the perpendicular to this base through a vertex as the y-axis?

12. Compute the lengths of the sides of the triangle whose vertices are (2, 1), (6, 4), and (7, 1). *Ans.* 5, 5, $\sqrt{10}$.

13. Plot the points A (−1, −2) and B (2, 3). Let the horizontal line through A cut the vertical line through B in the point C. What are the coördinates of C? Find the area of the triangle ABC and the length of AB.

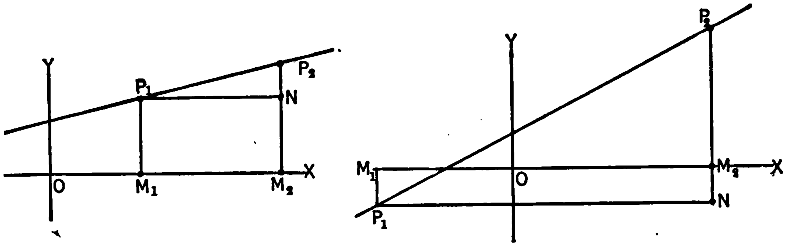
14. Plot the points A (3, 2) and B (6, 6) and compute the distance between them. *Ans.* 5.

15. If two points A $(x_1, 0)$ and B $(x_2, 0)$ are located, show that $AB = x_2 - x_1$ whether A and B lie on the same side of the origin or on opposite sides.

6. Distance between two points. — *The distance d between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by the formula*

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \qquad (1)$$

Proof. Let P_1 and P_2 represent any two given points, and let d represent the distance between them.



Draw the ordinates M_1P_1 and M_2P_2, and through P_1 draw P_1N parallel to the x-axis, meeting at N the ordinate M_2P_2 (produced if necesary).

In the right triangle P_1P_2N, $P_1N = M_1M_2 = OM_2 - OM_1 = x_2 - x_1$ and $NP_2 = M_2P_2 - M_1P_1 = y_2 - y_1$.

Substituting in $P_1P_2 = \sqrt{\overline{P_1N}^2 + \overline{NP_2}^2}$ we get the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Note. — The student should notice that the above demonstration applies equally to the two figures given, and should satisfy himself that the proof holds good when the points are located in other positions. Since results remain the same if the positions of the points are changed, in future demonstrations points will be located in the simplest position (usually in the first quadrant).

ILLUSTRATIVE EXAMPLES

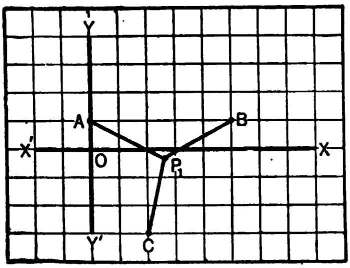
1. Find the distance between the points $A(-2, 2)$ and $B(3, 4)$.

Solution. — Here $x_1 = -2$, $y_1 = 2$, $x_2 = 3$, $y_2 = 4$. Substituting in the above formula, we have

$$AB = \sqrt{(3 + 2)^2 + (4 - 2)^2} = \sqrt{29}.$$

2. Find a point equidistant from the three points A (0, 1), B (5, 1), and C (2, −3).

Solution. — Let P_1 (x_1, y_1) be the required point.



From formula (1)

$$P_1A = \sqrt{(0 - x_1)^2 + (1 - y_1)^2},$$
$$P_1B = \sqrt{(5 - x_1)^2 + (1 - y_1)^2},$$
$$P_1C = \sqrt{(2 - x_1)^2 + (3 + y_1)^2}.$$

Since these distances are all equal, we can make the two equations:

$$\sqrt{x_1^2 + y_1^2 - 2y_1 + 1} = \sqrt{x_1^2 - 10x_1 + y_1^2 - 2y_1 + 26},$$
$$\sqrt{x_1^2 + y_1^2 - 2y_1 + 1} = \sqrt{x_1^2 - 4x_1 + y_1^2 + 6y_1 + 13}.$$

Squaring each and collecting,

$$10x_1 = 25,$$
$$2x_1 - 4y_1 = 6.$$

Whence $x_1 = \frac{5}{2}$, $y_1 = -\frac{1}{4}$.

It is evident that P_1 is the center of the circle passing through the three points A, B, and C..

After working each example, the student should examine his figure carefully and satisfy himself that his answer is reasonable.

EXERCISES

1. Find the lengths of the lines joining the following points:

(*a*) (−1, −4), (2, 1). *Ans.* $\sqrt{34}$.
(*b*) (3, 2), (0, −2).
(*c*) (a, b), $(-a, -b)$.
(*d*) $(a + b, a)$, $(b, a + b)$.

2. Find the lengths of the sides of the following triangles:

(*a*) (1, 1), (−2, 2), (−3, −3).
(*b*) (4, 2), (−3, 4), (2, −6).
(*c*) $(a, 0)$, $(0, -a)$, $(a + b, a)$.
(*d*) $(c + d, 0)$, (d, c), $(d, -c)$.

3. Prove that the points (−3, 1), (3, 1), and $(0, 1 + 3\sqrt{3})$ are the vertices of an equilateral triangle.

4. Prove that the points (4, 1), (−1, −4), and $(3, 2\sqrt{2})$ are equidistant from the origin.

5. Prove that (3, 1), (2, 4), and (−2, 1) are the vertices of an isosceles triangle.

6. Prove that the points (4, −3), (5, 4), and (−2, 5) all lie on a circle whose center is (1, 1). Find the radius.

7. Prove that (1, 1), (3, 4), and (−5, 5) are the vertices of a right triangle.

8. Prove that (1, 2), (−5, −3), (1, −11), and (7, −6) are the vertices of a parallelogram.

9. Prove that (0, −1), (3, 2), (0, 5), and (−3, 2) are the vertices of a square.

10. Find a point on the y-axis which is equidistant from (4, 0) and (−2, −2). *Ans.* (0, 2).

11. One end of a line whose length is five is at (4, 2); the abscissa of the other end is 1. Find the ordinate. *Ans.* 6 or −2.

12. Find the point equidistant from (0, 2), (3, 3), and (6, 2). *Ans.* (3, −2).

13. The point (x, y) is equidistant from (2, −1) and (7, 4). Write the equation which x and y must satisfy. *Ans.* $x + y = 6$.

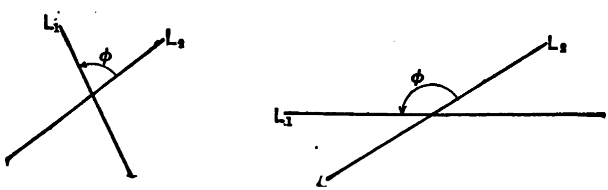
14. Express algebraically that the distance of the point (x, y) from the point (2, 3) is equal to 4.

15. The angle between oblique axes is 60°. Find the distance between the points (3, 5) and (5, 1). *Ans.* $2\sqrt{3}$.

Hint. — Locate the points, draw their coördinates and apply the law of cosines from trigonometry.

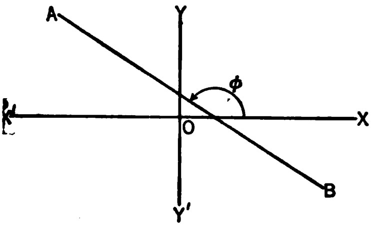
16. The angle between oblique axes is ω. Find the distance between the points $P_1 (x_1, y_1)$ and $P_2 (x_2, y_2)$.

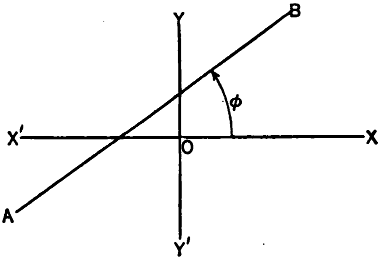
7. Inclination and Slope. — *The angle which one line makes with another is the angle not greater than* 180° *measured counter-clockwise from the second to the first.*



Thus, the angle which the line L_1 makes with another line L_2 is the angle ϕ in the figure.

The **inclination** *of a line is the angle which it makes with* the x-axis. This angle is always measured from the positive direction of the x-axis. Thus ϕ in the figures below represents the inclination of the line AB.



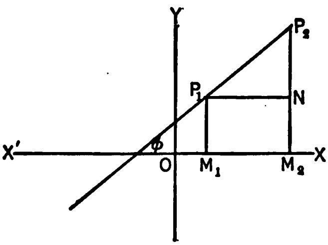


The **slope** *of a line is the tangent of its inclination.*

Formula for slope. — *The slope m of a line joining the two points P_1 (x_1, y_1) and P_2 (x_2, y_2) is given by the formula*

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \tag{2}$$

Proof. — Let P_1 (x_1, y_1) and P_2 (x_2, y_2) be two points on a line whose inclination is ϕ. It is required to find m the slope of the line.

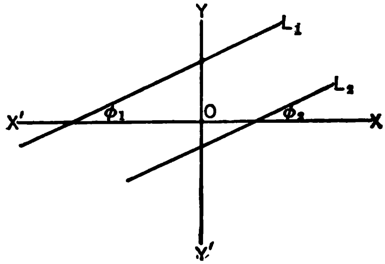


Draw the ordinates M_1P_1 and M_2P_2 and through P_1 draw P_1N parallel to the x-axis cutting M_2P_2 in N. Then angle $\phi = NP_1P_2$ (why?).

From the figure, it is seen that $m = \tan\phi = \tan NP_1P_2$ $= \dfrac{NP_2}{P_1N} = \dfrac{y_2 - y_1}{x_2 - x_1}.$

Parallel lines. — *If two lines are parallel, their slopes are equal, and conversely.*

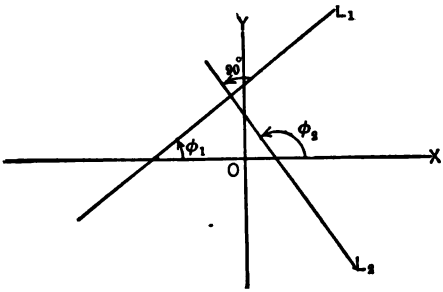
Proof. — Let ϕ_1 and ϕ_2 be the inclinations and m_1 and m_2 the slopes of the parallel lines L_1 and L_2.



Then $\phi_1 = \phi_2$ (why?) and therefore $m_1 = m_2$.

The proof of the converse is left to the student.

Perpendicular lines. — *If two lines are perpendicular, the slope of one is the negative reciprocal of the slope of the other, and conversely.*



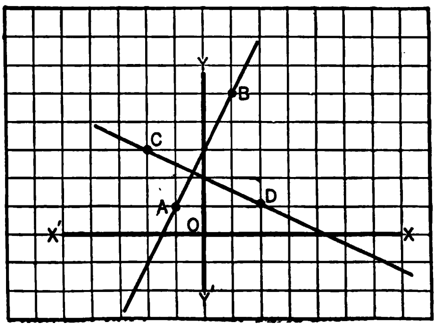
Proof.—Let ϕ_1 and ϕ_2 be the inclinations and m_1 and m_2 the slopes of the perpendicular lines L_1 and L_2.

Then $\phi_2 = 90° + \phi_1$ (why?).

Therefore

$$\tan\phi_2 = \tan(90° + \phi_1) = -\cot\phi_1 = \frac{-1}{\tan\phi_1}, \text{ whence } m_2 = \frac{-1}{m_1}.$$

A similar proof applies when ϕ_1 is obtuse. The proof of the converse is left to the student.



ILLUSTRATIVE EXAMPLE

Prove that the line joining A $(-1, 1)$ and B $(1, 5)$ is perpendicular to the line joining C $(-2, 3)$ and D $(2, 1)$.

Solution. — From formula (2), the slope of $AB = \frac{5-1}{1+1} = 2$, and the slope of $CD = \frac{1-3}{2+2} = \frac{-1}{2}$.

Since either slope is the negative reciprocal of the other, the lines are perpendicular.

EXERCISES

1. Find the slopes of the lines joining

(*a*) (2, 5) and (−3, −3);
(*b*) (3, 2) and (7, −7);
(*c*) (4, 3) and (−2, 5);
(*d*) (a, *b*) and (−*a*, 2 *b*).

2. Find the inclination of the lines joining

(*a*) (3, 2) and (−1, −2);
(*b*) $(\sqrt{3}, 0)$ and (0, 1);
(*c*) (0, 0) and $(1, \sqrt{3})$;
(*d*) (−4, 0) and $(-5, \sqrt{3})$;
(*e*) (5, 6) and (4, 7).

3. Find the slopes of the sides of the triangle whose vertices are (3, 5) (6, 2), and (5, 7). *Ans.* −1, 1, and −5.

4. The inclination of AB is 40°. If CD makes an angle of 20° with AB, find the slope of CD. *Ans.* $\sqrt{3}$.

5. Solve Ex. 7 and 8 in Art. 6 by means of formula 2.

6. Prove that the diagonals of the square in Ex. 9, Art. 6, cut at right angles.

7. Prove that (6, −5), (2, −1), (−3, −4), and (−2, −5) are the vertices of a trapezoid.

8. What is the slope of the line joining (2, 5) and (2, −4)? What is the slope of any line parallel to the y-axis?

9. Prove that the line joining (5, 2) and (6, 4) is parallel to the line joining (2, 5) and (4, 9) and perpendicular to the line joining (8, 1) and (6, 2).

10. Prove that the triangle whose vertices are (0, 0), (3, 1), and (2, 4) is a right isosceles triangle.

11. Prove by means of slopes that the figure whose vertices are (2, 1), (1, 3), (3, 4), and (4, 2) is a rectangle.

12. Prove by means of slopes that the three points (1, 1), (−2, −2), and (3, 3) lie in the same straight line.

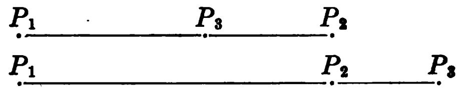
13. Three vertices of a parallelogram are (−1, −2), (2, 0), and (8, 6) joined in the order named. Find the fourth vertex. *Ans.* (5, 4).

14. A line with an inclination of 60° passes through the origin. If the ordinate of a point on the line is 6, what is the abscissa of the point? *Ans.* $2\sqrt{3}$.

15. A point is 4 units from the origin and the inclination of the line joining it to $(1, \sqrt{3})$ is 60°. Find its coördinates. *Ans.* $(2, 2\sqrt{3})$ and $(-2, -2\sqrt{3})$.

16. A point is equidistant from the two points $(2, -4)$ and $(4, 6)$, and the slope of the line joining it to $(-1, 5)$ is -2. Find its coördinates. *Ans.* $(\frac{7}{9}, \frac{13}{9})$.

8. Point of division. — If the point P_3 is taken anywhere on the line P_1P_2, P_3 divides the line into two segments P_1P_3 and P_3P_2. It will be understood that the segment P_1P_3 starts at P_1 and terminates at P_3 and that the segment P_3P_2 starts at P_3 and terminates at P_2. If both segments extend in the same direction, the segments are said to have



the same sign, if in opposite directions, opposite signs. Thus in the first figure above, the ratio of P_1P_3 to P_3P_2 is positive, while in the second figure the ratio is negative.

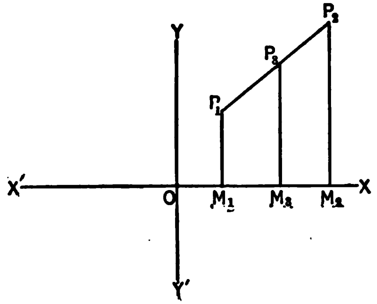
The division is called internal or external according as the point P_3 falls between P_1 and P_2 or on the line produced.

Formula for point of division. — *The coördinates* (x_3, y_3) *of the point* P_3 *which divides the line joining the two points* $P_1(x_1, y_1)$ *and* $P_2(x_2, y_2)$ *into segments such that the ratio* $\frac{P_1P_3}{P_3P_2} = \frac{r_1}{r_2}$, *are given by the formulas*

$$x_3 = \frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \qquad y_3 = \frac{r_1y_2 + r_2y_1}{r_1 + r_2}. \tag{3}$$

Proof. — Let P_1 and P_2 be the two given points, and P_3 the point which divides the line joining P_1 and P_2 in the ratio of r_1 to r_2. It is required to find the coördinates of P_3.

Draw the ordinates M_1P_1, M_2P_2, and M_3P_3. Then since the three parallels M_1P_1, M_2P_2, and M_3P_3 are cut by the two transversals M_1M_2 and P_1P_2, the corresponding segments are proportional.



Therefore,

$$\frac{M_1M_3}{M_3M_2} = \frac{P_1P_3}{P_3P_2} = \frac{r_1}{r_2}.$$

Whence

$$\frac{x_3 - x_1}{x_2 - x_3} = \frac{r_1}{r_2}.$$

Solving for x_3, the abscissa of the point of division is found to be

$$x_3 = \frac{r_1x_2 + r_2x_1}{r_1 + r_2}.$$

Similarly, by drawing the abscissas of the three points P_1, P_2, and P_3, the student is asked to derive

$$y_3 = \frac{r_1y_2 + r_2y_1}{r_1 + r_2}.$$

Coördinates of the middle point of a line. — *The coördinates* (x_3, y_3) *of the middle point,* P_3, *of the line joining the two points* P_1 (x_1, y_1) *and* P_2 (x_2, y_2) *are given by the formulas*

$$x_3 = \frac{x_1 + x_2}{2}, \qquad y_3 = \frac{y_1 + y_2}{2}. \tag{4}$$

Proof. — These results are derived immediately from formula (3) by substituting $r_1 = r_2$.

ILLUSTRATIVE EXAMPLES

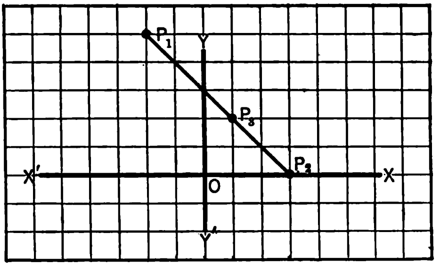
1. Find the coördinates of the point dividing the line joining $(-2, 5)$ and $(3, 0)$ in the ratio $\frac{3}{2}$.

Here $x_1 = -2$, $x_2 = 3$, $y_1 = 5$, $y_2 = 0$, $r_1 = 3$, and $r_2 = 2$.

Substituting in formula (3),

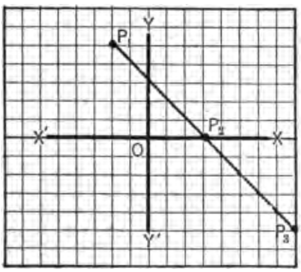
$$x_3 = \frac{3 \cdot 3 + 2 \cdot (-2)}{2 + 3} = 1,$$

$$y_3 = \frac{3 \cdot 0 + 2 \cdot 5}{2 + 3} = 2.$$



2. Divide the line joining the two points $(-2, 5)$ and $(3, 0)$ in the ratio $2 : -1$.

Here $x_1 = -2$, $x_2 = 3$, $y_1 = 5$, $y_2 = 0$, $r_1 = 2$, $r_2 = -1$.



Substituting in formula (3),

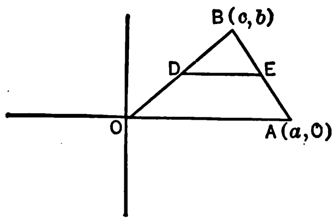
$$x_3 = \frac{2 \cdot 3 + (-1) \cdot (-2)}{2 - 1} = 8,$$

$$y_3 = \frac{2 \cdot 0 + (-1) \cdot 5}{2 - 1} = -5.$$

The figure shows that P_1P_3 is twice P_3P_2 and opposite in sign, or $P_1P_3 : P_3P_2 :: 2 : -1$.

3. Prove analytically that the line joining the middle points of two sides of a triangle is parallel to the third side.

In examples of this class, all points chosen should be represented by literal quantities so that the proof applies equally to all figures of the class. The position of the origin and axes should be taken so as to simplify the work as much as possible.



Let the base of the given triangle be represented by a and the altitude by b. Taking the side a of the triangle as the x-axis and one extremity of this base as the origin, the figure is as shown, and the vertices are $(0, 0)$, $(a, 0)$, and (c, b). Let D and E, the middle points of OB and BA respectively, be joined by the line DE. The coördinates of D, the middle point of OB, are found by formula (4) to be $\left(\frac{c}{2}, \frac{b}{2}\right)$, and those of E, the middle point of BA, to be $\left(\frac{c + a}{2}, \frac{b}{2}\right)$.

The slope of DE is shown by formula (2) to be

$$m = \frac{\frac{b}{2} - \frac{b}{2}}{\frac{c+a}{2} - \frac{c}{2}} = 0$$

Whence DE is parallel to the x-axis which coincides with OA the base of the triangle.

EXERCISES

1. Divide the line joining (3, -5) and (6, 2)

(a) in the ratio of $\frac{2}{5}$;

(b) in the ratio of $\frac{-2}{5}$.

Plot figure and discuss the position of points in the result.

2. Find the coördinates of the point C which divides the line joining A $(-3, 4)$ and B (7, 9) in the ratio $\frac{2}{3}$. Check the work by showing that the distance from A to C is $\frac{2}{3}$ of the distance from C to B.

3. Find the middle points of the sides of the triangle $(-1, 3)$, $(-3, -5)$ and $(3, -1)$ and compute the lengths of the medians.

4. If the point P_3 divides the line joining the points P_1 and P_2 in a negative ratio numerically greater than one, will the point P_3 be nearer P_1 or P_2? If the ratio is negative and numerically less than one, discuss the position of P_3.

Find the coördinates of the point which divides the line joining $(-1, 4)$ to (8, 1) in the ratio $\frac{-3}{2}$. *Ans.* $(26, -5)$.

5. Prove that in the parallelogram whose vertices are (1, 2), $(-5, -3)$, $(1, -11)$, and $(7, -6)$ the diagonals bisect each other.

6. Prove that in the trapezoid whose vertices are $(6, -5)$, $(2, -1)$, $(-3, -4)$, and $(-2, -5)$, the line joining the middle points of the non-parallel sides is parallel to the bases and equal to half their sum.

7. Find the points of trisection of the line joining $(-2, -2)$ and (7, 4). *Ans.* (1, 0) and (4, 2).

8. In what ratio does the point $(3, -2)$ divide the line joining $(-1, 2)$ and $(5, -4)$? *Ans.* 2 : 1.

9. The middle point of a line is at the point $(3, -2)$. One extremity is $(-1, -4)$, what is the other extremity? *Ans.* (7, 0).

10. The line joining $(-4, -2)$ and (4, 6) is divided in the ratio $\frac{-1}{2}$. Find the distance of the point of division from $(2, -3)$. *Ans.* $7\sqrt{5}$.

11. Prove that the lines joining the middle points of the adjacent

sides of the quadrilateral whose vertices are $(-3, -2)$, $(-1, 4)$, $(3, 6)$, and $(5, -4)$ form a parallelogram.

12. One extremity of a line is at the point $(-2, 3)$ and the line is divided by the point $(3, -2)$ in the ratio $\frac{5}{4}$. Find the other extremity. *Ans.* $(7, -6)$.

13. Find the center of gravity of the triangle whose vertices are $(-1, -2)$, $(3, 4)$, and $(5, -6)$.

Hint. — The center of gravity is the point of intersection of the medians and was shown in geometry to be two-thirds of the distance from any vertex to the middle of the opposite side. *Ans.* $(\frac{7}{3}, -\frac{2}{3})$.

14. The line AB is produced to C so that BC is equal to twice AB. A is $(5, -4)$ and B is $(3, -2)$, what are the coördinates of C? *Ans.* $(-1, 2)$.

15. The line joining P_1 $(-1, 3)$ and P_2 $(2, 4)$ crosses the y-axis at P_3. Find the ratio into which P_3 divides P_1P_2. Find the ordinate of P_3. *Ans.* $\frac{1}{2}$; $\frac{10}{3}$.

16. Three vertices of a parallelogram are $(-1, -2)$, $(2, 0)$, and $(8, 6)$, joined in the order named. Find the fourth vertex by drawing the diagonals and applying the formulas of this article. *Ans.* $(5, 4)$.

17. Prove analytically that the middle point of the hypotenuse of any right triangle is equidistant from each vertex.

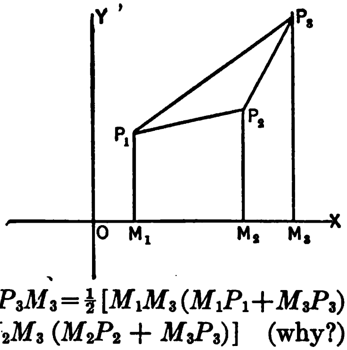
18. Prove analytically that the diagonals of any parallelogram bisect each other.

19. Prove analytically that the line joining the middle points of the non-parallel sides of a trapezoid is equal to half the sum of the parallel sides.

9. Area of a triangle. *The area of a triangle whose vertices are P_1 (x_1, y_1), P_2 (x_2, y_2), and P_3 (x_3, y_3) is given by the formula*

Area triangle $P_1P_2P_3$ =

$$\tfrac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]. \qquad (5)$$



Proof. — Locate the triangle whose vertices are P_1, P_2, and P_3, and draw the ordinates M_1P_1, M_2P_2, and M_3P_3.

Then triangle $P_1P_2P_3$ =

$$M_1P_1P_3M_3 - M_1P_1P_2M_2 - M_2P_2P_3M_3 = \tfrac{1}{2}[M_1M_3(M_1P_1 + M_3P_3) - M_1M_2(M_1P_1 + M_2P_2) - M_2M_3(M_2P_2 + M_3P_3)] \quad \text{(why?)}$$

$$= \tfrac{1}{2}[(x_3 - x_1)(y_1 + y_3) - (x_2 - x_1)(y_1 + y_2) - (x_3 - x_2)(y_2 + y_3)].$$

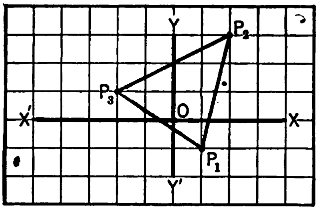
Expanding and collecting, this reduces to

$$\tfrac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right].$$

ILLUSTRATIVE EXAMPLE

Find the area of the triangle whose vertices are $(1, -1)$, $(2, 3)$, and $(-2, 1)$.

Denote $(1, -1)$ by P_1, $(2, 3)$ by P_2, and $(-2, 1)$ by P_3.

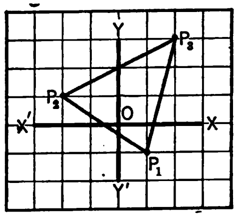


Then from formula (5),

$$\text{area } P_1P_2P_3 = \tfrac{1}{2}[1(3-1) + 2(1+1) - 2(-1-3)] = 7.$$

It will be noticed that in passing from P_1 to P_2 to P_3 we go in a counter-clockwise direction and that the area lies on the left. In this case the area is found to be positive.

If the same three points had been lettered differently, thus, $P_1(1, -1)$, $P_2(-2, 1)$, and P_3 $(2, 3)$, the formula would have given the result in the form



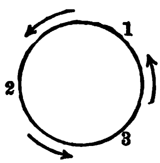
$$\text{Area } P_1P_2P_3 = \tfrac{1}{2}[1(1-3) - 2(3+1) + 2(-1-1)] = -7.$$

That is, if we pass through the points in a clockwise direction, keeping the area on the right, the formula gives a negative value to the area.

In any example, in order to obtain a positive result, the points should be taken in counter-clockwise order.

It is of decided advantage in remembering the formula

$$\text{Area } P_1P_2P_3 = \tfrac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$



to notice the cyclic order of the subscripts. If the numbers 1, 2, 3 are arranged in a circle as shown in the figure it will be observed that the subscripts of x in the formula follow the cyclic order, that is the order determined by following the arrow heads on the circle. Also the three subscripts in each term follow this order, starting however with 1 in the first, 2 in the second, and 3 in the third.

EXERCISES

1. Find the area of the triangle whose vertices are

(a) $(-1, 1), (1, 2), (-1, 3)$. *Ans.* 2.

(b) $(0, 0), (2, -1), (3, 4)$.

(c) $(a, 0), (a, b), (c, d)$.

(d) $(6, 6), (-2, 3), (-5, -1)$.

2. Find the area of the quadrilateral whose vertices are $(2, 3)$, $(-4, 1)$, $(-5, -2)$, $(3, -6)$. *Ans.* 42.

3. Prove by means of slopes that the quadrilateral whose vertices are $(2, 4)$, $(3, 0)$, $(5, 3)$, $(4, 7)$ is a parallelogram and find its area.

4. Prove that the area of the triangle whose vertices are $(2, 3)$, $(-4, -3)$, and $(-1, 0)$ is zero and hence show that these points all lie on a straight line.

5. The vertices of a triangle are $(-2, -2)$, $(4, 7)$, and $(4, -1)$. Lines are drawn from the vertex $(4, -1)$ trisecting the opposite side. Find the area of one of the three equivalent triangles formed. *Ans.* 8.

6. Are the three points $(1, 3)$, $(-1, -1)$, and $(3, 7)$ in the same straight line?

7. Prove that the lines joining the middle points of the adjacent sides of any rectangle form a rhombus whose area is one-half the area of the rectangle.

8. In a triangle whose vertices are $(1, 2)$, $(3, -4)$, $(-5, 6)$, lines are drawn joining the middle points of the sides. Prove that area of the first triangle is four times that of the second.

9. Find the area of the triangle whose vertices are $(-1, 5)$, $(2, 1)$, and $(4, 5)$.

Prove the triangle isosceles, compute the altitude, and determine the area as one-half the product of the base and altitude, thus checking the first result.

10. Find the area of the trapezoid whose vertices are $(0, 0)$, $(a, 0)$, (b, c), and (d, c). Show that this area is the product of the altitude by one-half the sum of the parallel sides.

CHAPTER II

LOCI

10. Equation of a locus. — One of the most important functions of analytic geometry is the application of algebra to geometry.

The two fundamental problems are

(1) To find the equation of a locus, having given certain geometric conditions.

(2) To plot and discuss the geometric figure or locus which corresponds to a given equation.

The first of these two problems will be considered in this article.

The equation of a locus is an equation which is satisfied by the coördinates of all points on the locus and not satisfied by the coördinates of points not on the locus.

Sometimes the equation of a locus can be written immediately from the above definition.

Thus, if a line is parallel to the y-axis and 2 units to the right of it, its equation is $x = 2$, for the equation is satisfied by the coördinates of every point on the line and by the coördinates of no point off the line.

EXERCISES

1. What is the equation of a line parallel to the x-axis and 3 units above it? Parallel to the x-axis and 5 units below it?

2. What is the equation of the x-axis? Of the y-axis?

3. What is the equation of a line parallel to the y-axis and 4 units to the left?

4. What is the equation of a line half way between the lines $y = 2$ and $y = 8$?

5. Find the equation of the line half way between the lines $x = -1$ and $x = 6$.

6. Find the equation of a line parallel to $y = -2$ and 5 units above it.

7. What is the equation of the line joining $(-2, 5)$ and $(3, 5)$? Is the point $(7, 5)$ on the line?

From the definition of the equation of a locus, it is evident that *a point whose coördinates satisfy the equation of a locus lies on that locus, and one whose coördinates do not satisfy the equation is not on the locus.*

The steps in finding the equation of a locus are, in general, as follows:

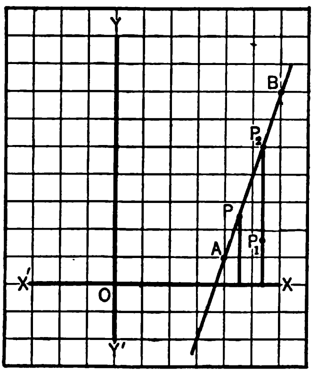
1st. Construct a figure in which all the given data are located and let $P(x, y)$ represent the coördinates of *any point* on the locus.

2nd. From the figure or from given data, equate two geometric magnitudes which are known to be equal.

3rd. Replace the geometric magnitudes by equivalent algebraic values expressed in terms of x, y and given constants.

4th. Simplify the result.

5th. Discuss why the coördinates of all points on the given locus satisfy the equation obtained and why the coördinates of all points off the locus fail to satisfy it.



ILLUSTRATIVE EXAMPLES

1. Find the equation of the straight line through the points $(4, 1)$ and $(6, 7)$.

1st. Plot the known points A and B and draw the straight line through them. Choose $P(x, y)$ *any point* on the required locus.

2nd. Since APB is a straight line, it is evident that

$$\text{the slope of } AP = \text{the slope of } AB.$$

3rd. The slope of $AP = \frac{y-1}{x-4}$, formula (2).

The slope of $AB = \frac{7-1}{6-4} = 3$, formula (2).

Whence, $\frac{y-1}{x-4} = 3.$

4th. Clearing of fractions and simplifying,

$$3x - y - 11 = 0.$$

5th. Since the point $P(x, y)$ was taken as *any point* on the desired locus, it is evident that the first condition of the equation of a locus is fulfilled, *viz.*, that the equation is satisfied by the coördinates of all points on the locus.

To prove the second condition, *viz.*, that any point not on the locus does not satisfy the equation, choose any point, P_1, not on the locus and draw its ordinate crossing the given line at P_2. Since P_2 is on the given line, its coördinates satisfy the equation, $3x - y - 11 = 0$, and we have after solving for y and substituting,

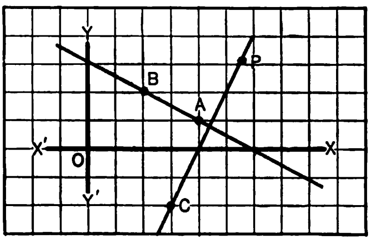
$$y_2 = 3x_2 - 11.$$

If the coördinates of P_1 are substituted in the same equation we obtain

$$y_1 = 3x_1 - 11.$$

The second members of these two equations are equal since $x_1 = x_2$, while the ordinate y_1 is either greater or less than y_2 according as P_1 is above or below the line. Hence the equation $3x - y - 11 = 0$ is not satisfied by the coördinates of $P_1(x_1, y_1)$.

Since we have shown that the equation is satisfied by the coördinates of every point on the locus and by the coördinates of no other points, it is the desired equation of the locus.



2. Find the equation of a line through the point $(3, -2)$ and perpendicular to the line joining $(4, 1)$ and $(2, 2)$.

1st. Plot the points $A(4, 1)$ and $B(2, 2)$, and draw through them the line AB. Plot $C(3, -2)$ and through it draw a line perpendicular to AB. Choose $P(x, y)$ any point on this line.

2nd. Since the lines CP and AB are perpendicular,

$$\text{the slope of } CP = -\frac{1}{\text{slope of } AB} \text{ (Art. 7).}$$

3rd. Slope of $CP = \dfrac{y+2}{x-3}$, formula (2).

Slope of $AB = \dfrac{2-1}{2-4} = -\frac{1}{2}$, formula (2).

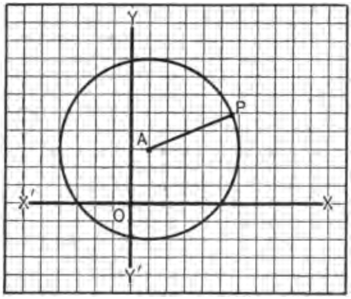
Therefore $\dfrac{y+2}{x-3} = 2$.

4th. Simplifying,

$$y - 2x + 8 = 0.$$

5th. The proof of this step is left to the student, being in general similar to that given in illustrative example 1.

3. Find the equation of the locus of the point which moves so that it is always at a distance 5 from the point (1, 3).



1st. Plot the point A (1, 3).

It is evident that the locus is a circle with center A and radius 5. Let P (x, y) represent any point on this circle.

2nd. Then $AP = 5$.

3rd. By formula (1),

$$AP = \sqrt{(x-1)^2 + (y-3)^2},$$

whence

$$\sqrt{(x-1)^2 + (y-3)^2} = 5.$$

4th. Squaring, expanding, and collecting,

$$x^2 - 2x + y^2 - 6y - 15 = 0.$$

5th. The proof of this step is left to the student.

EXERCISES

1. Find the equation of the locus of the point for which the ordinate is always three times the abscissa.

2. Find the equation of the line through the point (2, 3) and with inclination of 120°. Is the point (5, 6) on the line?

3. Find the equation of the line through (1, 2) and $(-3, -4)$. Check work by showing that the coördinates of these points satisfy the equation.

4. A point moves so that its distance from the point $(-1, 2)$ is always equal to its distance from the origin. Find its equation.

5. Find the equation of the straight line passing through the middle point of the line joining $(2, -7)$ and $(10, -5)$ and making an angle of 45° with the x-axis.

6. Find the equation of the straight line through the point $(-1, 5)$ and parallel to the line joining $(1, 3)$ and $(-5, 5)$.

7. Find the equation of the straight line perpendicular to the line joining the two points $(2, 1)$ and $(5, 4)$ and dividing the distance between them in the ratio of 2 to 1.

8. Find the equation of the straight line through the point $(1, 2)$ and with slope $\frac{1}{3}$. Find the ordinate of the point on the line for which the abscissa is 0, and thus find where the line crosses the y-axis. Similarly, find where the line crosses the x-axis.

9. Find the equation of the straight line perpendicular to the line joining the points $(-2, 1)$ and $(6, -3)$ and passing through its middle point.

10. Find the equation of the locus of a point which moves so as to be always equidistant from the two points $(-2, 1)$ and $(6, -3)$. Prove that this is the perpendicular bisector by showing that its equation is the same as that in Ex. 9.

11. Find the equations of the following circles:

(*a*) center $(0, 0)$, radius 4;
(*b*) center $(3, 2)$, radius 5;
(*c*) center (a, b), radius c.

12. Find the equation of the circle whose center is $(2, 3)$ and which is tangent to the x-axis.

13. Find the equation of a circle whose radius is 5 and whose center is the middle point of the line joining $(-1, -3)$ and $(3, 7)$.

14. Find the equation of the circle in the first quadrant which is tangent to both axes and whose radius is 2.

15. Find the equation of the circle whose center is $(1, 3)$ and whose circumference passes through the point $(-3, 0)$.

16. Find the equation of the circle of radius 3 which is tangent to the y-axis at the origin.

17. Find the equation of the circle whose diameter is the line joining the points $(5, -7)$ and $(3, -1)$.

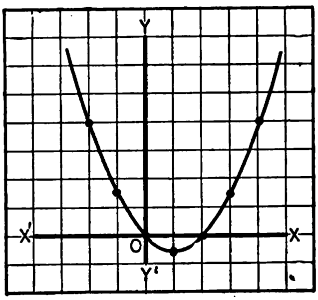
18. Find the equation of the circle whose center is the middle point of the line joining $(-1, 6)$ and $(5, 2)$ and whose circumference passes through $(1, 1)$.

11. The locus of an equation. — In the last article, equations of loci were derived from geometric data given. The second problem of analytic geometry, *viz.*, to plot and discuss the geometric figure which corresponds to a given equation, will now be considered.

This second problem divides itself into two parts, *plotting the locus of an equation and discussing an equation.*

12. Plotting the locus of an equation. — A pair of coördinates x and y locate definitely one point in a plane. If, however, these two coördinates must always satisfy a given equation, then a series of points may be chosen, the coördinates of each of which satisfy the given equation, for to each value of one variable corresponds one or more values of the other and hence an infinite number of points may be located.

Thus, if in the equation $y = \dfrac{x^2 - 2x}{2}$, we give to x a series of values differing by unity, we obtain



$x = -2,\quad y = 4;$
$x = -1,\quad y = \frac{3}{2};$
$x = 0,\quad y = 0;$
$x = 1,\quad y = -\frac{1}{2};$
$x = 2,\quad y = 0;$
$x = 3,\quad y = \frac{3}{2};$
$x = 4,\quad y = 4.$

Plotting, the points are as shown above.

It will be noticed that these points are not located indiscriminately over the plane, but apparently all lie on a curve as drawn.

More points on the curve may be obtained by giving fractional values to x, between those already used, and thus a more perfect approximation to the correct curve be obtained. This curve is called the locus of the equation.

The locus of an equation is a curve which contains all the points whose coördinates satisfy the equation and no other points.

In examples like the preceding, it is generally best to solve for y in terms of x, but in particular examples, it may be convenient or even necessary to solve for x in terms of y.

If two variables are so related that when the first is given, the value of the second is determined, then the second is said to be a **function** of the first.

That variable to which values are arbitrarily assigned is called the **independent variable** and the other the **dependent variable.**

If the two variables are connected by an algebraic equation, that is, by one which contains functions which are the result of a finite number of algebraic operations, such as addition, subtraction, multiplication, division, involution, and evolution, either function is said to be an **algebraic function** of the other.

An illustration is given by $y = x^4 - 8x^2 + 1$ or by $x^2y^3 + 2xy^2 = 7$.

In many cases of great importance, the equation connecting the variables is not algebraic, in which case one variable is said to be a **transcendental function** of the other.

Examples are, $y = \log x$, $y = e^x$, $y \tan^{-1} x = 3$.

The present chapter will be concerned with plotting and discussing *algebraic equations* only. Transcendental equations will be considered in a later chapter.

The locus is sometimes evident directly from its equation. For example, find the locus of the equation $x = 2$. Since no mention is made of y, the ordinate is unrestricted. Our problem then is to find a locus for which x is always 2, while y may have any value whatever. Such a locus is the line parallel to the y-axis and 2 units to the right of it.

EXERCISES

1. What is the locus of $y = 5$?

2. What is the locus of $x = -8$?

3. What is the locus of $y = x$?

4. What is the locus of $y = -x$?

In general, the locus of an equation will be determined by the process called **plotting.** The steps are as follows:

1*st.* *Solve for one variable in terms of the other.*

2*nd.* *Assign values to the independent variable and compute those of the dependent variable, expressing the results in the form of a table.*

3*rd.* *Plot the points thus obtained and connect with a smooth curve.*

ILLUSTRATIVE EXAMPLES

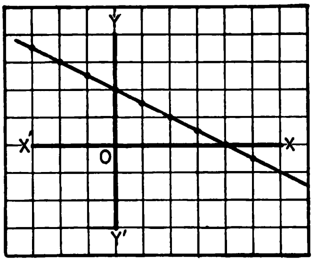
1. Plot the locus of the equation $x + 2y = 4$.

1*st.* Solving for y in terms of x,

$$y = \frac{4 - x}{2}.$$

2*nd.* It is convenient when assigning values to x and computing the corresponding values of y, to state the result in the form of a table.

x	y	x	y
0	2	0	2
1	$\frac{3}{2}$	-1	$\frac{5}{2}$
2	1	-2	3
3	$\frac{1}{2}$	-3	$\frac{7}{2}$
4	0	etc.	
5	$-\frac{1}{2}$		



3*rd.* Locating the points obtained and connecting by a smooth curve, the figure is approximately as shown.

It will later be demonstrated that every equation of first degree between two variables represents a straight line, a fact which corresponds with the appearance of this figure.

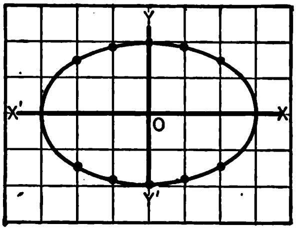
2. Plot the locus of the equation

$$4x^2 + 9y^2 = 36.$$

1*st.* Solving for y in terms of x,

$$y = \pm \tfrac{2}{3}\sqrt{9 - x^2}.$$

2*nd.* It will be noted that for values of x numerically greater than 3, y is imaginary and therefore no points of the curve can be constructed for which x is greater than 3 or less than -3. In making our table, values are taken between -3 and $+3$,



x	y	x	y
0	± 2	0	± 2
1	$\pm \frac{4}{3}\sqrt{2} = \pm 1.8$	-1	$\pm \frac{4}{3}\sqrt{2} = \pm 1.8$
2	$\pm \frac{2}{3}\sqrt{5} = \pm 1.4$	-2	$\pm \frac{2}{3}\sqrt{5} = \pm 1.4$
3	0	-3	0

3rd. Plotting the points and constructing a smooth curve through them, the locus is approximately as drawn.

3. Plot the locus of $y^3 - 2x - y = 0$.

1st. While, in general, it is better to solve for y in terms of x, in this example it is necessary to solve for x in terms of y. This gives

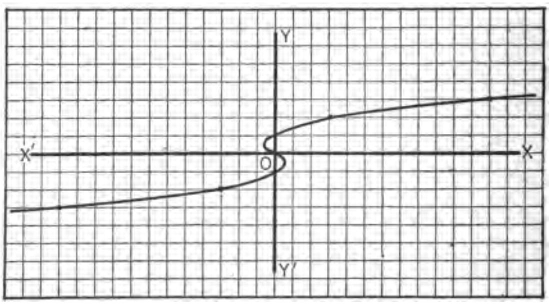
$$x = \frac{y^3 - y}{2}.$$

2nd. Assigning values to y and computing x,

y	x	y	x
0	0	0	0
1	0	-1	0
2	3	-2	-3
3	12	-3	-12
etc.		etc.	

It will be noticed that in the table above there are three values of y which give $x = 0$, and therefore three points are located on the y-axis.

In order to draw the curve more accurately in the vicinity of these points it is advisable to give to y the fractional values $-\frac{1}{2}$ and $+\frac{1}{2}$. Two additional points $(\frac{3}{16}, -\frac{1}{2})$ and $(-\frac{3}{16}, \frac{1}{2})$ are thus determined.



3rd. Locating the points and drawing a smooth curve through them, the figure is as shown.

Sometimes, as in the equation $x^2 + y^2 + 15 = 0$, there are no real values which satisfy the equation. In this case there are no real points on the locus.

Again, an equation may be satisfied by the coördinates of one point only, in which case there is only one real point on the locus. Such loci are called point-loci. An illustration is the equation $x^2 + y^2 = 0$, the locus of which is the origin.

EXERCISES

Plot the locus of each of the following equations:

1. $y + 2x - 3 = 0$.
2. $x^2 + y^2 = 4$.
3. $x^2 - y^2 = 4$.
4. $y = x^3$.
5. $y^2 = x^3$.
6. $x^2 + 4y^2 = 16$.
7. $x^2 - 4y^2 = 16$.
8. $x^2 + 4y^2 = 0$.
9. $4x^2 + 8x = 4y - 5$.
10. $2x - y = 12$.
11. $y^2 = 8x + 8$.
12. $x^2 + y^2 = 16$.
13. $y = x^3 - 2$.
14. $x^2 = 8y$.
15. $x^2 + 2x - 1 = y$.
16. $4y^2 - 9x^2 = 36$.
17. $4x^2 = y^3$.
18. $x^2 + y^2 = 25$.
19. $y^2 + 6x = 0$.
20. $2y = x^3 - x$.
21. $y^2 = (1 - x)(x + 3)$.
22. $2x + 5y + 2 = 0$.
23. $y^2 = x(3 - x)$.
24. $y^2 = 8 - 8x$.
25. $x^2 + 4x + 3 + 4y = 0$.
26. $y^2 = x^3 - 1$.

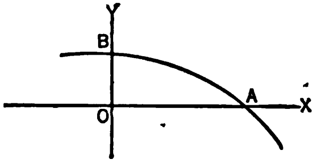
13. Discussion of an Equation. — The method of determining loci by plotting separate points is in general satisfactory in simple examples, but in those in which the equations are somewhat complicated the work is often long and the results more or less inaccurate. These difficulties are lessened in many cases, by making a study of the properties of the curve by means of the **discussion** of its equation.

The properties which will be discussed are as follows:

1. Intercepts.
2. Symmetry.
3. Extent.
4. Asymptotes.

Intercepts. — *The intercepts of a locus are the distances from the origin to the points where it cuts the axes*

Thus in the figure, the x-intercept is OA, which is the abscissa of the point A, *i.e.*, of the point on the locus whose ordinate is 0. It is then evident that to find the x-intercept, substitute $y = 0$ in the equation, and solve for x.



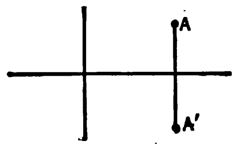
Likewise to find the y-intercept, substitute $x = 0$ and solve for y.

ILLUSTRATIVE EXAMPLE

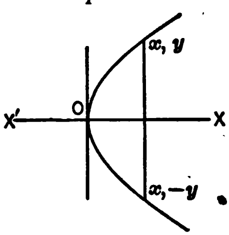
Find the x and y intercepts of the curve $2y - x^2 + 4x - 3 = 0$.
1st. Let $y = 0$, then $x^2 - 4x + 3 = 0$.
Whence, $x = 1$ or 3. The x-intercepts are therefore 1 and 3.
2nd. Let $x = 0$, then $2y - 3 = 0$, or $y = \frac{3}{2}$, the y-intercept.

Symmetry. — *Two points are symmetrical with respect to a line if that line is the perpendicular bisector of the line joining the two points.*

If A and A' are symmetrical with respect to the x-axis, then if the coördinates of A are (x, y) the coördinates of A' are $(x, -y)$. Similarly, the point symmetrical to A with respect to the y-axis is $(-x, y)$.



A curve is symmetrical with respect to a line if the curve is made up of pairs of points symmetrical with respect to the line.

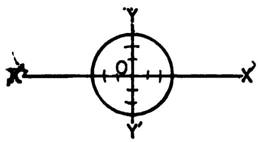


If a locus is symmetrical with respect to the x-axis, there is a point $(x, -y)$ on the curve corresponding to every point (x, y) on the curve. The coördinates $(x, -y)$ must therefore satisfy the equation of the curve, *i.e.*, y can be replaced by $-y$ and the equation remain unchanged.

The equation of the curve here sketched is $y^2 = 4x$. Replacing y by $-y$ in this equation, the result is $(-y)^2 = 4x$, which is the same as the given equation, $y^2 = 4x$.

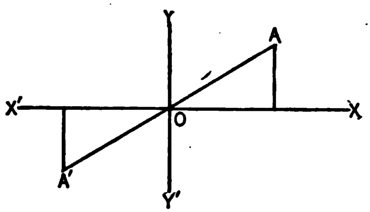
By a similar discussion, it may be shown that whenever x can be replaced by $-x$ without causing any change in the equation, then the locus is symmetrical with respect to the y-axis.

In the equation $x^2 + y^2 = 9$, x can be replaced by $-x$, and y by $-y$, therefore the curve is symmetrical with respect to both axes.



The method of replacing x by $-x$ and y by $-y$ applies equally well in testing for symmetry in either algebraic or transcendental equations. It is evident, however, that in case of *algebraic equations*, if there are no odd powers of y, then y can be replaced by $-y$, and the locus is symmetrical with respect to the x-axis, while if there are no odd powers of x, then x can be replaced by $-x$, and the locus is symmetrical with respect to the y-axis.

Two points are symmetrical with respect to the origin if the origin bisects the line joining the two points.



If A and A' are symmetrical with respect to the origin, then when the coördinates of A are (x, y) the coördinates of A' are $(-x, -y)$.

A curve is symmetrical with respect to the origin if the curve is made up of pairs of points symmetrical with respect to the origin.

From this definition, it is readily seen that the curve is symmetrical with respect to the origin if the equation remains unchanged when x and y are replaced by $-x$ and $-y$ respectively.

An algebraic equation always represents a locus symmetrical with respect to the origin if each term is of odd degree or if each term is of even degree. A constant term is considered of even degree.

Discuss each of the following equations for intercepts and symmetry and plot the loci:

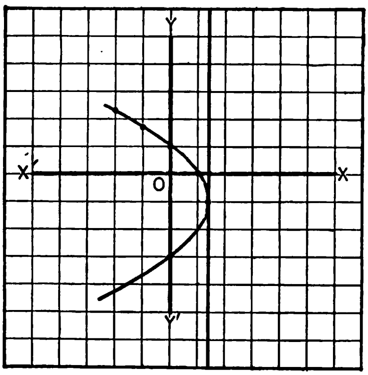
1. $y - x = 3$.
2. $y^2 = 4x$.
3. $x^2 = 4y + 4$.
4. $y = x^3$.

Extent. — In order to find how far the curve extends left and right from the origin, the equation is solved for y in terms of x and the values of x which make y real are then determined. If y is equal to an integral expression in x, or if the radicals involved are all of odd index, y is real for all values of x and the curve extends indefinitely left and right from the origin.

For example, the loci of $y = 2x + 1$ and $y = 5 + \sqrt[3]{5x^2 + x}$ extend indefinitely left and right from the origin.

When y involves a radical with even index, values of x which make the quantity under the radical negative must be determined, as for these values y is imaginary, and consequently there are no points corresponding to such abscissas.

Similarly, the extent of the curve above and below the x-axis may be determined by solving for x in terms of y.



The method of determining the extent of a curve is made clear in the following.

ILLUSTRATIVE EXAMPLES

1. Discuss for extent

$$y^2 + 2y + 3x = 3.$$

Solving for y,

$$y = -1 \pm \sqrt{4 - 3x}.$$

It is often helpful to make the coefficient of x in factors of first degree either $+1$ or -1, thus

$$y = -1 \pm \sqrt{3\left(\tfrac{4}{3} - x\right)}.$$

It can now be readily seen that if x is greater than $\frac{4}{3}$, y is imaginary and therefore there are no points on the curve to the right of the line

$x = \frac{4}{3}$. Since y is real for all values of x less than $\frac{4}{3}$, therefore the curve extends indefinitely to the left of that line.

Solving for x in terms of y,

$$x = -\frac{y^2 + 2y - 3}{3}.$$

Since x is real for all values of y, therefore the curve extends indefinitely above and below the x-axis.

Plotting a few points the curve is found as shown.

2. Discuss for extent $y^2 + 4x^2 - 16x + 12 = 0$.

Solving for y, $y = \pm 2\sqrt{-x^2 + 4x - 3}$.

Factoring the expression under the radical,

$$y = \pm 2\sqrt{(x-1)(3-x)}.$$

It is evident that the first factor is negative when x is less than 1, and positive when x is more than 1, also that the second factor is negative when x is more than 3 and positive when x is less than 3.

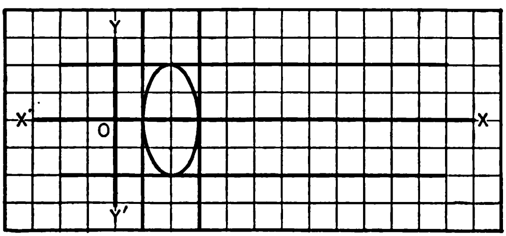
The product then is positive when x is greater than 1 and less than 3 and therefore y is real for $1 \leqq x \leqq 3$. The product is negative when x is less than 1 or more than 3 and y is then imaginary.

The whole curve therefore lies between the lines $x = 1$ and $x = 3$.

Solving for x in terms of y,

$$x = \frac{4 \pm \sqrt{4 - y^2}}{2}.$$

Since x is real when $-2 \leqq y \leqq 2$ and is imaginary for all other values, therefore the whole curve lies between the lines $y = -2$ and $y = +2$.



In plotting points it is only necessary to use values of x from 1 to 3. The figure is found to be as shown.

3. Discuss for extent $y^2 + 4y - 2x^2 + 6x + 1 = 0$.

Solving for y, $y = -2 \pm \sqrt{2x^2 - 6x + 3}$.

If the same plan were followed as in Ex. 2, the expression $2x^2 - 6x + 3$ should now be resolved into factors. These factors are not evident. From a theorem in algebra, $ax^2 + bx + c = a(x - x_1)(x - x_2)$, in which x_1 and x_2 are the roots of the equation $ax^2 + bx + c = 0$.

To apply this method here, solve $2x^2 - 6x + 3 = 0$, obtaining $x = \frac{3 \pm \sqrt{3}}{2} = 2.3+$ or $.6+$.

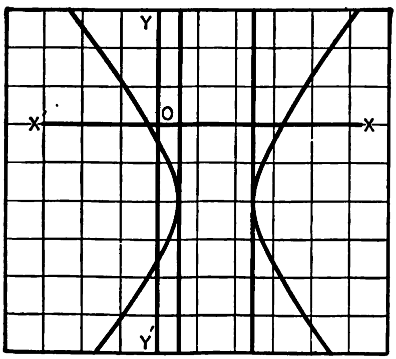
Whence, the factors are $2(x - 2.3+)(x - .6+)$, an expression which can readily be seen to be negative for *every value* of x between $.6+$ and $2.3+$ and positive for all other values. Hence y is imaginary when $.6+ < x < 2.3+$ and real when $x \leqq .6+$ or $x \geqq 2.3+$.

The fact observed here is universally true, *viz.*, that a quadratic expression has the same sign for every value between its roots, and the opposite for all other values.

This fact is of importance as by its use the process of determining the sign of a quadratic expression may be shortened.

In order, then, to determine the sign of a quadratic expression, find its roots, then substitute in it some value between the roots and determine the sign. The expression has this sign for all values between the roots and the opposite sign for all other values.

Thus, in the above problem, substitute in $2x^2 - 6x + 3$ any number between the roots $.6+$ and $2.3+$ such as $x = 1$. The sign is found to be $-$. Hence the expression is negative for all values of x between the roots.



Solving for x,

$$x = \frac{3 \pm \sqrt{2y^2 + 8y + 11}}{2}.$$

The roots of $2y^2 + 8y + 11 = 0$ are found to be imaginary. From a principle in algebra it is known that when the roots of the quadratic equation $ax^2 + bx + c = 0$ are imaginary, the expression $ax^2 + bx + c$ is positive for all values of x (if a is a positive number).

Hence x is real for every value of y.

That $2y^2 + 8y + 11$ is always positive can also be shown as follows:

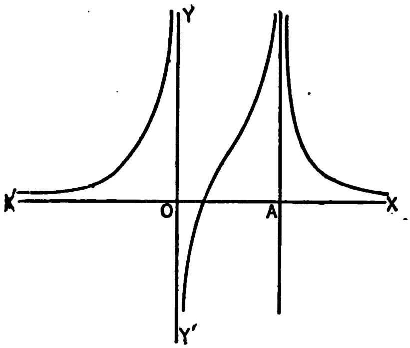
$$2y^2 + 8y + 11 = 2(y^2 + 4y + \tfrac{11}{2}) = 2[(y + 2)^2 + \tfrac{3}{2}],$$

an expression which is always positive.

Discuss each of the following equations for intercepts, symmetry, and extent and plot the loci:

1. $x^2 + y^2 = 36$.
2. $4x^2 + 9y^2 = 36$.
3. $4x^2 - 9y^2 = 36$.
4. $x^2 - 4x + 4y - 8 = 0$.
5. $x^2 - 2x + 4y^2 - 8y + 1 = 0$.
6. $x^2 + y^2 + 2x + 2y - 1 = 0$.

Asymptotes. — If, as a point generating a curve recedes indefinitely, the curve approaches coincidence with a fixed straight line, the line is called an asymptote to the curve.



At this time, only those asymptotes which are parallel to the axes will be considered.

If $OA = a$, then the asymptotes in the adjoining figure are $x = 0$, $x = a$, and $y = 0$.

It will be noticed that as x approaches either zero or a, y increases indefinitely; also, that as y approaches zero, x increases indefinitely. This fact leads readily to the method of finding vertical and horizontal asymptotes, *viz.*, solve for one variable in terms of the other and determine those values of the second variable for which the first is infinite.

Thus, find the vertical and horizontal asymptotes of $xy + 2x - y = 0$.

Solving for y,

$$y = \frac{2x}{1-x}.$$

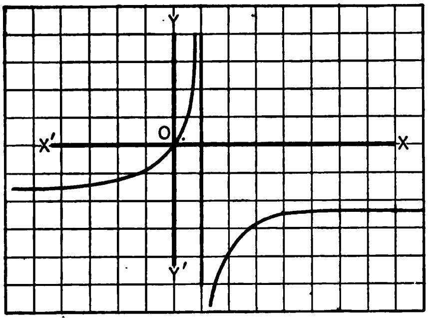
As x approaches 1, y approaches infinity, therefore $x = 1$ is a vertical asymptote.

Solving for x,

$$x = \frac{y}{y+2}.$$

As y approaches -2, x approaches infinity, therefore $y = -2$ is an asymptote.

In plotting the locus, care must be taken in determining points near the asymptotes. Thus one or more points should be plotted between $x = 0$ and $x = 1$, also between $x = 1$ and $x = 2$. It will be observed that in the locus of an algebraic equation, there can be no asymptotes parallel to the axes, unless when one variable is expressed in terms of the other the result is a fraction with a variable denominator.



The process of determining intercepts, symmetry, extent, and asymptotes involves the following:

Steps in discussion of an equation

1*st. Let $y = 0$, solve for x, thus finding the x-intercept.*

Let $x = 0$, solve for y, thus finding the y-intercept.

2*nd. In an algebraic equation observe:*

If no odd powers of y are present the locus is symmetrical with respect to the x-axis.

If no odd powers of x are present the locus is symmetrical with respect to the y-axis.

If every term is of odd degrēe or if every term is of even degree the locus is symmetrical with respect to the origin.

3*rd. Solve for y in terms of x and find what values of x make y imaginary. Points having these values as abscissas are excluded from the locus. Find what values of x make y real. Points having these values as abscissas are on the locus.*

Similarly, solve for x in terms of y and determine the values of y which make x real or imaginary.

4th. Determine asymptotes parallel to the axes by finding those finite values of either variable which make the other infinitely great.

ILLUSTRATIVE EXAMPLES

1. Discuss the equation $y^2 = (x + 2)(x + 1)(x - 2)$ and plot the locus.

1st. Let $y = 0$, then $x = -2, -1$, and 2, the intercepts on the x-axis. Let $x = 0$, y is imaginary and hence the curve does not cut the y-axis.

2nd. No odd powers of y are present, hence the curve is symmetrical with respect to the x-axis.

Odd powers of x are present, hence the curve is not symmetrical with respect to the y-axis.

The terms are partly of odd degree and partly of even degree, therefore the curve is not symmetrical with respect to the origin.

3rd. Solving for y in terms of x,

$$y = \pm\sqrt{(x+2)(x+1)(x-2)}.$$

Placing the quantity under the radical equal to zero, the roots are found to be -2, -1, and $+2$. For values of $x < -2$, the quantity under the radical is negative and hence y is imaginary. Points to the left of the line $x = -2$ are therefore excluded. For values of x more than -2 but less than -1, the quantity under the radical is positive and hence y is real. Part of the curve then lies between the lines $x = -2$ and $x = -1$. For values of x more than -1 but less than $+2$, the quantity under the radical is negative and y is imaginary. Points between the lines $x = -1$ and $x = +2$ are therefore excluded. For values of x greater than $+2$, y is real and the curve extends indefinitely to the right of the line $x = +2$.

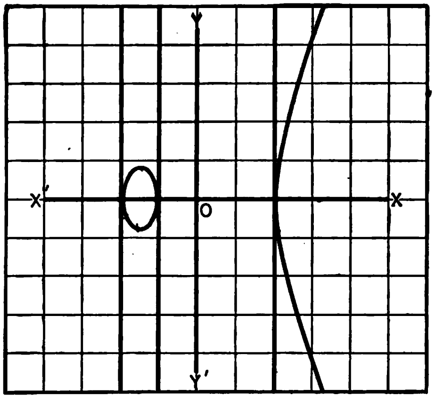
In attempting to solve for x in terms of y, it is observed that the equation is of third degree in x. It is not usually convenient to solve an equation of third or higher degree. By remembering, however, that every equation of odd degree has at least one real root, it is seen that this curve extends indefinitely both above and below the x-axis.

4th. There are no asymptotes parallel to the y-axis, since the variable does not appear in the denominator of the equation in the third step.

In plotting the locus, it is noticed that the intercepts have already determined three points of the curve on the x-axis. From the third step of the discussion, it is observed that it is only necessary to assign values to x between -2 and -1, and values greater than $+2$.

Thus,

x	y
-2	0
$-\frac{3}{2}$	$\pm 0.93+$
-1	0
$+2$	0
$+3$	$\pm 4.4+$
$+4$	$\pm 7.7+$



In every problem of this article, care should be taken to see that the locus agrees fully with the discussion.

2. Discuss the equation $(x - 2)\,y^2 = x$ and plot the locus.

1st. Let $y = 0$, then $x = 0$, the intercept on the x-axis.

Let $x = 0$, then $y = 0$, the intercept on the y-axis.

2nd. No odd powers of y are present, hence the curve is symmetrical with respect to the x-axis.

Odd powers of x are present, hence the curve is not symmetrical with respect to the y-axis.

Terms are partly of odd and partly of even degree, hence the curve is not symmetrical with respect to the origin.

3rd. Solving for y,

$$y = \pm\sqrt{\frac{x}{x - 2}}.$$

It is seen that the numerator is positive when x is positive and negative when x is negative, also that the denominator is positive when x is greater than 2 and negative when x is less than 2, and hence y is imaginary for values of x between 0 and 2 and real for all other values. Therefore no part of the curve lies between the lines $x = 0$ and $x = 2$.

Solving for x,

$$x = \frac{2\,y^2}{y^2 - 1}.$$

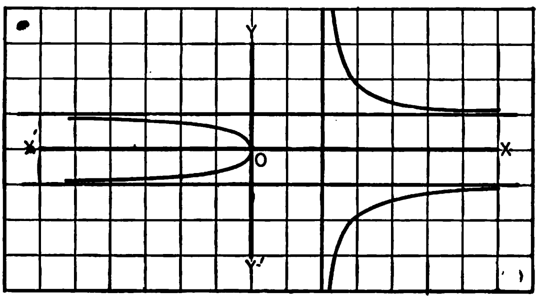
Whence x is real for every value of y.

4th. In the equation $y = \pm\sqrt{\dfrac{x}{x-2}}$, if $x = 2$, $y =$ infinity, therefore $x = 2$ is an asymptote.

In the equation $x = \dfrac{2y^2}{y^2 - 1}$, if $y = \pm 1$, $x =$ infinity, therefore $y = -1$ and $y = +1$ are asymptotes.

Plotting the locus:

x	y	x	y
2	$\pm\infty$	0	0
$\frac{5}{2}$	$\pm\sqrt{5}$	-1	$\pm\sqrt{\frac{1}{3}}$
3	$\pm\sqrt{3}$	-2	$\pm\sqrt{\frac{1}{2}}$
4	$\pm\sqrt{2}$	-3	$\pm\sqrt{\frac{3}{5}}$



EXERCISES

Discuss and plot the following:

1. $x^2 = 4y$.
2. $x^2 + y^2 = 6$.
3. $y = 2x^3$.
4. $(x-1)y = 2$.
5. $25x^2 + 9y^2 = 225$.
6. $9x^2 - 16y^2 = 144$.
7. $xy = 16$.
8. $xy^2 = 9$.
9. $x^2 = 4y^3$.
10. $x^2y - y = x$.
11. $y^2 = 3x - 9$.
12. $y = (x-2)(x-3)$.
13. $y^2 = x^3 + x^2$.
14. $y^2 = x^2 - x^3$.
15. $y = x^3 - x$.
16. $x^2y - y + 6 = 0$.

17. $y^2x^2 - y^2 = x.$
18. $xy - x - y = 0.$
19. $y(x - 2)^2 = 1.$
20. $y^2 = x(x - 3)(x - 5).$
21. $x^2 = y(y - 1)(y - 2).$
22. $x^2 + 2x + 10y - 8 = 0.$
23. $(y - 1)^2 = (x - 2)^3.$
24. $x^2 + xy + y^2 = 3.$
25. $y(3x - 2) = 2x + 4.$
26. $y(x - 1)(x - 4) = 1.$
27. $y + yx^2 = x.$
28. $x^2 - 4y^2 + 4x = 0.$
29. $x^2 + y^2 - 4x = 0.$
30. $y^2 = 12(4 - x).$
31. $y^2 = x(x - 1)(x - 2)(x - 3).$
32. $x^4 + 3 = 6xy.$
33. $x^2 + 2x + y^2 + 4y - 20 = 0.$
34. $x^2 + 2y^2 - 4x + 4y - 10 = 0.$
35. $y(x - 1)(x - 3) = x + 1.$
36. $x^2y - y + 1 = 0.$
37. $y^2(4 + x^2) = 1.$
38. $y = (x - 1)(x - 2)(x - 3).$
39. $x^2 - 4y^2 - 2x + 8y - 7 = 0.$
40. $x^2 - xy + 2x - 9 = 0.$

In some equations, the constants are represented by letters instead of by figures. There will be a different locus for each value given the constant, but it will be seen that these loci have properties common to all.

Discuss the equation $ay^2 = x(x - 2a)^2$ and plot the locus.

1*st*. Let $x = 0$, then $y = 0$, y intercept.

Let $y = 0$, then $x = 0$ and $2a$, x intercepts.

2*nd*. No odd powers of y are present, therefore the locus is symmetrical with respect to the x-axis.

Odd powers of x are present, therefore the locus is not symmetrical with respect to the y-axis.

Terms are partly of odd and partly of even degree therefore the locus is not symmetrical with respect to the origin.

3*rd*. Solving for y,

$$y = \pm (x - 2a)\sqrt{\frac{x}{a}}.$$

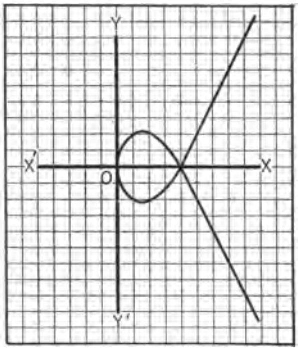
y is imaginary when x has a sign opposite to a, and real when x has the same sign as a, therefore the curve is entirely to the right of the y-axis when a is positive and entirely to the left when a is negative.

It is not convenient to solve the third degree equation for x in terms of y, but as every equation of odd degree has at least one real root, the curve extends indefinitely above and below the x-axis.

4th. There are no asymptotes parallel to the axes.

In computing the coördinates of the points, it is best to assign to x values which are multiples of a, and to assign to a any convenient length when plotting. The figure for a positive and equal to 2 is shown.

x	y
0	0
a	$\pm a$
$2a$	0
$3a$	$\pm a\sqrt{3}$
$4a$	$\pm 4a$



EXERCISES

1. Discuss the following equations and plot the loci:

(*a*) $y^2 = 4ax$.
(*b*) $4x^2 + 9y^2 = 36a^2$.
(*c*) $ay^3 = x^4$.
(*d*) $y^2 = 4a(a - x)$.
(*e*) $y^2 = x^2(a^2 - x^2)$.
(*f*) $y + (x - 3a)^3 = 0$.
(*g*) $x^2 + y^2 = 4a^2$.
(*h*) $x^2 - y^2 = 4a^2$.
(*i*) $y = (x - 2a)(x - 3a)$.
(*j*) $a^4y^2 = a^2x^4 - x^6$.
(*k*) $y^2(a^2 - x^2) = a^2x^2$.
(*l*) $a^2y^2 = x^3(2a - x)$.
(*m*) $a^3x = y^3(y - 2a)$.
(*n*) $y^2(2a - x) = x^3$.
(*o*) $y(x^2 + 4a^2) = 8a^3$.
(*p*) $9ay^2 = (x - 2a)(x - 5a)^2$.
(*q*) $27ay^2 = 4(x - 2a)^3$.
(*r*) $x^2y^2(x^2 - a^2) = a^6$.

2. Show that the following equations either represent point-loci or have no loci.

(*a*) $x^2 + 1 = 0$.
(*b*) $y^2 + 4 = 0$.
(*c*) $x^2 + 8y^2 + 2 = 0$.
(*d*) $x^2 + 4y^2 = 0$.
(*e*) $(x - 2)^2 + (y - 3)^2 + 1 = 0$
(*f*) $(x + 1)^2 + (y + 3)^2 = 0$.
(*g*) $x^2 + 2x + y^2 + 6y + 16 = 0$.
(*h*) $(x - 1)^2 + y^2 + 4 = 0$.

3. Find the equations of the following loci, and discuss and plot them.

(*a*) A point moves so as to be always equidistant from the x-axis and the point (0, 3).

(*b*) A point moves so that the square of its distance from the origin is four times its ordinate.

(*c*) A point moves so that its distance from the y-axis is equal to its distance from the point $(-2, -4)$.

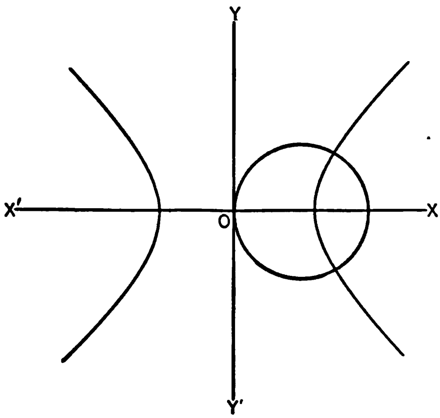
(*d*) A point moves so that its distance from the line $x = 2$ is equal to its distance from the point (4, 1).

(*e*) A point moves so that its distance from the point $(-1, -3)$ is twice its distance from the point (2, 1).

(*f*) A point moves so that the sum of its distances from (4, 0) and $(-4, 0)$ is equal to 10.

(*g*) A point moves so that the difference of its distances from (0, 5) and $(0, -5)$ is equal to 6.

14. Points of intersection of two curves. — If two equations are given, the loci will, in general, be two distinct curves. These may or may not intersect. If the curves intersect, they have one or more points in common. It follows from the definition of the locus of an equation that a point lies on two curves if and only if its coördinates satisfy each equation. In order then to find the coördinates of the points of intersection, it is necessary to find those values of x and y which satisfy the two equations, that is, the two equations must be solved simultaneously.



If there are no real roots, the curves do not intersect.

For example, find the intersection of the curves $x^2 - 4x + y^2 = 0$ and $x^2 - y^2 = 6$.

Solving simultaneously, $x = -1$ or 3. When $x = 3$, $y = \pm\sqrt{3}$, when $x = -1$, y is imaginary. Thus, there are but two real intersections $(3, +\sqrt{3})$ and $(3, -\sqrt{3})$. See figure on opposite page.

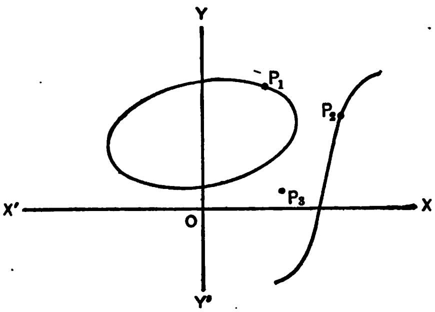
EXERCISES

Find the points of intersection and plot the following curves:

1. $x^2 + y^2 = 100$ and $2x - y = 4$.
2. $y^2 = 4x$ and $x^2 + y^2 = 5$.
3. $4x + y - 5 = 0$ and $7x - 3y - 4 = 0$.
4. $x + y = 6$ and $y^2 = 8x$.
5. $xy = 12$ and $x^2 + y^2 = 25$.
6. $y = x^3$ and $y = x$.
7. $4x^2 - y^2 = 7$ and $3x + 2y = 12$.
8. $x^2 = y + 2$ and $2x + 3y = 10$.
9. $x^2 + y^2 = 25$ and $3x^2 - 2y^2 = 30$.
10. $y^2 = x^2 - x^4$ and $2y = x$.
11. $y^2 = 4ax$ and $x + y = 3a$.
12. $b^2x^2 + a^2y^2 = a^2b^2$ and $bx + ay = ab$.
13. Find the distance between the points of intersection of $x^2 = 4y + 4$ and $x = y + 1$. *Ans.* $4\sqrt{2}$.
14. Do the curves $x^2 + y^2 = 9$ and $y^2 = x - 4$ intersect?
15. Find the points of intersection of the loci $x^2 + y^2 = 9$ and $y = x + b$. For what values of b are these intersections real and distinct? imaginary? coincident?
16. Find the area of the triangle whose sides have the equations $2y - 3x + 1 = 0$, $4y + 3x + 11 = 0$, and $y + 3x - 4 = 0$. *Ans.* 9.
17. Find the area of the polygon whose sides have the equations $x = -4$, $y = 3$, $y = 2x + 1$, and $y = -2$.
18. Show that the three loci $x^2 + y^2 = 25$, $y - x + 1 = 0$, and $y - 2x = 2$ pass through a common point.
19. Find the slopes of the sides of the triangle formed by the lines, $x = 8$, $x + y = 3$, $x - 2y = 6$.
20. Prove that the quadrilateral whose sides have the equations $y = 4$, $y = -2$, $x - 2y = 6$, and $x - 2y = -6$ is a parallelogram.
21. The equations of the sides of a triangle are $2x + 4y = 2$, $x - 3y = 6$, and $12y + x = 6$. Find the lengths of the medians.

15. Locus by factoring. — *If an equation whose second member is zero has for its first member the product of variable factors, then the locus of the equation is found by setting each factor equal to zero and plotting the result.*

Proof. — Let u and v represent any two functions of x and y. The given equation can be written



$$uv = 0, \qquad (1)$$

and the equations formed by setting the factors separately equal to zero are

$$u = 0 \qquad (2)$$

and $$v = 0. \qquad (3)$$

Assume the loci of (2) and (3) to be the figures as shown above.

To prove the proposition it will be necessary to show:

1*st.* that the coördinates of any point on the locus of equation (2) or (3) satisfy equation (1);

2*nd.* that the coördinates of no other points satisfy equation (1).

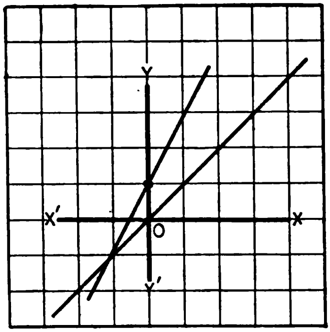
Let $P_1\,(x_1, y_1)$ represent any point on the locus of (2), then the coördinates (x_1, y_1) must satisfy the equation $u = 0$. If the same coördinates are substituted in the equation $uv = 0$, the equation will be satisfied, since one of the factors is zero and consequently the product is zero.

Similarly, the coördinates of $P_2\,(x_2, y_2)$ any point on the locus of equation (3) can be shown to satisfy equation (1).

Let $P_3\,(x_3, y_3)$ represent any point not on either locus. The coördinates of this point will not satisfy equation (1) since neither factor is zero and consequently the product is not zero.

It has therefore been shown that $uv = 0$ has for its locus the combined loci of $u = 0$, and $v = 0$, since the locus of an equation is the curve which contains all the points which satisfy the equation and no other points.

ILLUSTRATIVE EXAMPLES

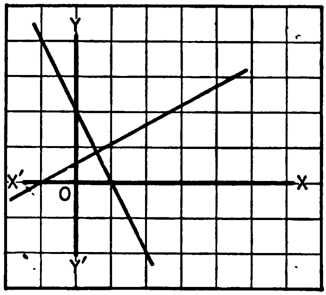


1. Plot the locus of the equation $2x^2 - 3xy + x + y^2 - y = 0$.

Grouping the second degree terms together, the equation can readily be factored thus:

$$(2x^2 - 3xy + y^2) + (x - y) = 0,$$
$$(x - y)(2x - y) + (x - y) = 0,$$
$$(x - y)(2x - y + 1) = 0.$$

Plotting the loci represented by the equations $x - y = 0$ and $2x - y + 1 = 0$, the locus is found to be as shown.



2. Plot the locus of the equation $2x^2 - 3xy - 2y^2 + 5y - 2 = 0$.

Since the factors of this equation are not readily found, the principle is used that $ax^2 + bx + c = a(x - x_1)(x - x_2)$ where x_1 and x_2 are the roots of the equation $ax^2 + bx + c = 0$.

Solving for x,

$$x = \frac{3y \pm \sqrt{25y^2 - 40y + 16}}{4} = 2y - 1 \quad \text{or} \quad \frac{-y + 2}{2}.$$

The equation therefore may be written

$$2(x - 2y + 1)\left(x - \frac{-y + 2}{2}\right) = 0,$$

or

$$(x - 2y + 1)(2x + y - 2) = 0.$$

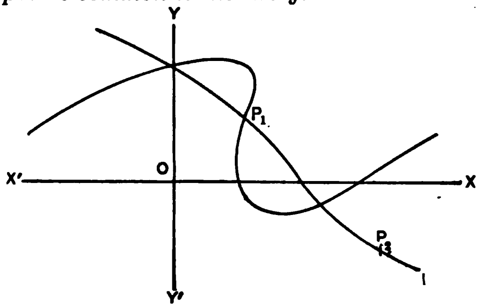
Plotting separately the loci represented by $x - 2y + 1 = 0$ and $2x + y - 2 = 0$, the figure is found as shown.

EXERCISES

Plot the loci represented by the following equations:

1. $y^2 + 6y = 0.$
2. $x^2 - 9y^2 = 0.$
3. $x^2 + 2xy = 0.$
4. $4x^2 - 11xy - 3y^2 = 0.$
5. $x^2 - 9y^2 + 2x - 6y = 0.$
6. $x^2 - xy - 2y^2 + 3y - 1 = 0.$
7. $(x^2 - 2y)(y + x - 1) = 0.$
8. $2x^2 - 7xy + 3y^2 + 5y - 2 = 0.$
9. $(x^2 + y^2 - 9)(x^2 + 4y^2 - 9) = 0.$
10. Write the single equation which represents:
    (*a*) the two coördinate axes;
    (*b*) the two lines parallel to the x-axis and at distances 2 and 4 units respectively above it;
    (*c*) the two lines which bisect the angles between the axes.
11. Show that the locus of $x^2 - 7x + 12 = 0$ is a pair of parallel lines.
12. Plot the locus of $(y^2 - 2y - 8)(x^2 - 2x - 3) = 0$, and show that the lines enclose a parallelogram.

16. Loci through intersections of two given loci. — *If an equation whose second member is zero is multiplied by any constant and added to another equation whose second member is zero, the resulting equation represents a locus through all points common to the two given loci and through no other points on either locus.*



Proof. — Let

$$u = 0 \quad (1)$$

and

$$v = 0 \quad (2)$$

represent the equations of two given loci, also let $P_1\ (x_1, y_1)$ represent any point common to the two given loci and $P_2\ (x_2, y_2)$ any point on one of the loci but not on the other.

To prove that

$$u + kv = 0, \tag{3}$$

in which k represents any constant, positive or negative, is satisfied by the coördinates of P_1 but not by those of P_2.

Since P_1 is a point on the locus of equation (1), its coördinates must satisfy the equation $u = 0$. Since P_1 is a point on the locus of equation (2), its coördinates must satisfy the equation $v = 0$. Wherefore the equation $u + kv = 0$ is satisfied by the coördinates of P_1.

The coördinates of P_2 will cause one term of $u + kv$ to equal zero, but not the other, therefore $u + kv = 0$ is not satisfied by the coördinates of P_2.

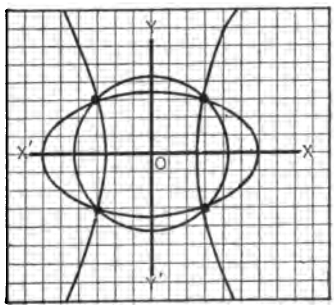
Hence $u + kv = 0$ represents a locus through the points common to $u = 0$ and $v = 0$ and through no other points on either locus.

ILLUSTRATIVE EXAMPLE

Find the equation of a system of loci through all the intersections of the two loci whose equations are $x^2 + y^2 = 18$ and $4x^2 - y^2 = 27$. Find the particular curve of the system which passes through the point (6, 0). Check the accuracy of the result by plotting the curves.

Multiplying the first equation by k and adding to the second,

$$(k+4)x^2 + (k-1)y^2 = 18k + 27.$$



Since the point (6, 0) is on the locus, its coördinates must satisfy the above equation, whence

$$36k + 144 = 18k + 27.$$

Solving, $k = -\frac{13}{2}$.

Substituting this value instead of k in the equation of the system, we get

$$x^2 + 3y^2 = 36.$$

The three curves are shown in the figure above.

EXAMPLES

Find the equation of a system of loci through the intersections of the following loci:

1. $x + 2y + 1 = 0$ and $3x - 4y - 8 = 0$.

Ans. $(k + 3)x + (2k - 4)y + k - 8 = 0$.

2. $x^2 + y^2 = 9$ and $x - 4y = 8$.

3. $2x^2 - y^2 = 7$ and $x^2 + 3y^2 = 10$.

4. Write the equation of a system of loci through the intersections of $y^2 + 4x = 0$ and $y - 2x = 0$.

Test the accuracy of the work by finding the coördinates of the points of intersection of the given curves and substituting in the resulting equation.

5. Write the equation of a system of loci through the intersections of the curves whose equations are $x^2 - y^2 - 9 = 0$ and $x + y = 6$. So determine k that the resulting curve shall pass through the origin. Factor the resulting equation and plot.

6. Write the equation of a system of loci through the intersections of the loci whose equations are $x^2 - 4y = 0$ and $y - x = 0$. Give k such a value that the resulting equation shall not contain y. *Ans.* $x^2 - 4x = 0$.

7. Write the equation of a system of loci through the points of intersection of $x^2 + y^2 - 4 = 0$ and $x^2 + y^2 - 4x = 0$, and by giving k the value -1, determine the equation of first degree, the locus of which passes through the common points of the two loci. Plot the loci of the three equations.

8. Find the equation of first degree which represents a locus through the intersections of $x^2 - 2x + y^2 = 0$ and $x^2 + y^2 = 1$.

CHAPTER III

THE STRAIGHT LINE

17. This chapter will be concerned with a study of the equations and properties of straight lines. Later chapters will consider other well-known curves.

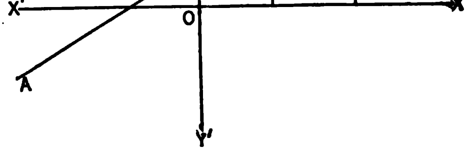
It was observed in plane geometry that a straight line was fully determined if two conditions regarding the line were known; for example, two points on the line or one point and the direction.

Similarly, it is found that the *equation* of a straight line can always be found if the two conditions which fix the line are given.

18. First standard equation of a line. In terms of point and slope. — *The equation of a straight line passing through a given point P_1 (x_1, y_1) and having a given slope m is*

$$y - y_1 = m(x - x_1). \qquad (6)$$

Proof. — Construct the given line AB whose slope is m and which passes through the point P_1 (x_1, y_1). Let the point P (x, y) represent *any* point on the line.



The slope of the line $PP_1 = m$, by hypothesis. From formula (2), the slope of $PP_1 = \dfrac{y - y_1}{x - x_1}$.

Hence, $\frac{y - y_1}{x - x_1} = m$ or $y - y_1 = m(x - x_1)$.

This is the equation of the line with slope m and passing through the point $P_1 (x_1, y_1)$, since it fulfils the two requirements of the definition. For, $P (x, y)$ was taken as *any* point on the line, therefore the equation is satisfied by the coördinates of *every* point on the line. Moreover, that the equation is not satisfied by the coördinates of any point not on the line, can be shown in a manner identical to that given in the first illustration of Art. 10. This step is so similar in all examples that the student will not be required to give it, unless called for, but he should never lose sight of the fact that this is one of the essential conditions in the determining of the equation of a locus.

19. Second standard equation of a line. In terms of slope and y-intercept. — *The equation of a straight line of slope m and y-intercept b is*

$$\boldsymbol{y = mx + b}. \tag{7}$$

Proof. — Since the y-intercept determines the point whose coördinates are $(0, b)$, this is a particular case of equation (6). Substituting in that equation $x_1 = 0$ and $y_1 = b$, the equation becomes

$$y = mx + b.$$

This equation can also be derived from a figure in a manner similar to that used in deriving equation (6).

In deriving many equations, the student may either locate his given data in a figure and derive the equation according to the method outlined in Art. 10, or he may substitute the data in any standard equation previously derived.

Since m and b may have any values, positive, negative or zero, the equation $y = mx + b$ represents any line which cuts the y-axis, that is, the locus of this equation includes all straight lines except those parallel to the y-axis. The equation of such a line has been shown in Art. 10 to be of the form $x = a$, in which a represents the constant distance of the line from the y-axis. The two equations $y = mx + b$ and $x = a$ represent all straight lines.

ILLUSTRATIVE EXAMPLE

Find the equation of a line through the point (2, 1) and perpendicular to the line joining (−3, 1) and (1, 5).

The slope of the line joining the two points is 1 by formula (2), therefore the slope of the required line is −1. Substituting $m = -1$, $x_1 = 2$, and $y_1 = 1$ in equation (6), the equation of the line through the point (2, 1) and perpendicular to the line joining (−3, 1) and (1, 5) is

$$y - 1 = -1\,(x - 2) \quad \text{or} \quad x + y = 3.$$

EXERCISES

1. Find the equations of the lines:

(*a*) through (−2, −1), inclination 60°. *Ans.* $y = \sqrt{3}\,x + 2\sqrt{3} - 1$.
(*b*) through (−3, −2), slope 2. *Ans.* $y - 2x = 4$.
(*c*) through (−2, 5), inclination 90°.
(*d*) through (2, −5), inclination 135°.
(*e*) through (−1, 1), parallel to the line joining (2, 3) and (5, 2).
(*f*) through (2, 6), and perpendicular to the line joining (5, 5) and (−1, 3).
(*g*) through (4, 2), with equal and positive intercepts on the axes.
(*h*) x-intercept 5 and slope −3.
(*i*) y-intercept −2 and slope −4.

2. Find the equation of the line with slope −2 through the intersection of the lines $2y + x - 3 = 0$ and $x - 3y + 2 = 0$. *Ans.* $2x + y = 3$.

3. In the equation $y = mx + b$, what is the relation between the lines if b remains constant and m changes? If m remains constant and b changes?

4. What is the sign of m if both intercepts on the axes are positive? If both negative? If of opposite signs?

5. Find the equation of a line perpendicular to the line joining (−1, −2) and (3, 6), through its middle point. *Ans.* $2y + x - 5 = 0$.

6. Find the equations of the perpendicular bisectors of the sides of the triangle whose vertices are $(-2, -1)$, $(4, 1)$, and $(0, -3)$. Prove that these bisectors meet in a point.

7. Find the equation of the line parallel to the y-axis through the middle point of the line joining $(2, 3)$ and $(4, -3)$. *Ans.* $x = 3$.

8. The vertices of a triangle are $(5, -3)$, $(3, 7)$, and $(-3, 1)$. Find the equations of the line through the vertices and parallel to the sides.

9. An isosceles right triangle has its hypotenuse along the x-axis and its vertex at the point $(2, 3)$. Find the equations of its sides.

10. The vertices of a triangle are $(7, 1)$, $(5, -3)$, and $(-3, 5)$. Find the equations of the perpendiculars dropped from the vertices on the opposite sides. Prove these lines meet in a point.

11. Two lines are drawn through $(2, 4)$ with inclinations $30°$ and $60°$. Find the equations of the two lines which bisect the angles between the two given lines.

12. If $\tan \theta = 3$, find the equations of the lines through the origin whose inclinations are

(*a*) $\theta - 45°$;
(*b*) $\theta + 45°$;
(*c*) $\theta + 30°$.

20. Third standard equation of a line. In terms of two given points. — *The equation of a straight line passing through two given points* $P_1 (x_1, y_1)$ *and* $P_2 (x_2, y_2)$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1). \tag{8}$$

Proof. — The slope of the line through the two given points $P_1 (x_1, y_1)$ and $P_2 (x_2, y_2)$ is, by formula (2), $m = \dfrac{y_2 - y_1}{x_2 - x_1}$; also it is given that the line passes through the point $P_1 (x_1, y_1)$. Therefore, applying equation (6), the result becomes

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1).$$

As an exercise, the student is asked to derive the equation by the method outlined in Art. 10.

21. Fourth standard equation of a line. In terms of the intercepts. — *The equation of a straight line whose x-intercept is a and whose y-intercept is b is*

$$\frac{x}{a} + \frac{y}{b} = 1. \qquad (9)$$

Proof. — The intercepts determine two points $(a, 0)$ and $(0, b)$ on the line. Substituting these results in (8), the equation is

$$y - 0 = \frac{b - 0}{0 - a}(x - a).$$

Simplifying,

$$\frac{x}{a} + \frac{y}{b} = 1.$$

The student is asked to derive this equation by applying the method of Art. 10.

EXERCISES

1. Write the equations of the lines through the following pairs of points:

(*a*) (3, 2) and (4, 5). (*c*) (1, 6) and (−2, 4).
(*b*) (2, −3) and (−3, −2). (*d*) $(a, 2a)$ and $(3a, -a)$.

2. Write the equations of the lines which make the following intercepts on the x and y axes respectively:

(*a*) 1 and 5. (*c*) −4 and −4.
(*b*) 8 and −3. (*d*) $-a$ and $+a$.

3. Write the equation of the line through the points (5, −1) and (−4, −2), and check the result by showing that the coördinates of the given points satisfy the equation.

4. Is the point (5, −6) on the straight line joining (2, 4) and (−3, −2)?

5. Find the equations of the sides of the triangle whose vertices are (3, −1), (−4, 2), and (−1, −1).

6. Find the equation of the line whose y-intercept is −5 and which passes through the intersection of the two lines $2x + y + 5 = 0$ and $4x - y + 7 = 0$.

7. The vertices of a triangle are (1, 3), (4, −3), and (−3, −2). Find the equations of the lines from (−3, −2) trisecting the opposite side.

8. Determine whether the following sets of points lie on a straight line:

(*a*) (2, 3), (−1, −2), and (3, 2).

(*b*) (2, 4), (1, 2), and (−2, −4).

9. Find the equations of the medians of the triangle whose vertices are (4, 1), (2, −3), and (−1, 5).

10. Prove that the medians in example 9 meet at a point $\frac{2}{3}$ of the distance from any vertex to the middle of the opposite side.

11. Find the equations of the lines joining the middle points of the sides of the triangle whose vertices are (−2, −3), (4, 1), and (2, −5).

12. Find the equation of a line whose x-intercept is 4 and which passes through the intersection of the lines $x + y = 6$ and $3x - 2y = 8$.

22. Locus of equation of first degree. — It was shown in Art. 19 that any straight line can be represented by either $y = mx + b$, or by $x = a$, both of which are equations of first degree. It will now be shown that the converse is true, namely:

Every equation of first degree represents a straight line.

Proof. — If A, B, and C may have any values, positive, negative, or zero, then the equation $Ax + By + C = 0$ includes all equations of first degree. If B is not zero, the equation may be divided by B, and after transposing and solving for y, the result is $y = -\frac{A}{B}x - \frac{C}{B}$.

This equation is of the form $y = mx + b$, in which $m = -A/B$ and $b = -C/B$. Therefore the equation $Ax + By + C = 0$ represents a straight line of slope $-A/B$ and y-intercept $-C/B$.

If $B = 0$, the equation becomes $Ax + C = 0$, and may be written $x = -C/A$, a straight line parallel to the y-axis at a distance $-C/A$ from it.

Hence all equations of first degree represent straight lines.

The method just outlined of changing a general equation to a standard form is one of great practical use in analytic geometry, not only for the straight line, but also for all the other curves which follow. After an equation has been put into one of the standard forms, it is only necessary to compare the constants in order to write down many facts of importance regarding the locus which any given equation represents.

Thus, given the equation $2x + 3y = 6$.

Solving for y,

$$y = -\tfrac{2}{3}x + 2.$$

This is in the form $y = mx + b$, in which $m = -\frac{2}{3}$ = the slope of the line and $b = 2$ = the y-intercept.

Again, dividing the given equation by 6,

$$\frac{x}{3} + \frac{y}{2} = 1.$$

This is in the form $\dfrac{x}{a} + \dfrac{y}{b} = 1$, in which $a = 3 = x$-intercept and $b = 2 = y$-intercept.

23. Plotting straight lines. — Since every equation of first degree has been shown to be a straight line, therefore in plotting the locus of a first degree equation it is sufficient to locate two points and then draw the indefinite straight line through them. The most convenient points are usually those determined by the intercepts on the axes.

If the intercepts are both zero, the line passes through the origin and it is necessary to locate another point on the line.

EXERCISES

1. Find the slopes of the following lines:

(*a*) $2x - 6y = 6$. (*c*) $7x + 4y = 8$.
(*b*) $x + 3y - 5 = 0$. (*d*) $3y - x = 12$.

2. Find the slopes of the following lines and determine which of them are parallel and which perpendicular to each other. Plot the loci.

(*a*) $3x + y - 7 = 0$. (*c*) $3y - x = 2$.
(*b*) $6x + 2y - 1 = 0$. (*d*) $2x - 6y = 4$.

3. Find the equation of the line through the point $(-1, 5)$ and perpendicular to the line $2y - 3x = 7$. *Ans.* $2x + 3y = 13$.

4. Find the slopes of the two lines $Ax + By + C = 0$ and $A'x + B'y + C' = 0$ and show that if the lines are parallel,

$$\frac{A}{A'} = \frac{B}{B'}.$$

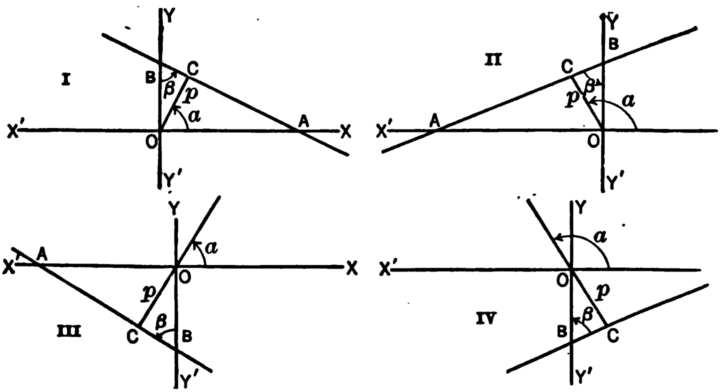
5. Prove that if the lines in Ex. 4 are perpendicular, then $AA' = -BB'$.

6. Find the equation of a line through (x_1, y_1) parallel to $Ax + By$ $C' = 0$. *Ans.* $Ax + By - (Ax_1 + By_1) = 0$.

7. The equations of two sides of a parallelogram are $y - x = 2$ a $2x + y = 4$. What are the equations of the other sides if they interse at the point $(0, -4)$? *Ans.* $y + 4 = x$ and $y + 2x + 4 = 0$.

24. Normal equation of a straight line. — As has bee previously stated, whenever two conditions which determi a straight line are known, the equation of the line can found.

In the case now to be considered, the line is determine by its distance from the origin and the inclination of th line perpendicular to the given line through the origin.



It will be recalled that inclination is always taken to be less than 180° and consequently in Figs. III and IV, th line OC must be produced through the origin before the inclination can be determined.

The inclination α in Figs. I and III, in which the line AB crosses the first and third quadrants respectively, is seen to be acute, while in Figs. II and IV, in which the line crosses the second and fourth quadrants, it is obtuse.

In each figure, $OC = p$ and is positive when above the x-axis and negative when below it.

The equation of the line which is determined by the conditions of this article is called the **normal equation of a straight line.**

The equation of a straight line in terms of p, the length of the perpendicular from the origin to the line, and α, the inclination of that perpendicular, is

$$x \cos \alpha + y \sin \alpha = p. \tag{10}$$

Proof. — In each of the above figures, if a and b represent the intercepts on the x-axis and y-axis respectively, then

from the triangle AOC, $p = a \cos \alpha$, and
from the triangle BOC, $p = b \sin \beta = b \sin \alpha$,

since β is either equal to α or to $180° - \alpha$ (why?) and therefore $\sin \beta = \sin \alpha$.

Computing the intercepts a and b, $a = \dfrac{p}{\cos \alpha}$ and $b = \dfrac{p}{\sin \alpha}$.

Substituting these in the intercept form of the equation,

$$\frac{x}{a} + \frac{y}{b} = 1,$$

the equation becomes,

$$\frac{x}{\dfrac{p}{\cos \alpha}} + \frac{y}{\dfrac{p}{\sin \alpha}} = 1.$$

Simplifying,

$$x \cos \alpha + y \sin \alpha = p.$$

EXERCISES

1. Write the equations of the lines, having given:

(*a*) $p = 3,\ \alpha = 120°$. (*c*) $p = -4,\ \alpha = 30°$.
(*b*) $p = 5,\ \alpha = 180°$. (*d*) $p = 0,\ \alpha = 60°$.

2. What system of lines is given by $x \cos \alpha + y \sin \alpha - p = 0$, when α is constant and p varies? when p is constant and α varies?

3. Given $p = 5$ and ϕ the inclination of the line $= 120°$. Compute α and write the equation of the line.

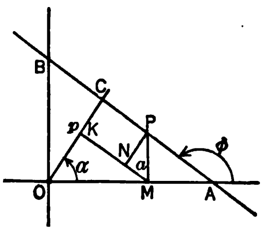
4. Write the equations of the lines, having given:

(*a*) $\phi = 135°,\ p = 5$. (*b*) $\phi = 135°,\ p = -5$.

5. Draw each of the following lines, find p and α and write their equations:

(*a*) intercepts each equal 3.
(*b*) $\phi = 120°$, y-intercept $= 3$.
(*c*) intercepts each $= -3$.
(*d*) x-intercept $= 3$, y-intercept $= -3$.
(*e*) $\phi = 30°$, x-intercept $= -5$.

6. For what values of p and α will $x \cos \alpha + y \sin \alpha - p = 0$ be parallel to the x-axis? to the y-axis? pass through the origin?



7. Write the equation of the line through the point $(3, 0)$ if $\alpha = 60°$.

8. For the equation $y - x = 4$, find slope, inclination, α and p.

9. Derive equation (10) by the method of Art. 10, using the figure here given.

Hint. — $p = OK + KC$.

10. Derive equation (10) by computing m and b in terms of p and α and substituting in equation (7).

25. Reduction to normal form. — It is required to reduce the general equation of a straight line, $Ax + By + C = 0$, to the normal form $x \cos \alpha + y \sin \alpha = p$.

Equating the x-intercepts in each,

$$\frac{p}{\cos \alpha} = \frac{-C}{A}. \tag{1}$$

Equating the y-intercepts in each,

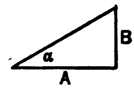
$$\frac{p}{\sin\alpha} = \frac{-C}{B}. \tag{2}$$

Dividing (1) by (2),

$$\tan\alpha = \frac{B}{A}.$$

Whence, $\cos\alpha = \pm\dfrac{A}{\sqrt{A^2+B^2}}$ and $\sin\alpha = \pm\dfrac{B}{\sqrt{A^2+B^2}}$.

This is readily seen by drawing a right triangle with leg B opposite angle α and A adjacent to it. The hypotenuse is then $\pm\sqrt{A^2+B^2}$.



From (1), $p = \dfrac{-C}{A}\cos\alpha = \mp\dfrac{C}{\sqrt{A^2+B^2}}$.

Since α is always less than 180°, $\sin\alpha$ is always + and therefore the sign of the radical is always the same as the sign of B.

Substituting these values of $\sin\alpha$, $\cos\alpha$, and p in the normal form, the general equation becomes,

$$\pm\frac{A}{\sqrt{A^2+B^2}}x \pm \frac{B}{\sqrt{A^2+B^2}}y = \pm\frac{-C}{\sqrt{A^2+B^2}}.$$

These results can be summarized as follows:

To reduce an equation in the form $Ax + By + C = 0$ to the normal form, divide the equation by $\pm\sqrt{A^2+B^2}$, in which the sign of the radical is the same as the sign of B. If B is missing, choose the sign of the radical the same as A.

Example. — Reduce $4x - 3y - 15 = 0$ to normal form and decide from the signs of $\sin\alpha$, $\cos\alpha$, and p, which quadrant is crossed by the line.

$A = 4, B = -3, C = -15, \pm\sqrt{A^2+B^2} = \pm\sqrt{16+9} = \pm 5.$

Dividing by -5,

$$-\tfrac{4}{5}x + \tfrac{3}{5}y + 3 = 0.$$

Comparing with

$$x \cos \alpha + y \sin \alpha - p = 0,$$

it is seen that $\cos \alpha$ is negative and $\sin \alpha$ positive, therefore α is obtuse; also p is negative and thus the perpendicular falls below the x-axis. The line then crosses the fourth quadrant. Check by plotting the line.

EXERCISES

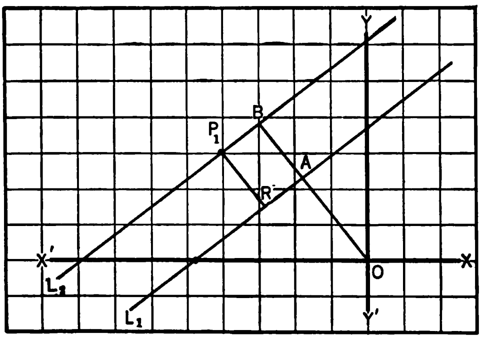
1. Reduce the following equations to normal form and determine p and α:

(a) $3x - 4y = 25$.	(e) $3x + 4y = 25$.
(b) $3x + y - 10 = 0$.	(f) $y - 3x + 4 = 0$.
(c) $y + 2 = 0$.	(g) $y - 2 = 0$.
(d) $3x - 4y = 0$.	(h) $x - 2 = 0$.

2. A line passes through $(-2, -1)$ and is perpendicular to $2x + y + 3 = 0$. Find its equation and distance from the origin.

3. A line passes through $(-4, -5)$ and has its intercepts equal and both negative. Find its equation and distance from the origin.

26. Perpendicular distance from a line to a point. — The solution of a particular case of this problem will be illustrated in the following example:



Find the distance from the line $3x - 4y + 15 = 0$ to the point $P_1(-4, 3)$.

Let L_1 in the figure represent the given line and P_1 the given point.

Through P_1, draw L_2 parallel to L_1 and RP_1 perpendicular to L_1. Then RP_1 is the required distance since it is measured *from* the line *to* the point. Draw the perpendiculars $OB = p_2$ to L_2 and $OA = p_1$ to L_1. Then from the figure,

$$RP_1 = OB - OA = p_2 - p_1.$$

The slope of the given line L_1 is $\frac{3}{4}$, whence the equation of the line L_2 through P_1 and parallel to L_1 is by standard equation (6),

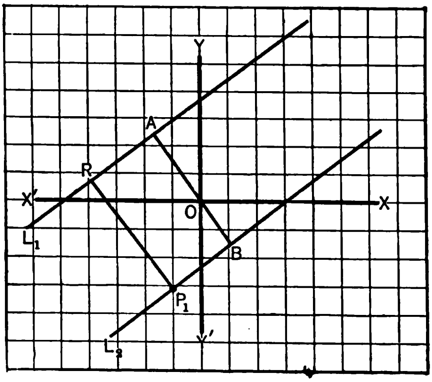
$$y - 3 = \tfrac{3}{4}(x + 4) \quad \text{or} \quad 3x - 4y + 24 = 0.$$

Reducing to normal form, the equations of L_1 and L_2 become respectively,

$$-\tfrac{3}{5}x + \tfrac{4}{5}y - 3 = 0 \text{ and } -\tfrac{3}{5}x + \tfrac{4}{5}y - \tfrac{24}{5} = 0.$$

Whence $p_1 = 3$ and $p_2 = \frac{24}{5}$, and therefore $RP_1 = \frac{24}{5} - 3 = \frac{9}{5}$. It is observed that this result is positive. This checks with the figure as RP_1 has the same direction as OA which is positive. The distance from a line to a point is always positive if the point is above the line, and negative if below the line.

The point and the line may lie on opposite sides of the origin as in the accompanying figure. The same process as that used in the preceding example will lead to the correct result, but care must be taken to give the correct signs to the perpendiculars.



Thus, find the distance of the point $P_1(-1, -3)$ from the line L_1 of the preceding example.

Make the construction as before. The distance required is

$$RP_1 = AB = OB - OA = p_2 - p_1.$$

The equation of L_2 is $3x - 4y - 9 = 0$, or in normal form,

$$-\tfrac{3}{5}x + \tfrac{4}{5}y + \tfrac{9}{5} = 0.$$

Whence $p_2 = -\frac{9}{5}$, $p_1 = 3$, as in Ex. 1.

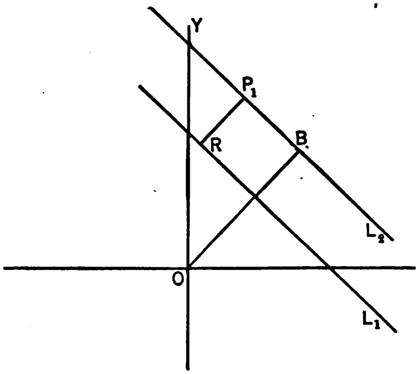
$$RP_1 = p_2 - p_1 = -\tfrac{9}{5} - 3 = -\tfrac{24}{5}.$$

The minus sign indicates that P_1 is below the line L_1. The general formula for the distance from a line to a point will be determined in a manner similar to that used in the examples above.

The distance d, from the line $Ax + By + C = 0$ to the point $P_1\,(x_1, y_1)$, is given by the formula

$$d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}, \tag{11}$$

the sign of the radical being taken the same as that of B.



Proof. Let L_1 represent the given line and P_1 the given point. Through P_1 draw L_2 parallel to L_1 and let p_1 and p_2 represent the perpendicular distances from O to L_1 and L_2 respectively.

Then $d = p_2 - p_1$.

Slope of L_2 = slope of $L_1 = \dfrac{-A}{B}$

Whence the equation of L_2 is

$$y - y_1 = \frac{-A}{B}(x - x_1)$$

or $$Ax + By - (Ax_1 + By_1) = 0.$$

Reducing equations of L_1 and L_2 to normal form,

$$p_1 = \frac{-C}{\pm\sqrt{A^2+B^2}}$$

and
$$p_2 = \frac{Ax_1 + By_1}{\pm\sqrt{A^2+B^2}},$$

where the sign of the radical in each case is the same as that of B.

Hence,
$$d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2+B^2}},$$

the sign of the radical being the same as the sign of B.

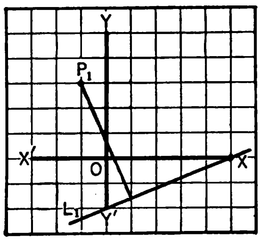
ILLUSTRATIVE EXAMPLE

Find the distance from the line $5x - 12y = 25$ to the point $(-1, 3)$.

Substituting in the formula,

$$d = \frac{-5 - 36 - 25}{-13} = \frac{66}{13}.$$

This positive value of d checks with the figure since it is measured upward from the line.



EXERCISES

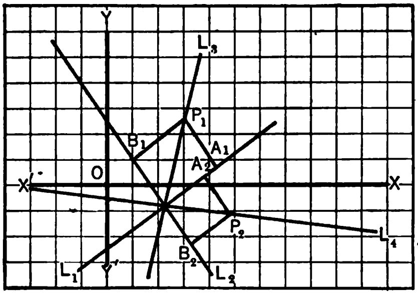
1. Find the distance from the line $3x + 4y = 5$ to the point $(-1, -1)$.

2. Find the distance from the line $5x + 12y = 13$ to the point of intersection of the lines $y - x + 1 = 0$ and $2y - x = 1$.

3. Find the altitudes of the triangle whose vertices are $(1, 1)$, $(-3, 4)$, and $(-3, -2)$.

27. Bisectors of the angles between two lines. — Since the bisector of an angle is the locus of a point which moves so as to be numerically equidistant from the sides, the equation may always be readily found as in the following example.

Find the equations of the bisectors of the angles between the lines $3x - 4y = 10$ and $4x + 3y = 7$.



Draw the given lines L_1 and L_2. There are two bisectors L_3 and L_4.

Let P_1 (x_1, y_1) be any point on the bisector L_3.

Then the perpendiculars A_1P_1 and B_1P_1 are equal in length. They are each positive, being measured upwards from L_1 and L_2 respectively.

Then

$$A_1P_1 = B_1P_1.$$

$$A_1P_1 = \frac{3x_1 - 4y_1 - 10}{-5}, \text{ formula (11)},$$

and

$$B_1P_1 = \frac{4x_1 + 3y_1 - 7}{5}.$$

Therefore,

$$\frac{3x_1 - 4y_1 - 10}{-5} = \frac{4x_1 + 3y_1 - 7}{5}.$$

Since P_1 (x_1, y_1) was taken as any point on the line L_3, the subscripts may be dropped and the equation of L_3 is $7x - y - 17 = 0$.

Similarly, let P_2 (x_2, y_2) be any point on L_4. Then $A_2P_2 = -B_2P_2$, since A_2P_2 is negative and B_2P_2 is positive.

Then

$$\frac{3x_2 - 4y_2 - 10}{-5} = -\frac{4x_2 + 3y_2 - 7}{5}.$$

Hence the equation of L_4 is $x + 7y + 3 = 0$.

EXERCISES

1. Find the equations of the bisectors of the angles between the lines $4x + 3y = 6$ and $4x - 3y = 6$ and show that they are perpendicular.

2. A line is drawn through $(0, 0)$ perpendicular to $3x + 4y = 6$. Find the equations of the bisectors of the angles between these two lines.

MISCELLANEOUS EXERCISES

1. How far from the origin is the line through (1, 6) parallel to $y + 4x = 7$? *Ans.* $10/\sqrt{17}$.

2. Show that α is the same for all parallel lines. Find the equation of a line parallel to $3x + 4y = 25$ and nearer the origin by two units. *Ans.* $3x + 4y = 15$.

3. Find the equations of the lines halfway between the parallels:

(*a*) $4x - 3y = 15$, $4x - 3y = -15$.   (*b*) $5x + 12y = 13$, $5x + 12y + 39 = 0$.

4. Find the equation of a line parallel to $12x - 5y + 13 = 0$:

(*a*) at a distance of 3 from it;
(*b*) at a distance of -3 from it.

5. Find the equation of a line with slope 2 at a distance of 5 units from the origin.

6. Find the distance from the line to the point in the following examples constructing a figure for each:

(*a*) $4x - 3y + 6 = 0$ to $(2, 1)$.   (*c*) $5x - 12y + 6 = 0$ to $(3, 4)$.
(*b*) $3x + 4y - 5 = 0$ to $(-1, -5)$.   (*d*) $6x + 2y + 7 = 0$ to $(-1, 5)$.

7. Find the area of the triangle whose vertices are the points $(3, -2)$, $(4, 3)$, and $(-2, 1)$ by finding the lengths of a side and the corresponding altitude.

8. Find the altitudes of the triangle formed by the lines $y + x = 3$, $y - 5x = 9$, and $y = -1$.

9. Find the point which is equidistant from the points (1, 3) and (5, 5) and is at a distance of 2 from the line $3x + 4y - 10 = 0$.

10. Find the equations of the bisectors of the angles of the following triangles and prove that these bisectors meet in a point, the equations of the sides being:

(*a*) $3x - 4y = 12$, $4x + 3y = 12$, $3x + 4y + 12 = 0$.
(*b*) $5x - 12y = 24$, $12x + 5y = 24$, $5y - 12x = 20$.
(*c*) $y = 4$, $x = -4$, $3x - 4y = 4$.

11. A triangle has sides $4x + 3y = 24$, $3y - 4x = 24$, and $y = -4$. Prove that

(*a*) the triangle is isosceles;

(*b*) the bisector of the exterior angle formed by the first two sides is parallel to the third side.

12. Given the triangle whose sides are $4x + 3y = 24$, $4x - 3y + 12 = 0$ and $y + 4 = 0$. Prove that the bisector of the angle formed by the first two lines divides the opposite side into segments which are proportional to the sides adjacent to the angle.

13. Find the locus of all points which are twice as far from $3x - 4y + 12 = 0$ as from $5x - 12y = 30$.

14. Find the distances between the parallel lines

(*a*) $3x + 2y = 13$ and $3x + 2y + 26 = 0$.

(*b*) $x + 2y = 5$ and $x + 2y + 10 = 0$.

28. The angle which a line makes with another line. — In Art. 7, the ang e which one straight line makes with another was defined as the angle less than 180° measured counter-clockwise from the second to the first.

Thus, in both figures 1 and 2, θ is the angle which L_1 makes with L_2.

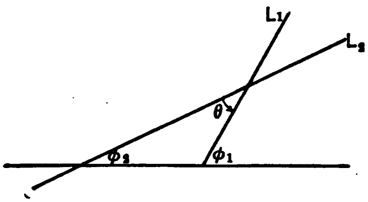


Fig. 1

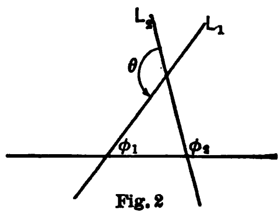


Fig. 2

If m_1 and m_2 are he slopes of two lines and θ is the angle which the first line makes with the second,

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}. \qquad (12)$$

Proof. — Let ϕ_1 and ϕ_2 represent the inclinations of the two lines L_1 and L_2 respectively, then $\tan\phi_1 = m_1$ and $\tan\phi_2 = m_2$.

In Fig. 1, $\phi_1 = \phi_2 + \theta$. (Why?)

Whence $\theta = \phi_1 - \phi_2$,

therefore $\tan\theta = \tan(\phi_1 - \phi_2) = \dfrac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \tan\phi_2}$.

But $\tan\phi_1 = m_1$ and $\tan\phi_2 = m_2$;

therefore $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$.

In Fig. 2, $\phi_2 = \phi_1 + (180° - \theta)$.

Whence $\theta = 180° + (\phi_1 - \phi_2)$,

and therefore

$$\tan\theta = \tan[180° + (\phi_1 - \phi_2)] = \tan(\phi_1 - \phi_2) = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

The student should not fail to fix in mind that the angle θ is always measured *from* the second line *to* the first.

ILLUSTRATIVE EXAMPLES

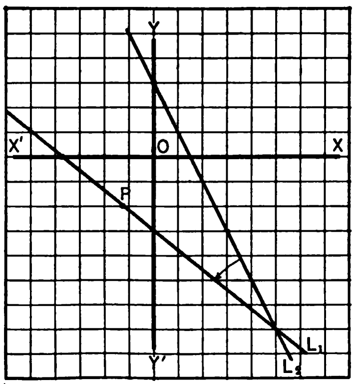
1. Find the angle which the line $y - 3x + 2 = 0$ makes with the line $2y - x = 0$.

Reducing each equation to slope form, $m_1 = 3$, $m_2 = \frac{1}{2}$.

Substituting in formula (12), $\tan\theta = \dfrac{3 - \frac{1}{2}}{1 + \frac{3}{2}} = 1$.

Therefore $\theta = 45°$.

2. Find the equation of the line through $(-1, -2)$ making the angle $\tan^{-1}\frac{1}{2}$ with the line $2x + y - 3 = 0$.



The facts given are sufficient to determine the slope of the line.

In the formula

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2},$$

if any two quantities are known the third may be found. Here $\tan\theta = \frac{1}{2}$. Since the angle $(\tan^{-1}\frac{1}{2})$ is to be measured *from* the given line *to* the required line, and the slope of the given line is -2, therefore $m_2 = -2$.

Substituting in formula (12),

$$\frac{1}{2} = \frac{m_1 + 2}{1 - 2m_1}, \quad \text{whence} \quad m_1 = -\frac{3}{4}.$$

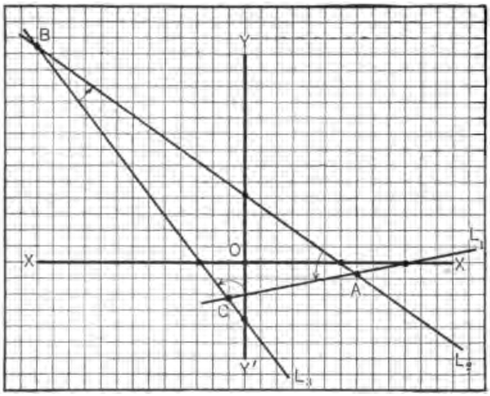
The equation of L_1 may then be written by substituting in standard equation (6) and is

$$y + 2 = -\tfrac{3}{4}(x + 1)$$

or

$$3x + 4y + 11 = 0.$$

3. Compute the angles of the triangle formed by the intersection of the lines whose equations are $x - 5y = 10$, $2x + 3y = 12$, and $11x + 10y + 33 = 0$.



Since the angles must always be taken in a counter-clockwise direction, the angle A is measured from L_2 to L_1, B from L_3 to L_2 and C from L_1 to L_3.

The slopes of L_1, L_2, and L_3 are respectively $\frac{1}{5}$, $-\frac{2}{3}$, and $-\frac{11}{10}$.

Substituting in the formula

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2},$$

being careful to use for m_2 the slope of the line *from* which the angle is measured, the results are:

$$\tan A = \frac{\frac{1}{5} + \frac{2}{3}}{1 - \frac{2}{15}} = 1. \qquad \text{Therefore } A = 45°.$$

$$\tan B = \frac{-\frac{2}{3} + \frac{11}{10}}{1 + \frac{22}{30}} = \frac{1}{4}. \qquad \text{Therefore } B = \tan^{-1}\tfrac{1}{4}.$$

$$\tan C = \frac{-\frac{11}{10} - \frac{1}{5}}{1 - \frac{11}{50}} = -\frac{5}{3}. \qquad \text{Therefore } C = \tan^{-1}\left(-\tfrac{5}{3}\right).$$

EXERCISES

1. Find the tangent of the angle which the first line makes with the second in the following:

(*a*) $3x - y + 4 = 0$ and $2x + 4y + 5 = 0$.
(*b*) $y - 2x = 3$ and $y + 3x = 5$. *Ans.* -1.
(*c*) $y - 2 = 0$ and $x - y = 1$.
(*d*) $x + 2y = 3$ and $y - 2x = 4$.
(*e*) $ax + by = ab$ and $ax - by = ab$.

2. Find the equation of the line passing through the point $(-2, -4)$ and making an angle $\tan^{-1} \frac{3}{4}$ with the line $y - 2x = 7$. *Ans.* $11x + 2y + 30 = 0$.

3. Find the equation of the line through the origin making an angle of 120° with the line $y = x/\sqrt{3}$. Prove that the x-axis bisects the angle between these lines.

4. Find equation of the line through $(-2, -1)$ making an angle of 135° with the line $x - y = 2$. *Ans.* $y = -1$.

5. A right isosceles triangle has the extremities of the hypotenuse at the points $(1, 2)$ and $(-3, 4)$ and the vertex of the right angle below the line joining the points. Find the equations of the three sides.

6. Find the interior angles of the triangle whose sides are the lines $x + y = 2$, $y - x = 2$, and $y - 2x = 4$.

7. Given the triangle formed by the three lines $x - 2y = 2$, $x + 3y + 3 = 0$, and $y = 2$. Prove that the exterior angle formed by the first two lines is equal to the sum of the two opposite interior angles.

8. The vertex of an isosceles triangle is $(2, 3)$ and the base is along the line $x + y = 0$. Given that the vertex angle is 120°, find the equations of the other two sides.

9. Prove that the line through the origin which makes the angle $(\tan^{-1} \frac{1}{3})$ with the line $2y = x + 3$ bisects the angle between the coordinate axes.

10. Two opposite vertices of a rhombus are $(2, 1)$ and $(-2, -3)$. Find the equations of the sides if the interior angles at these vertices are 60°.

11. Two opposite vertices of a square are $(2, 3)$ and $(-1, -3)$. Find the equations of the sides.

12. The base of an equilateral triangle lies in the line $y - 3x = 6$. The opposite vertex is at $(4, 3)$. Find the equations of the other two sides.

29. Systems of straight lines. — Each of the standard equations of a straight line contains certain arbitrary constants. An arbitrary constant is represented by a letter to which different values may be assigned. For example, in the equation $y = mx + b$, m and b are arbitrary constants.

Often a fixed value is given to one of these constants while the other is left arbitrary. Thus, in the equation $y = 2x + b$, b is an arbitrary constant and by assigning different values to b, an infinite number of lines is obtained each having slope 2. Such a set of lines is called a **system of lines,** and the arbitrary constant is called the **parameter** of the system. Other illustrations are $y = mx + 2$ which represents a system of straight lines whose y-intercept is 2, and $x \cos \alpha + y \sin \alpha = 5$ which represents a system of lines each five units from the origin.

The equation of a straight line can always be found at once if the two facts determining the line give the values of the arbitrary constants in one of the standard equations. Thus, if slope $= 2$ and y-intercept $= 3$, the equation of the line, by substituting in equation (7), is $y = 2x + 3$. When the two conditions determining the line do not give the values of the arbitrary constants in any one standard equation, either of two methods may be used.

First, from the data given, find the values of the arbitrary constants in some standard equation, then substitute these constants in that equation; or second, write the equation of the system of lines satisfying one of the conditions given. This equation will contain one parameter. Determine the value of this parameter from the other condition.

ILLUSTRATIVE EXAMPLE

Find the equation of the straight line whose slope is $-\frac{3}{4}$ and which is 3 units from the origin.

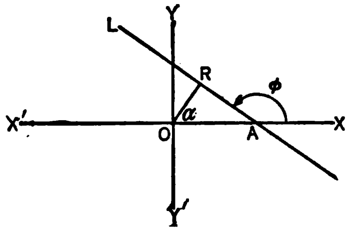
It is given that the line L is 3 units from the origin, with inclination $\phi = \tan^{-1}(-\frac{3}{4})$. The problem will be first worked by finding the constants for substitution in equation (10).

Let α represent the inclination of the perpendicular from the origin to the line. Then $\phi = \alpha + 90°$ and $\tan\phi = \tan(\alpha + 90°) = -\cot\alpha = -\frac{3}{4}$. Therefore $\tan\alpha = \frac{4}{3}$, $\sin\alpha = \frac{4}{5}$, and $\cos\alpha = \frac{3}{5}$.

Substituting in equation (10), the equation of L is

$$\tfrac{3}{5}x + \tfrac{4}{5}y - 3 = 0$$

or $$3x + 4y = 15.$$



The problem might also be worked by finding the coördinates of the point A and substituting in equation (6).

In the triangle ARO, $OA = 3\csc(180° - \phi) = 3\csc\phi = 5$.

Substituting in (6), the equation of L is $y - 0 = -\frac{3}{4}(x - 5)$ or $3x + 4y = 15$.

The second method of solving problems of this class is to first write the equation of the system of lines satisfying one of the given conditions.

Thus, in the equation

$$y = -\tfrac{3}{4}x + b, \tag{1}$$

which represents the system of lines with slope $-\frac{3}{4}$, the parameter b must be so determined that the line shall be 3 units from the origin. Reducing equation (1) to normal form, $\frac{3}{5}x + \frac{4}{5}y - \frac{4}{5}b = 0$.

Whence, $\frac{4}{5}b = 3$, or $b = \frac{15}{4}$, and the equation of L is found to be $3x + 4y = 15$.

Another application of the second method is to use

$$x\cos\alpha + y\sin\alpha - 3 = 0, \tag{2}$$

which represents the system of lines 3 units from the origin. Here α must be so determined that the slope of the line shall be $-\frac{3}{4}$. Reducing (2) to slope form, $y = -\cot\alpha\, x + \dfrac{3}{\sin\alpha}$, whence, $-\cot\alpha = -\frac{3}{4}$, $\sin\alpha = \frac{4}{5}$, and $\cos\alpha = \frac{3}{5}$. Substituting in (2), the required equation is found to be $3x + 4y = 15$.

EXERCISES

1. Write the equations of the systems of straight lines which satisfy the following conditions:

(*a*) distance from the origin $= 5$. *Ans.* $x\cos\alpha + y\sin\alpha = 5$.

(*b*) x-intercept $= 3$.

(*c*) slope $= 5$.

(*d*) passes through $(1, 4)$.

(*e*) inclination of perpendicular from origin to line is 60°.
(*f*) slope of perpendicular from origin to line is $\frac{3}{4}$.
(*g*) sum of intercepts on axes = 6.

2. Write the equation of the system of lines 3 units from the origin and so determine α that a line of the system shall pass through the point (2, 3).

3. Find the equation of the line with slope 2 and which in addition satisfies the following condition:

(*a*) distance from the origin = 5.
(*b*) x-intercept = 5.
(*c*) sum of intercepts = 6.
(*d*) distance from the origin = -3.

4. Find the equation of the straight line through $(-4, -2)$ and satisfying in addition the following condition:

(*a*) distance from origin $= \sqrt{10}$.
(*b*) parallel to $2x - 5y = 6$.
(*c*) sum of intercepts = 3.
(*d*) inclination of perpendicular from origin to line = 45°.
(*e*) portion of line intercepted by axes is bisected by given point.

5. Find the equation of the straight line 4 units from the origin and satisfying in addition the following condition:

(*a*) perpendicular to the line $2x - y = 3$.
(*b*) through the point (2, 4).
(*c*) y-intercept = 5.
(*d*) product of intercepts = 32.

6. A line through the point (3, 1) intersects the x- and y-axes at A and B respectively. The line AB is divided by the point in the ratio $\frac{1}{3}$. Find its equation.

7. Find the equations of the two lines through (1, 4) and making the product of the intercepts 18.

8. Find the equation of the line through $(-3, -4)$ and making the y-intercept twice the x-intercept.

9. Find the equations of the two lines in which the inclination of the perpendicular from the origin on the line is 45° and the product of the intercepts 8.

MISCELLANEOUS EXAMPLES ON CHAPTER III

1. The equations of two sides of a parallelogram are $2x - y = 3$ and $3x + 2y = 1$. Find the equations of the other two sides if they intersect at (2, 5).

2. Find the equations of the lines through the point $(-1, -2)$ trisecting that portion of the line $2y + 6 = x$ which is intercepted between the axes.

3. A tangent to a circle with center $(-3, 5)$ is $3x - 4y - 6 = 0$. Find the length of the radius. *Ans.* 7.

4. One side of an equilateral triangle has its extremities at $(3, -4)$ and $(3, 2)$. Find the equations of the other sides.

5. The line joining A $(1, 3)$ and B $(3, 0)$ is cut by the line $y - x + 8 = 0$. In what ratio does the point of intersection divide AB? *Ans.* $(-2/1)$.

6. Find the center and radius of the circle circumscribing the triangle whose vertices are $(0, 2)$, $(3, 3)$, and $(6, 6)$. *Ans.* Center $(-1, 10)$, $r = \sqrt{65}$.

7. Find the center and radius of the circle inscribed in the triangle the equations of whose sides are $3x + 4y = 6$, $4y - 3x = 6$, and $y = -2$. *Ans.* Center $(0, -\frac{4}{9})$, $r = \frac{14}{9}$.

8. An isosceles right triangle is constructed with its hypotenuse along $4x - 2y = 3$ and the vertex of its right angle at $(-1, 3)$. Find the equations of the equal sides and the coördinates of the other vertices.

9. Find the equations of the following loci. Prove that they are straight lines and construct the lines.

(*a*) A point moves so as to be always equidistant from the points $(-1, 2)$ and $(3, 4)$. *Ans.* $2x + y = 5$.

(*b*) A point moves so that the sum of its distances from $y - 2 = 0$ and $5x + 12y - 26 = 0$ is equal to 7.

(*c*) A point moves so that its distance from the line $3x + 4y - 6 = 0$ is one-half its distance from the line $5x - 12y + 13 = 0$.

(*d*) A point moves so that the square of its distance from $(-2, 3)$ minus the square of its distance from $(1, 4)$ is equal to 10.

10. A point moves so that five times its distance from the x-axis is three times its distance from the origin. Find the equation of the locus and prove it represents a pair of straight lines.

11. The base of an isosceles triangle is the line joining $(-3, 2)$ and $(4, 3)$. Its vertex is on the line $y = -2$. Find the equations of its sides.

12. Show that $6x^2 + 5xy - 6y^2 - x + 5y - 1 = 0$ represents a pair of perpendicular lines.

13. The sides of a quadrilateral are given by the equations $x^2 + 4xy + 4y^2 + 3x + 6y = 0$ and $y^2 + y - 6 = 0$. Prove that the figure is a parallelogram.

14. Prove analytically that the perpendiculars drawn from the vertices of any triangle to the opposite sides meet in a point.

15. Find what relation must hold among the coefficients in the general equation of a line $Ax + By + C = 0$ in order that

(*a*) the x-intercept shall $= 3$.
(*b*) the given line shall be perpendicular to $2x + 3y = 5$.
(*c*) the slope shall $= 5$.
(*d*) the perpendicular from the origin to the line shall $= 5$.
(*e*) the line shall be parallel to the x-axis.
(*f*) the line shall pass through the point $(3, 5)$.

16. Write the equation of the set of lines through the point of intersection of the two lines $3x + 2y + 8 = 0$ and $x - 3y = 1$ and so determine the parameter of the system that the line shall pass through the point $(1, 2)$.

17. Prove that the two lines whose equations are $xy + 2x - 4y - 8 = 0$ are the bisectors of the angles between lines whose equations are $x^2 - y^2 - 8x - 4y + 12 = 0$.

18. Find the equation of the line perpendicular to the line $2x + 3y - 12 = 0$ and bisecting the portion of the line intercepted by the axes.

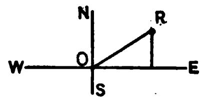
19. A vertical section of a weir (or dam) is in the form of an isosceles trapezoid, one of the parallel sides being 100 feet long and lying in the surface of the water. The other parallel side is 60 feet long and the height of the weir is 15 feet. Taking the origin at the mid-point of the 100 foot side, find the equation of each of the non-parallel sides.

20. In a right triangle the sides are 6, 8, and 10. Find the equations of the sides if the x-axis is parallel to the hypotenuse and the origin is at the center of gravity.

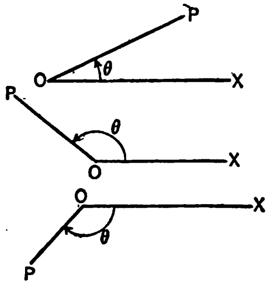
CHAPTER IV

POLAR COÖRDINATES

30. Definition. — A second method of locating a point in a plane is by means of **polar coördinates.** These often lead to simpler results than those obtained by rectangular coördinates. A comparison of the two systems of coördinates is shown by the following illustration. If in a country where roads follow section lines, the question were asked how to reach R from O, the answer would be of the form, go 4 miles east and 3 miles north. If the same question were asked in an open country, the *direction* would probably be pointed out and the questioner told to go 5 miles in that direction. The first is an illustration of rectangular and the second of polar coördinates.



In order to locate a point in any system of coördinates, two fixed things are necessary. In rectangular coördinates these are two intersecting perpendicular lines. In the polar system a fixed directed straight line called the **polar axis** or **initial line** and a fixed point on that line called the **pole** or **origin** are given.



In the figure, OX is the polar axis or initial line and O the pole or the origin.

The line OP from the pole to the point is called the **radius vector** and is represented by ρ. The angle which OP makes with the polar axis is called the **vectorial angle** and is represented by θ. In the figure, $OP = \rho =$ radius vector, $XOP =$

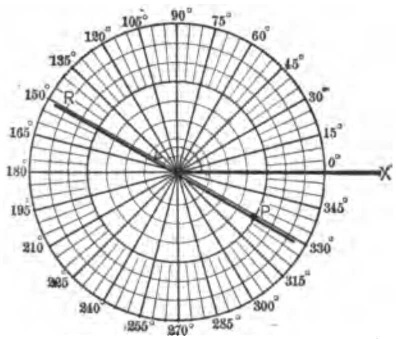
θ = vectorial angle. These two quantities are called the polar coördinates of the point and the point is represented by $P(\rho, \theta)$. The radius vector is positive when measured on the terminal line of the angle and negative when measured on that line produced through the origin. The vectorial angle is positive when measured counter-clockwise and negative when measured in clockwise direction. As in trigonometry the angle θ may have an unlimited number of values differing by 2π, since it is any angle whose initial line is OX and whose terminal line is OP.

The position of a point in a plane is definitely determined if its polar coördinates are given. The same point may, however, be expressed in many different ways. Thus, in the first figure above, if the least value of $\theta = 30°$ and $\rho = 5$, then P may be written $(5, 30°)$ $(5, -330°)$, $(-5, 210°)$, $(-5, -150°)$, $(5, 390°)$, etc.

The steps in plotting a point P in polar coördinates are as follows:

From the polar axis OX construct an angle equal to θ.

If ρ is positive, lay off $OP = \rho$ on the terminal line of the angle. If ρ is negative, produce the terminal line through O and lay off on it OP equal to the numerical value of ρ.



Thus, locate the point $P(-5, 150°)$ The angle $XOR = 150°$ is first constructed in a positive direction from OX. Since ρ is negative, the terminal line of the angle is produced through O to P making OP 5 units in length. P then represents the point $(-5, 150°)$. Show that $(-5, -210°)$ represents the same point.

EXERCISES

1. Plot the points $(-3, 30°)$, $(3, -150°)$, $(-5, 180°)$, $(-2, \frac{3}{4}\pi)$, $(-3, -\frac{2}{3}\pi)$, $(-1, 330°)$.

2. Write three other pairs of coördinates of each of the points $(-3, 20°)$, $(2, \frac{2}{3}\pi)$, $(-4, 240°)$, $(3, 330°)$.

3. A side of a square is 3 inches. A diagonal is taken as the polar axis and one extremity of that diagonal as pole. Find the coördinates of the vertices.

4. Each side of a rhombus is 4 inches. One side is on the polar axis and a vertex is at the pole. Find the coördinates of the vertices if the angle at the pole is 60°.

5. Prove that the three points $(0, 0)$, $(3, 30°)$, and $(3, -30°)$ are the vertices of an equilateral triangle.

6. Show that $(2, 30°)$ and $(2, -30°)$ are symmetrical with respect to the polar axis, that $(2, 30°)$ and $(-2, 30°)$ are symmetrical with respect to the pole and that $(2, 30°)$ and $(2, 150°)$ are symmetrical with respect to a perpendicular to the polar axis through the pole.

7. What point is symmetrical to $(4, -30°)$

(*a*) with respect to the polar axis?
(*b*) with respect to the pole?
(*c*) with respect to the perpendicular to the polar axis through the pole?

8. What point is symmetrical to (ρ, θ)

(*a*) with respect to the polar axis?
(*b*) with respect to the pole?
(*c*) with respect to the perpendicular to the polar axis through the pole?

9. Where do the points lie

(*a*) for which $\theta = 45°$?
(*b*) for which $\theta = 0$?
(*c*) for which $\rho = 5$?
(*d*) for which $\rho = 0$?

10. Find the distance between the points $(2, 30°)$ and $(-3, 150°)$.

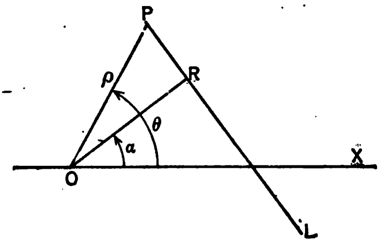
Hint. — Use law of cosines in trigonometry. *Ans.* $\sqrt{7}$.

11. If θ is a positive angle less than 360°, in how many ways can the following points be expressed:

(*a*) $(3, 30)$?
(*b*) $(-3, 240°)$?
(*c*) the pole?

31. The equation of a locus: polar coördinates. — The definition of the equation of a locus in polar coördinates is the same as that given in Art. 10, and the steps in finding the equation are identical to those stated in that article except that ρ and θ are used instead of x and y.

Thus, find the equation of a line such that the perpendicular from the pole upon it is p and the angle which the perpendicular makes with the polar axis is α.



1*st*. Given the line L such that $OR = p$ and $XOR = \alpha$.

Let $P(\rho, \theta)$ represent any point on the line.

2*nd*. $\text{Cos } ROP = \dfrac{OR}{OP}$, from trigonometry.

3*rd*. $\text{Cos } (\theta - \alpha) = p/\rho$,

4*th*. Clearing of fractions, the required equation is

$$\rho \cos (\theta - \alpha) = p.$$

EXERCISES

1. Prove that the equation of a line

(*a*) perpendicular to the polar axis and at a distance of four units to the right of the pole is $\rho \cos \theta = 4$.

(*b*) parallel to the polar axis and two units above it is $\rho \sin \theta = 2$.

2. Prove that the equation of a line through the pole with inclination $\pi/6$ is $\theta = \pi/6$.

3. Prove that the equation of the circle with center at the pole and radius 5 is $\rho = 5$.

4. Prove that the equation of the circle which passes through the pole and has its center on the polar axis a units to the right of the origin is $\rho = 2\, a \cos \theta$.

5. Prove that the equation of the circle which passes through the pole and has its center on the perpendicular to the polar axis through the pole and b units above it is $\rho = 2\, b \sin \theta$.

32. The locus of an equation: polar coördinates. — It is required to find a locus which contains all the points whose coördinates (ρ, θ) satisfy the equation and which contains no other points.

As in the case of rectangular coördinates, this can always be done by assigning values to one variable and finding the values of the other, then plotting the points and connecting by a smooth curve. It was found in that case, however, that the work was greatly facilitated by combining with the plotting a certain amount of discussion. The same is true in the case of polar coördinates.

The points in discussion which are particularly helpful are:

1. Intercepts on the polar axis.
2. Symmetry.
3. Extent.

Intercepts. — Placing $\theta = 0$ and solving for ρ, points are found at which the curve intersects the polar axis. Other intersections may be found by letting $\theta = 180°$, $360°$, etc., and finding the corresponding values of ρ.

The coördinates of the pole are $\rho = 0$, $\theta =$ any angle. Even though the pole is on the curve not all such values satisfy the equation. Placing $\rho = 0$, and solving for θ the particular angles are determined at which the curve passes through the origin.

Thus, in the equation $\rho^2 = a^2 \cos 2\theta$, if $\theta = 0$, $\rho = \pm a$. Two points on the polar axis are thus located. If $\theta = 180°$, $360°$, etc., no new points are found on the initial line. Placing $\rho = 0$, θ is found to be $45°$, $135°$, $225°$, and $315°$, which shows the pole is on the locus.

Symmetry. — The tests for symmetry ordinarily used, correspond closely to those of rectangular coördinates. It can be shown that in polar coördinates a curve is symmetrical with respect to (*a*) the polar axis if θ can be replaced by $-\theta$ without changing the equation. Why?

(*b*) the perpendicular to the polar axis through the pole if θ can be replaced by $\pi - \theta$ without changing the equation. Why?

(*c*) the pole if ρ can be replaced by $-\rho$ or θ by $\pi+\theta$ without changing the equation. Why?

In general, the test for symmetry with respect to the polar axis will be the only one used. This is of particular practical importance, since any part of the curve determined by giving θ values from 0° to 180° can be reproduced from 0° to $-180°$ by the principle of symmetry. Points should be plotted until it is certain that any further points found are the same as those obtained by symmetry.

While the above tests (*a*), (*b*), and (*c*) are universally true, their converse does not necessarily hold. A curve may, for example, be symmetrical with respect to the polar axis even though the equation is changed when θ is replaced by $-\theta$. This point is discussed in Art. 34.

The equation $\rho^2 = a^2 \cos 2\theta$ stands all the tests of symmetry mentioned in this article and hence the locus is symmetrical with respect to the perpendicular to the polar axis through the pole, to the pole and also to the polar axis.

Extent. — Under this head will be considered:

values of θ which make ρ imaginary;

values of θ which make ρ infinite;

values of θ which make ρ a maximum or minimum numerical value.

In those problems in which ρ enters the equation in even degree, it is possible that certain values of θ may make ρ imaginary. Such values of θ are excluded.

In some examples, there are values of θ which make ρ infinite. Such values are important as they show that the curve extends to infinity in that direction. In such cases it is well to determine values of ρ corresponding to values of θ a little less and a little greater than those which render $\rho = \infty$, as important changes often take place in the vicinity of such points.

Other important values of θ are those which give to ρ its maximum or minimum numerical values.

Consider again $\rho^2 = a^2 \cos 2\theta$ or $\rho = \pm a\sqrt{\cos 2\theta}$.

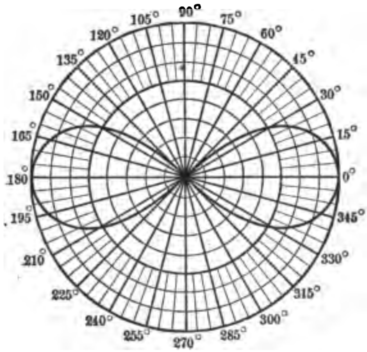
It is seen that values of θ between 45° and 135°, also between 225° and 315°, make ρ imaginary, and therefore these values of θ are excluded. There are no values of θ which make $\rho = \infty$, hence this curve has no infinite branch.

The greatest value of ρ corresponds to $\theta = 0°$ or 180° for which $\cos 2\theta = 1$ and $\rho = \pm a$.

It has already been shown that the curve passes through the pole, hence the least numerical value of ρ is 0.

Taking into account symmetry and excluded values of θ, the curve can be completely drawn by assigning to θ values from 0° to 45°. This curve is called the **lemniscate.**

θ	2θ	$\cos 2\theta$	ρ
Degrees	Degrees		
0	0	1	$\pm a$
15	30	.86	$\pm .93\,a$
30	60	.5	$\pm .7\,a$
45	90	0	0



If θ in the equation has coefficient unity, it is usually sufficient in plotting to take values of θ differing by 30°. If θ has an integer coefficient as in this problem, smaller intervals should be used, and when θ has a fractional coefficient, it is often sufficient to take much larger intervals between the values of θ.

In plotting curves, the student is advised to use polar coördinate paper. Such paper is usually accompanied by tables which facilitate the calculation.

ILLUSTRATIVE EXAMPLES

1. Plot and discuss $\rho = \dfrac{6}{1 - 2\cos\theta}$.

Intercepts. — If $\theta = 0°$, $\rho = -6$. If $\theta = 180°$, $\rho = 2$. $\theta = 360°$, $540°$, etc., give no additional values to ρ. No values of θ make $\rho = 0$. Hence the curve crosses the polar axis in two points only, one point 6 units to the left, and the other 2 units to the left, of the pole.

Symmetry. — The equation is unchanged if θ is replaced by $-\theta$, hence the curve is symmetrical with respect to the polar axis.

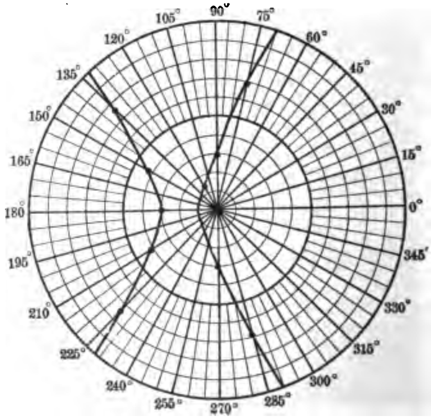
Extent. — There are no excluded values of θ, since the value of ρ contains no radical.

When $1 - 2\cos\theta = 0$ or $\cos\theta = \frac{1}{2}$, $\rho = \infty$, therefore the curve has infinite branches corresponding to $\theta = 60°$ and $300°$.

ρ will have the least value when $1 - 2\cos\theta$ is greatest, which will be when $\cos\theta = -1$. Then $\rho = 2$ is the minimum numerical value.

A table of values is here given in which the natural values of $\cos\theta$ are used. The figure proves to be an hyperbola.

θ	Cos θ	ρ
Degrees		
0	1	− 6
30	.866	− 8.2
45	.707	−14.5
60	.5	∞
75	.259	+12.4
90	0	6
120	− .5	3
135	− .707	2.5
150	− .866	2.2
180	−1	2



If values of θ greater than 180° were used, the same points would be obtained as those determined by applying the principle of symmetry with respect to the initial line.

2. Plot and discuss the locus of $\rho = a \cos^3 \frac{\theta}{3}$.

Intercepts. — If $\theta = 0$, $\rho = a$; if $\theta = 180°$, $\rho = a/8$; if $\theta = 360°$, $\rho = -a/8$; if $\theta = 540°$, $\rho = -a$. If $\rho = 0$, $\theta = 270°$.

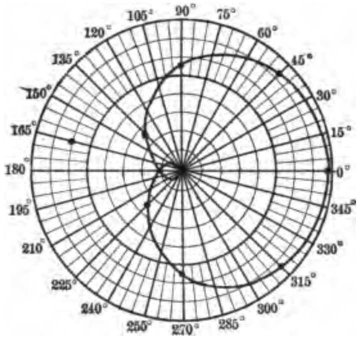
Symmetry. — Since θ can be replaced by $-\theta$ without changing the equation, the curve is symmetrical with respect to the polar axis.

Extent. — There are no excluded values of θ. ρ is never infinite. It is greatest when $\theta = 0°$, for which value $\rho = a$.

Making a table of values and plotting, the figure is found to be as shown below, a being taken as 8.

It should be noticed that in this curve it is not sufficient to plot from 0° to 180° but that points up to $\theta = 270°$ are necessary before the application of symmetry can be applied to complete the figure.

θ	$\frac{\theta}{3}$	$\cos\frac{\theta}{3}$	ρ
Degrees	Degrees		
0	0	1	a
45	15	.97	.91 a
90	30	.87	.65 a
135	45	.71	.35 a
180	60	.5	.13 a
225	75	.26	.02 a
270	90	0	0



EXERCISES

Discuss and plot the loci of the following:

1. $\rho = 5$.
2. $\rho = 5 \cos \theta$.
3. $\rho = 4 \sin \theta$.
4. $\rho \cos \theta = 4$.
5. $\rho \sin \theta = 4$.
6. $\theta = 10°$.
7. $\rho = \cos (\theta + 45°)$.
8. $\rho^2 = a^2 \sin 2\theta$.
9. $\rho = \dfrac{8}{1 + 2 \cos \theta}$.
10. $\rho^2 \cos 2\theta = a^2$.
11. The parabola $\rho = \dfrac{3}{1 + \sin \theta}$.

12. The cardioid $\rho = a\,(1 + \cos\theta)$.

13. The ellipse $\rho = \dfrac{8}{5 - 3\cos\theta}$.

14. The limaçon $\rho = 4\,(1 - 2\cos\theta)$.

15. $\rho = 4\,(2 - \cos\theta)$.

16. $\rho = a\,(1 + \sin\theta)$.

17. $\rho = a\sin^3\dfrac{\theta}{3}$.

18. $\rho^2 \sin 2\theta = a^2$.

19. $\rho = a\csc^2\dfrac{\theta}{2}$.

20. $\rho = a\sin^2\dfrac{\theta}{2}$.

33. Equations of the form $\boldsymbol{\rho = a \sin k\theta}$ and $\boldsymbol{\rho = a \cos k\theta}$, where k is any integer, are of frequent occurrence. A sketch of these curves sufficiently correct for many purposes can be constructed by making use of the following discussion.

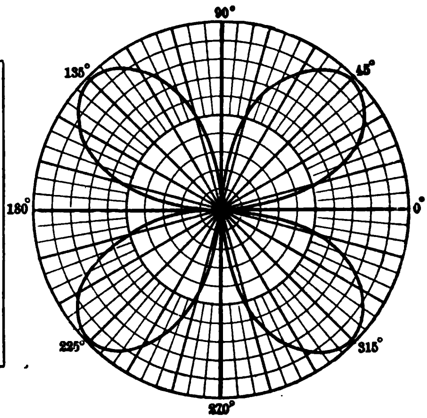
Draw a radial line corresponding to each value of θ which makes $\rho = 0$, also a radial line corresponding to each value of θ which makes ρ a numerical maximum. Discuss the changes which take place in ρ as θ increases through each interval determined by these radial lines.

Thus, plot the locus of $\rho = a\sin 2\theta$.

If $\rho = 0$, $\theta = 0°, 90°, 180°, 270°$.

ρ has a numerical maximum of $\pm a$ when $\theta = 45°, 135°, 225°, 315°$. Therefore radial lines are drawn at intervals of 45° beginning with the polar axis.

As θ increases from	ρ varies from	Quadrant occupied by curve
Degrees		
0 to 45	0 to a	1st
45 to 90	a to 0	1st
90 to 135	0 to $-a$	4th
135 to 180	$-a$ to 0	4th
180 to 225	0 to a	3rd
225 to 270	a to 0	3rd
270 to 315	0 to $-a$	2nd
315 to 360	$-a$ to 0	2nd



The plan here used will often be of advantage in other examples and should be kept in mind for use whenever practicable.

EXERCISES

Construct the following loci:

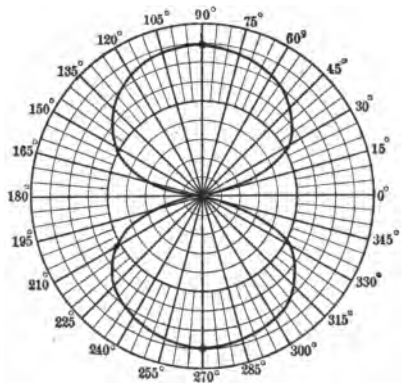
1. $\rho = a \cos 2\theta$.
2. $\rho = 4 \sin 3\theta$.
3. $\rho = a \cos 5\theta$.
4. $\rho = 8 \sin 4\theta$.
5. $\rho = 4 \cos 3\theta$.
6. $\rho = 4 \sin 5\theta$.
7. $\rho = a \cos \theta$.
8. $\rho = 5 \cos 4\theta$.
9. $\rho = 4 \sin 6\theta$.
10. $\rho = a \sin^2 \theta$.

34. Difficulties arising from the multiple representation of points in the polar system. — The fact that the same point may be expressed by more than one pair of coördinates often leads to confusion and sometimes to error unless great care is taken. In the rectangular system, where each point has one pair of coördinates, and each pair of coördinates corresponds to a single point, it is always safe to conclude that if the coördinates of a point fail to satisfy an equation then the point is not on the locus. This is not always the case in the polar system, for it often happens that if one pair of coördinates fails to satisfy an equation, another pair representing the same point may show the point to be on the locus.

Thus, in the equation $\rho = a \sin 2\theta$, if the point be taken whose coördinates are $(a/2, -15°)$ the equation is not satisfied; but the same point when considered as determined by $(-a/2, 165°)$ is found to be on the curve.

Care must be taken to hold this multiple representation of points in mind when considering the question of symmetry. If the curve is symmetrical to the polar axis, then corresponding to every point (ρ, θ) on the curve there must be a point $(\rho, -\theta)$ also on the curve. It has been shown, however, that any point as $(\rho, -\theta)$ may be on the curve even though its coördinates, in that particular form, do not satisfy the equation of the curve, and thus the locus is sometimes symmetrical with respect to the polar axis even though the equation is changed by the substitution of $-\theta$ for θ. This is shown in the case of the curve $\rho = a \sin 2\theta$, drawn in Art. 33, which is found to be symmetrical with respect to the polar axis, even though the usual test for symmetry fails.

Another case where confusion sometimes arises is that of excluded values. It often happens that certain values of θ make ρ imaginary and therefore these values of θ are excluded. It may be, however, that if the set of points corresponding to these values of θ were expressed by other pairs of coördinates, these coördinates would satisfy the equation, showing that the curve is found in that area from which a too hasty conclusion would have excluded it. Thus, in $\rho^2 = 4 \sin \theta$, ρ is imaginary for values of θ between 180° and 360°. This might seem to indicate that there is no part of the curve below the polar axis. In plotting points, however, it is found that for every value of θ in the first and second quadrants, ρ has two values, one positive and the other negative, showing that the curve is found in each of the four quadrants.



Thus, when $\theta = 90°$, $\rho = \pm 2$. The coördinates $(-2, 90°)$ satisfy the equation. Another pair of coördinates for the same point is $(2, 270°)$. These do not satisfy the equation.

EXERCISES

1. Show that the point $(-\frac{1}{2}, 150°)$ is on the curve $\rho = \cos 2\theta$ although its coördinates do not satisfy the equation. How may the given point be written in order that its coördinates shall satisfy the equation?

2. Determine whether the point $(1, 210°)$ is on the curve whose equation is $\rho = 2 \cos 4\theta$.

3. Discuss and plot $\rho^2 = 4 \cos \theta$.

4. Discuss and plot $\rho^2 = \cos 3\theta$.

5. Discuss and plot $\rho^2 = 1 - 2 \sin \theta$.

6. Discuss and plot $\rho = \sin 4\theta$.

35. Spirals. — A spiral is a curve traced by a point which, while it revolves about the pole, continually approaches or recedes from this point.

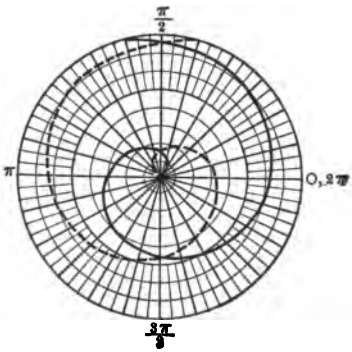
There are five principal spirals as follows:

The spiral of Archimedes, $\rho = a\theta$.
The reciprocal or hyperbolic spiral, $\rho = a/\theta$.
The parabolic spiral, $\rho^2 = a\theta$.
The lituus or trumpet, $\rho^2 = a/\theta$.
The logarithmic spiral $\rho = e^{a\theta}$ $(e = 2.718+)$.

Plot the locus $\rho = a\theta$ (where a is positive).

It is seen that when $\theta = 0$, $\rho = 0$, and as θ increases without limit, ρ also increases without limit. The curve

θ	ρ
0	0
$\pi/2 = 1.57$	$1.57\,a$
$\pi = 3.14$	$3.14\,a$
$3\pi/2 = 4.71$	$4.71\,a$
$2\pi = 6.28$	$6.28\,a$
$5\pi/2 = 7.85$	$7.85\,a$
$3\pi = 9.42$	$9.42\,a$



thus starts at the pole and winds around the pole indefinitely, receding from it with each revolution.

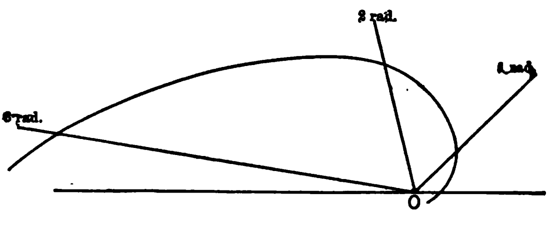
In plotting these curves, θ is expressed in circular measure. It is usually sufficient to determine only such points as correspond to values of θ differing by $\pi/2$ radians. In some examples it is more convenient to take the interval between the successive values of θ to be 1 radian.

The curve sketched in the figure with the heavy line corresponds to positive values of θ and that with the dotted line to negative values of θ. These two spirals constitute the complete locus of the equation.

Plot the locus $\rho = e^{a\theta}$.

Some definite value must be assigned to a. Suppose $a = 1$, the equation becomes $\rho = e^{\theta}$. Assigning to θ values differing by 1 radian, the following table is computed.

θ	ρ	θ	ρ
0	$e^0 = 1$	0	$e^0 = 1$
1	$e = 2.72$	-1	$e^{-1} = .37$
2	$e^2 = 7.39$	-2	$e^{-2} = .14$
3	$e^3 = 20.1$	-3	$e^{-3} = .05$
4	$e^4 = 54.6$	-4	$e^{-4} = .02$



It is seen that as θ increases from 0 radians to 4 radians, ρ increases from 1 to 54.6, also that as θ increases indefinitely, ρ also increases indefinitely.

As θ decreases from 0 to -4 radians, ρ decreases from 1 to .02, and as θ decreases indefinitely, ρ approaches 0 as a limit. Hence the curve winds around the pole indefinitely, coming closer and closer to it with each revolution, but not reaching it until an infinite number of revolutions in clockwise direction have been made.

EXERCISES

1. Plot the spiral $\rho = a/\theta$.
2. Plot the spiral $\rho^2 = a\theta$.
3. Plot the spiral $\rho^2 = a/\theta$.

36. Intersections of curves. — As in rectangular coördinates, if two equations are solved simultaneously, points are found whose coördinates satisfy both equations and hence such points are the intersections of the two loci. In polar coördinates, this process does not always give all the common points, for since the coördinates of a point may be written in a number of different ways, it may happen that one equation is satisfied by one pair of coördinates of the point of intersection, and the other equation by a different pair of coördinates of the same point.

To make sure that all intersections are obtained, the curves should always be drawn. These will show any additional common points. Care must always be taken to make sure whether the pole is on both curves.

ILLUSTRATIVE EXAMPLE

Find the points of intersection of the two curves $\rho = -1 - \cos\theta$ and $\rho = 1 + \cos\theta$.

Equating the two values of ρ,

$$-1 - \cos\theta = 1 + \cos\theta.$$

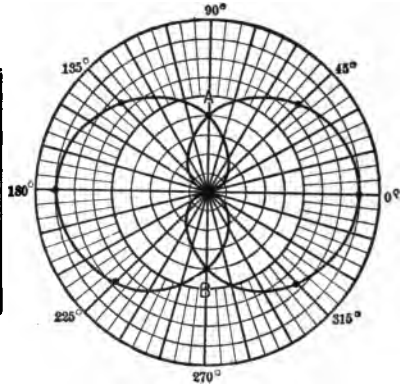
Hence $2\cos\theta = -2$, $\cos\theta = -1$, $\theta = 180°$.

Substituting in either equation, ρ is found to be 0. The pole then is a common point.

Plotting the loci, taking account of symmetry, the figure is as shown below.

$\rho = -1 - \cos\theta$	
θ	ρ
Degrees	
0	-2
45	-1.7
90	-1
135	$-\ .3$
180	0

$\rho = 1 + \cos\theta$	
θ	ρ
Degrees	
0	2
45	1.7
90	1
135	.3
180	0



The point marked A in the figure has coördinates $(1, 90°)$ for the right hand curve and $(-1, -90°)$ for the left hand curve.

The point marked B in the figure has coördinates $(1, -90°)$ for the right hand curve and $(-1, 90°)$ for the left hand curve.

The curves then intersect in three points.

EXERCISES

Find the points of intersection of the following pairs of curves and plot the loci.

1. $\rho = a.$
   $\rho = a\cos\theta.$
2. $\rho^2 = a^2 \sin 2\theta,$
   $\rho = a\sin\theta.$
3. $\rho = 1 + \cos\theta,$
   $2\rho = \sec^2 \frac{1}{2}\theta.$
4. $\rho = \sqrt{2},$
   $\rho = 2\sin\theta.$
5. $\rho = \sin 2\theta,$
   $\rho = \sin\theta.$
6. $\rho = 1 + \sin\theta,$
   $\rho(2 - \sin\theta) = 2.$
7. $\rho = 2\sin 3\theta,$
   $\rho = 2\sin\theta.$ *Ans.* $(0, 0)$, $(\sqrt{2}, 45°)$, $(\sqrt{2}, 135°)$.
8. $\rho^2 = 2a^2\cos 2\theta,$
   $\rho = a.$ *Ans.* $(a, \pm 30°)$, $(a, \pm 150°)$.
9. $\rho^2 = a^2\cos\theta,$
   $\rho = a.$

10. $\rho(3 - 2\cos\theta) = 1,$
$\rho = 1 - \cos\theta.$ *Ans.* $(\frac{1}{2}, 60°), (\frac{1}{2}, 300°).$

11. $\rho = 6 - \cos\theta,$
$\rho(1 - 2\cos\theta) = 6.$

12. $\rho = 2 - 2\sin\theta,$
$\rho = 2\cos 2\theta.$

13. $\rho = a\theta,$
$\rho = a/\theta.$

14. $\rho = 2a\sin\theta\tan\theta,$
$\rho = a\sin\theta.$

15. Show that $\theta = 60°$ and $\rho = a$ intersect in two points.

MISCELLANEOUS EXAMPLES

Discuss and plot the following:

1. $\rho = 2\sec\theta.$
2. $\rho = a\tan^2\theta\sec\theta$ (semi-cubical parabola).
3. $\rho = a^2\sin 4\theta.$
4. $\rho = 2a\sin\theta\tan\theta$ (cissoid).
5. $\rho = a\sec^2\theta.$
6. $\rho = 1 + \sin 2\theta.$
7. $\rho = a(\sin 2\theta + \cos 2\theta).$
8. $\rho = \dfrac{2a\sec\theta}{1 + \tan\theta}.$

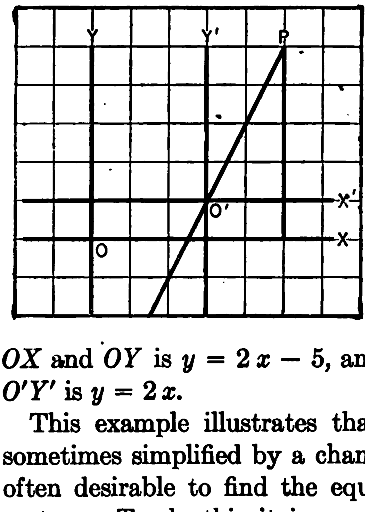
(*Hint.* Change to sine and cosine when calculating for $\theta = 90°$.)

9. $\rho = \dfrac{3a\tan\theta\sec\theta}{1 + \tan^3\theta}$ (folium of Descartes).
10. $\rho^2 = a^2(1 - \cos\theta).$
11. $\rho = 2\sin\theta + \cos\theta.$
12. $\rho^2 = \cos 4\theta.$
13. $\rho^2\cos\theta = a^2\sin 3\theta.$
14. $\rho = 4\sin 5\theta.$
15. $\rho^2 = \dfrac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}.$
16. $\rho = \cos\dfrac{\theta}{2} + \sin\dfrac{\theta}{2}.$

CHAPTER V

TRANSFORMATION OF COÖRDINATES

37. If a point is referred to a given system of axes, its coördinates are fixed. If the axes are changed, the coördinates of the point are also changed. Thus, if the point P when referred to OX and OY is (5, 5), it is seen that if referred to the parallel system, $O'X'$ and $O'Y'$ through O' (3, 1) the coördinates of P are changed to (2, 4). Similarly, the equation of the line $O'P$ when referred to OX and OY is $y = 2x - 5$, and when referred to $O'X'$ and $O'Y'$ is $y = 2x$.



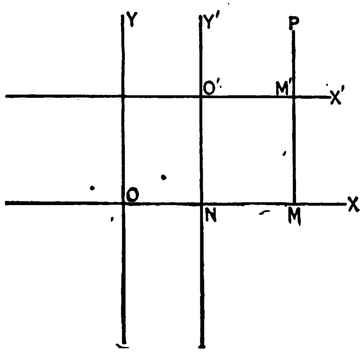
This example illustrates that an equation of a locus is sometimes simplified by a change of axes and it is therefore often desirable to find the equation of the curve in a new system. To do this it is necessary to determine the laws which connect the coördinates of a point in the given system with the coördinates of the same point in the new system.

Transformation of coördinates is the operation of changing the axes. There are two principal transformations in rectangular coördinates. When the new axes are respectively parallel to the old through a new origin the transformation is called **translation of axes.** When the origin is unchanged but the axes are each rotated through a given angle, the transformation is called **rotation of axes.**

38. Translation of axes. — *If x and y are the coördinates of any point before translation to a new origin (h, k) and x' and y' the coördinates of the same point after translation, then*

$$\begin{aligned} x &= x' + h, \\ y &= y' + k. \end{aligned} \tag{13}$$

Proof. — Let OX and OY be the given set of axes. Through O' having coördinates (h, k) when referred to the given axes, draw a new set $O'X'$ and $O'Y'$ parallel respectively to OX and OY. Let P be any point in the plane. Its coördinates in the given system are represented by x and y and in the new system by x' and y'. Draw the ordinate $MM'P$ and extend $O'Y'$ to meet the x-axis in N.



Then

$$x = OM, \qquad x' = O'M', \qquad h = ON,$$
$$y = MP, \qquad y' = M'P, \qquad k = NO'.$$

From the figure it is seen that

$$OM = ON + NM = ON + O'M'$$

and $$MP = MM' + M'P = NO' + M'P,$$

whence $$x = x' + h$$
$$y = y' + k.$$

In recalling all formulas of transformation it is well to hold the figure in mind as an aid to the memory.

To transform an equation referred to a given system of axes to another system parallel to the first through the point (h, k), replace x in the given equation by $x' + h$ and y by

$y' + k$ and simplify the result. This gives the new equation of the given locus in which x' and y' are the variable coördinates in the new system. It is customary to drop the primes when the work of transformation is finished.

EXERCISES

1. What are the new coördinates of the points $(3, -3)$, $(-4, 2)$, $(0, -2)$, $(4, 0)$ referred to parallel axes through $(1, 2)$?

2. Transform the equation $3x + 2y = 12$ when the axes are translated to a new origin $(-2, -3)$. Construct the two sets of axes and plot the locus of each equation, showing that they represent the same line. *Ans.* $3x + 2y = 24$.

3. Transform the equations $y - x = 3$ and $3y + 2x = 4$ when the axes are translated to a new origin at their point of intersection.

4. Transform the following equations to a new set of axes parallel to the old, the new origin as indicated. In each case draw both sets of axes and the curve.

(*a*) $x^2 - 2x - y^2 + 4y = 4$, $(1, -2)$.
(*b*) $4x^2 - 8x + 9y^2 - 36y + 4 = 0$, $(1, 2)$. *Ans.* $4x^2 + 9y^2 = 36$.
(*c*) $x^2 - 2hx + y^2 - 2ky + h^2 + k^2 = 0$, (h, k). *Ans.* $x^2 + y^2 = 0$.
(*d*) $y + 2 = (x + 1)^3$, $(-1, -2)$. *Ans.* $y = x^3$.
(*e*) $y^2 + 4y = (x - 1)^3$, $(1, -2)$.

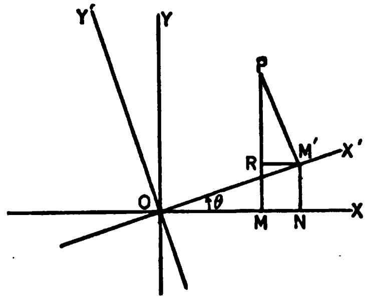
5. The equation of a curve after translation to a new origin $(-1, 2)$ is $x^2 + y^2 = 9$. What was the original equation?

Ans. $(x + 1)^2 + (y - 2)^2 = 9$.

39. Rotation of axes. — *If x and y are the coördinates of any point before rotation through an angle θ, and x' and y' the coördinates after rotation, then*

$$\begin{aligned} x &= x' \cos\theta - y' \sin\theta, \\ y &= x' \sin\theta + y' \cos\theta. \end{aligned} \tag{14}$$

Proof. — Let OX and OY be the given set of axes, and let OX' and OY' be the positions of the axes after they have been rotated about the origin through an angle θ. Take P any point in the plane whose coördinates in the given system are x and y, and in the new system x' and y'. Draw the ordinates MP and $M'P$.



Then $OM = x$, $MP = y$, $OM' = x'$ and $M'P = y'$. Draw through M' the lines RM' and NM' parallel to the x and y axes respectively. The angle RPM' is equal to θ. (Why?)

It is seen from the figure that

$$x = OM = ON - MN = ON - RM' = x' \cos \theta - y' \sin \theta,$$
$$y = MP = MR + RP = NM' + RP = x' \sin \theta + y' \cos \theta.$$

EXERCISES

1. Find the coördinates of the points (3, 1), (−5, 0), and (0, −2), after the axes have been rotated through 45°, also through 90°.

2. Transform the equation $x^2 + y^2 = 16$ when the axes are rotated through 60°. *Ans.* $x^2 + y^2 = 16$.

3. Show that the equation $x^2 + y^2 = a^2$ will be unchanged after rotation of the axes through any angle θ.

4. Transform the following equations when the axes are rotated through the angle given. Construct both sets of axes and the curve.

(*a*) $xy = 4$, $\pi/4$.
(*b*) $y^2 = 4x$, $\pi/2$.
(*c*) $x^2 + 2xy + y^2 - x - y = 0$, $\pi/4$.
(*d*) $3x^2 - 4xy + 8x - 5 = 0$, $\tan^{-1} 2$.
(*e*) $x/a + y/b = 1$, $\tan^{-1} a/b$.
(*f*) $3y^2 + 8xy - 3x^2 = 0$, $\tan^{-1} \frac{1}{3}$.

5. The equation of a locus after the axes have been rotated through −45° is $y - x = 1$; what was the equation before rotation?

6. Through what angle must the axes be rotated in order that the new x-axis shall pass through (3, 4)?

7. Transform the equation $xy - y + 2x - 6 = 0$ to new axes whose origin referred to given axes is (1, −2) and which make an angle of 45° with those axes. *Ans.* $x^2 - y^2 = 8$.

Hint. — First translate to the new origin, then rotate the axes.

8. Three sides of a triangle are $x - y = 4$, $x + y = 6$, and $y + 2x = 20$. If the first two lines are chosen as axes, what will be the equation of the third? *Ans.* $3x - y = 9\sqrt{2}$.

40. Degree of equation not changed by translation and rotation. — Since in each of the formulas of transformation the values of x and y are of first degree in x' and y', therefore the transformed equation will never be of higher degree than the given equation. That it cannot be of lower degree is shown by the fact that if this were the case, a transformation back to the original system of axes would have to raise the degree in order to give the original equation. This has been shown to be impossible.

41. Simplifications by transformation. — One of the principal advantages obtained from transformation is the simplification of equations. Some of these simplifications are best accomplished by translation, others by rotation.

By translation to a proper new origin it is often possible to remove the first degree terms, to make the constant term disappear, or to eliminate one first degree term and the constant term.

The methods by which these results are usually accomplished are illustrated in the following examples:

1. Simplify the equation $x^2 - 2x + y^2 - 6y = 15$ by translation to a new origin.

Substituting $x = x' + h$ and $y = y' + k$ in the equation

$$x^2 - 2x + y^2 - 6y = 15 \tag{1}$$

and collecting terms, the equation becomes

$$x'^2 + y'^2 + (2h - 2)x' + (2k - 6)y' + h^2 + k^2 - 2h - 6k - 15 = 0. \tag{2}$$

It is readily seen that it is possible to so choose h and k that the coefficients of x' and y' shall be 0. Thus

$$2h - 2 = 0, \quad h = 1, \tag{3}$$

$$2k - 6 = 0, \quad k = 3. \tag{4}$$

Substituting these values back for h and k, the equation becomes

$$x'^2 + y'^2 = 25. \qquad (5)$$

One advantage gained by the transformation is that the new equation shows that the locus is symmetrical with respect to both axes since there are no odd powers of x or y in the equation.

Another method of accomplishing the same result is to complete the squares of all x terms and of all y terms, thus

$$(x^2 - 2x + 1) + (y^2 - 6y + 9) = 15 + 1 + 9, \qquad (2)$$

or

$$(x - 1)^2 + (y - 3)^2 = 25. \qquad (3)$$

It is readily seen that if the axes are translated to a new origin at (1, 3) the equation will have no first degree terms.

Although often desirable, it is not always possible to remove the first degree terms. This is illustrated in the second example.

2. Simplify by translation $y^2 + 4y - 8x - 4 = 0$.

Substituting $x = x' + h$ and $y = y' + k$, in the equation

$$y^2 + 4y - 8x - 4 = 0 \qquad (1)$$

and collecting terms, the equation becomes

$$y'^2 + y'(2k + 4) - 8x' + k^2 + 4k - 8h - 4 = 0. \qquad (2)$$

It is evident that the coefficient of x' cannot be made equal to zero. The quantities h and k may, however, be determined so that the coefficient of y and the constant term shall be zero.

Thus $$2k + 4 = 0 \qquad (3)$$

and $$k^2 + 4k - 8h - 4 = 0. \qquad (4)$$

Whence $$k = -2, \; h = -1.$$

The equation then reads $$y'^2 = 8x'. \qquad (5)$$

This locus is symmetrical with respect to the new x-axis and passes through the new origin.

This problem can also be solved in a manner similar to the second method used for the first example.

By rotation through a proper angle it is possible to remove the xy-term from an equation of second degree as is shown in the following example.

3. Remove the xy-term from $3x^2 + 10xy + 3y^2 = 8$.

Substituting $x = x' \cos\theta - y' \sin\theta$ and $y = x' \sin\theta + y' \cos\theta$ in the equation

$$3x^2 + 10xy + 3y^2 = 8, \tag{1}$$

$$\begin{aligned} x'^2 (3\cos^2\theta + 10 \sin\theta\cos\theta + 3\sin^2\theta) \\ + x'y' (10\cos^2\theta - 10\sin^2\theta) \\ + y'^2 (3\sin^2\theta - 10\sin\theta\cos\theta + 3\cos^2\theta) = 8. \end{aligned} \tag{2}$$

The coefficients of all the terms can be changed to functions of 2θ. The equation then becomes

$$x'^2 (3+5\sin 2\theta) + x'y' (10\cos 2\theta) + y'^2 (3-5\sin 2\theta) = 8. \tag{3}$$

Since the new equation is to contain no $x'y'$ term, therefore $10\cos 2\theta = 0$, whence $\cos 2\theta = 0$, $2\theta = 90°$, and $\theta = 45°$.

Substituting $\theta = 45°$ in equation (3),

$$8x'^2 - 2y'^2 = 8. \tag{4}$$

EXERCISES

1. Simplify the following equations by translation of axes. Plot both pairs of axes and the curve.

(a) $x^2 + 4x + 9y^2 - 18y + 4 = 0$. *Ans.* $x^2 + 9y^2 = 9$.

(b) $x^2 + 2x - 9y^2 - 36y = 44$. *Ans.* $x^2 - 9y^2 = 9$.

(c) $x^2 - 6x + y^2 + 6y = 7$.

(d) $y^2 - 8y + 6x - 2 = 0$.

(e) $x^2 + 4x = 2y + 6$.

2. By rotating the axes, remove the xy-term from the following. Plot both pairs of axes and the curve.

(a) $x^2 + 2xy + y^2 = 9$.

(b) $xy = 4$.

(c) $11x^2 + 24xy + 4y^2 = 20$

(d) $8x^2 + 4xy + 5y^2 = 36$.

3. In the following, remove the xy-term by rotation of axes. Construct the two sets of axes and the curve in each example.

(a) $x^2 - xy + y^2 + 5x - y = 1.$
(b) $2xy - 2\sqrt{2}y = 4.$

4. To what new origin must the axes be translated in order that the two lines $2x - y - 3 = 0$ and $x + 2y + 1 = 0$, when referred to the new system, shall have no constant term? Find the equations referred to the new axes.

5. Through what angle must the axes be rotated in order that the new equation of the line $x - y = 4$ shall have no x-term? Check from the figure. *Ans.* 45°.

6. Transform the equation $x - y = 6$ to the form $y = 0$.

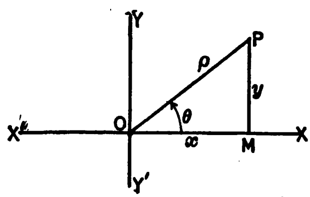
Hint. — First translate the axes to a new origin located anywhere on the given line and then rotate the axes.

42. Transformation from rectangular to polar coördinates and vice versa.

If x and y are the coördinates of a point in a rectangular system and ρ and θ the coördinates of the same point in a polar system, the origin and the x-axis coinciding respectively with the pole and the polar axis, then

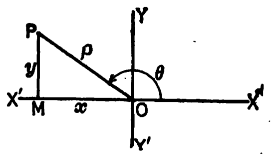
$$\begin{aligned} x &= \rho \cos \theta, \\ y &= \rho \sin \theta. \end{aligned} \tag{15}$$

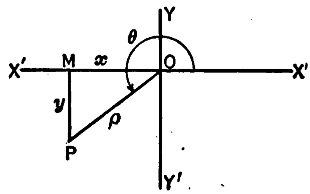
Proof. — Let OX and OY represent the rectangular axes, then O and OX are the pole and initial line respectively. Let P represent any point whose coördinates in the rectangular system are x and y and in the polar system ρ and θ. Draw MP perpendicular to OX. Then $x = OM$, $y = MP$, $\rho = OP$, $\theta =$ angle MOP.

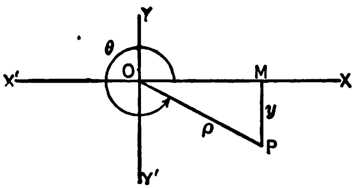


It is readily seen from trigonometry that $x = \rho \cos \theta$ and $y = \rho \sin \theta$.

It is seen from the following figures that if P is located in any other quadrant than the first, the proof is identical with that given above.







If ρ and θ are the coördinates of a point in a polar system and x and y the coördinates of the same point in a rectangular system, the pole and polar axis coinciding respectively with the origin and x-axis, then

$$\begin{aligned}\rho &= \pm\sqrt{x^2+y^2},\\ \theta &= \tan^{-1} y/x.\end{aligned} \qquad (16)$$

Proof. — These results can be read directly from the figures. It is also seen that

$$\cos\theta = \frac{x}{\rho} = \frac{x}{\pm\sqrt{x^2+y^2}}, \quad \sin\theta = \frac{y}{\rho} = \frac{y}{\pm\sqrt{x^2+y^2}}.$$

It is particularly helpful in this set of formulas, as has been suggested before in this chapter, that the student keep the figures in mind when recalling formulas of transformation.

EXERCISES

1. Find the polar coördinates of the points $(0, 3)$, $(-3, 3)$, $(-3, -4)$, $(5, -12)$.

2. Find the rectangular coördinates of the points $(3, \pi/4)$, $(4, \pi)$, $(5, -\pi/6)$, $(2, 5\pi/4)$.

3. Transform the following equations from rectangular to polar coördinates. Plot each curve.

(*a*) $x = a$. *Ans.* $\rho \cos\theta = a$.
(*b*) $y = 6$.
(*c*) $y = x$.
(*d*) $x^2 + y^2 = a^2$. *Ans.* $\rho^2 = a^2$.
(*e*) $x^2 - y^2 = 6$.
(*f*) $xy = 4$.
(*g*) $x^2 + y^2 + 4x = 0$.
(*h*) $y^2(2a - x) = x^3$.
(*i*) $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.
(*j*) $x^2 + y^2 + 2ax = a\sqrt{x^2 + y^2}$.

4. Transform the following equations from polar to rectangular coördinates. Plot each curve.

(*a*) $\theta = 45°$.
(*b*) $\rho \cos\theta = 2$.
(*c*) $\rho = 2a\cos\theta$.

Hint. — In (*c*) and similar examples it is sometimes best to multiply by ρ before transforming.

(*d*) $\rho^2 \sin 2\theta = 4$.
(*e*) $\rho^2 = a^2\cos 2\theta$.
(*f*) $\rho = a\cos\theta + b\sin\theta$.
(*g*) $\rho = a(1 - \cos\theta)$.
(*h*) $\rho = a(\cos 2\theta + \sin\theta)$.
(*i*) $\rho = 2a\tan\theta\sin\theta$.
(*j*) $\rho = 2 + 3\cos\theta$.
(*k*) $\rho = a(1 + \cos 2\theta)$.

5. Translate axes to new origin and then transform to polar coördinates:

(*a*) $x^2 + y^2 + 4x + 8y - 20 = 0$, new origin $(-2, -4)$.
(*b*) $x^2 - y^2 + 2x + 6y = 24$, new origin $(-1, 3)$.

CHAPTER VI

THE CIRCLE

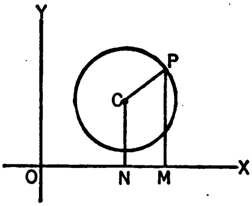
43. A **circle** is a locus traced by a point which is everywhere equidistant from a fixed point, called its **center.** The distance of any point from the center is called the **radius.**

A circle, therefore, is determined, and its equation can be written if its *center* and *radius* are known.

First standard equation of a circle. Center and radius known. — *The equation of a circle whose center is* $C(h, k)$ *and whose radius is* r *is*

$$(x-h)^2+(y-k)^2=r^2. \qquad (17)$$

Proof. — Let $P(x, y)$ represent any point on the circle. By the definition of a circle, $PC = r$. From the formula for the distance between two points, formula (1),



$$PC=\sqrt{(x-h)^2+(y-k)^2},$$

whence $\sqrt{(x-h)^2+(y-k)^2}=r.$

Squaring, $(x-h)^2+(y-k)^2=r^2.$

Second standard equation of circle. Center at origin, radius r. — *The equation of a circle whose center is at the origin and whose radius is* r *is*

$$x^2+y^2=r^2. \qquad (18)$$

Proof. — Substituting $h = 0$ and $k = 0$, in equation (17), it reduces to equation (18).

EXERCISES

1. Write the equations of the circles whose centers and radii are as follows:

(*a*) $C(1, 4)$, radius 5. (*d*) $C(-4, 0)$, radius 2.
(*b*) $C(0, 0)$, radius 2. (*e*) $C(-1, -2)$, radius 7.
(*c*) $C(-3, 4)$, radius 5. (*f*) $C(5, -1)$, radius 3.

2. Write the equations of the circles, having given:

(*a*) Center at the intersection of the lines $2x - y - 3 = 0$ and $x + 3y - 5 = 0$, and radius 5.
(*b*) Center at origin and passing through the point (5, 6).
(*c*) Line joining (1, 5) and (−3, 1) as diameter.
(*d*) Center at (5, 6) and tangent to x-axis.

44. General form of equation of circle. — Equation (17) when expanded becomes

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0. \tag{1}$$

It is thus seen that the equation of a circle is of second degree. If the constants are collected, equation (17) is seen to be of the form

$$x^2 + y^2 + Dx + Ey + F = 0. \tag{2}$$

It will be shown that every equation of this form represents a circle.

Completing the squares of the x-terms and of the y-terms, equation (2) becomes

$$\left(x + \frac{D}{2}\right)^2 + \left(y + \frac{E}{2}\right)^2 = \frac{D^2 + E^2 - 4F}{4}, \tag{3}$$

from which it is seen by comparison with $(x - h)^2 + (y - k)^2 = r^2$ that equation (2) represents a circle whose center is at $(-D/2, -E/2)$ and whose radius is

$$\tfrac{1}{2}\sqrt{D^2 + E^2 - 4F}.$$

If $D^2 + E^2 - 4F < 0$, the radius is imaginary and no circle is possible. If $D^2 + E^2 - 4F = 0$, the equation represents only one point, the center. The foregoing may be summarized as follows:

The equation $\boldsymbol{x^2 + y^2 + Dx + Ey + F = 0}$ **(19)**

represents a circle whose center is $(-D/2, -E/2)$ *and whose radius is* $\frac{1}{2}\sqrt{D^2 + E^2 - 4F}$, *provided* $D^2 + E^2 - 4F > 0$.

It should be noticed that equation (19) is not the most general form of the equation of second degree, this being

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

If, in this equation, $B = 0$ and $C = A$, it is possible to divide through by A and thus reduce it to the form of the general equation of the circle $x^2 + y^2 + Dx + Ey + F = 0$. Whence:

The general equation of second degree $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ *represents a circle if* $B = 0$ *and* $A = C$.

In plotting loci of equations of second degree, the student should look for the presence of the conditions which make a second degree equation a circle, as when these exist, he can save himself all the work of discussion and of plotting points, since a circle can be readily drawn as soon as its center and radius are known.

In determining center and radius, he can either complete the square of the x-terms and of the y-terms or can make use of the facts learned in connection with the general equation.

Thus, plot the locus of $2x^2 + 2y^2 - 18x + 16y + 60 = 0$. Since the coefficients of x^2 and y^2 are equal, this can be put in the form of the general equation of a circle by dividing by 2, giving

$$x^2 + y^2 - 9x + 8y + 30 = 0. \tag{2}$$

Completing the squares,

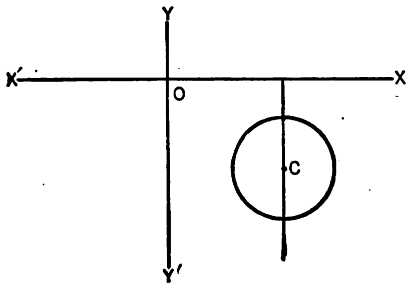
$$(x - \tfrac{9}{2})^2 + (y + 4)^2 = (\tfrac{5}{2})^2. \tag{3}$$

Whence by comparison with standard equation (17), the center is $(\frac{9}{2}, -4)$ and radius $\frac{5}{2}$.

Or, comparing (2) with the general equation (19),

$$h = -D/2 = \tfrac{9}{2},$$
$$k = -E/2 = -4,$$
$$r = \tfrac{1}{2}\sqrt{81 + 64 - 120} = \tfrac{5}{2},$$

whence the circle is as shown.



45. Radical axis. — In Art. 16, it was learned that if an equation is multiplied by any constant and added to any other equation, the result represents a locus through the points of intersection of the two given loci. If the equation of a circle is multiplied by k and added to the equation of another circle the resulting equation represents a system of circles, since for every value of the constant multiplier k the coefficients of x^2 and y^2 are the same.

If the equations of the two circles are put into general form (19), the terms of second degree will be eliminated if the constant multiplier is -1, or if the equations of the two circles are subtracted. This result being of first degree represents a straight line. When the circles intersect, this line is their **common chord.** When the circles touch at one point only, it is their **common tangent.** Whether the circles have any common points or not this line is called the **radical axis.**

This radical axis is the locus of points from which tangents to the two circles are of equal length as will be proved in Ex. 16 of the list which follows.

In finding the intersection of two circles, it is best to first find the radical axis and then find the intersection of this with either of the given circles.

Exercise. — Find the intersections of the circles:

(*a*) $x^2 + y^2 - 6x + 4 = 0$ and $x^2 + y^2 - 4x - 4y = 0$.

(*b*) $x^2 + y^2 - y = 0$ and $2x^2 + 2y^2 + x = 0$.

46. Circle determined by three conditions. Since the equation of the circle in either of the two forms

$$(x - h)^2 + (y - k)^2 = r^2,$$

or

$$x^2 + y^2 + Dx + Ey + F = 0,$$

has three arbitrary constants, therefore three conditions are necessary in order to determine its equation.

Sometimes it is best to use the data given to obtain three equations in h, k, and r and sometimes in D, E, and F. From the three equations, the three constants can be determined, and the required equation obtained by substituting their values back in the corresponding standard equation.

In other examples, it is better to determine more directly the center and radius by using the given data in connection with equations and formulas already derived. Thus, the center is often at the intersection of two lines whose equations can be found, and the radius the distance between two known points. Whenever the coördinates of the center and the radius are known or have been found, it is only necessary to substitute in standard equation (17).

ILLUSTRATIVE EXAMPLES

1. Find the equation of the circle through the three points (4, 6), (−2, −2), and (−4, 2).

Let the required equation be

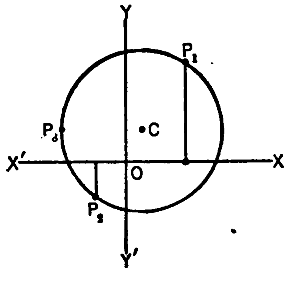
$$x^2 + y^2 + Dx + Ey + F = 0, \tag{1}$$

in which D, E, and F are unknown constants. Since each of the points is on the circle, therefore the coördinates of the three given points must satisfy equation (1), whence

$$16 + 36 + 4D + 6E + F = 0, \tag{2}$$

$$4 + 4 - 2D - 2E + F = 0, \tag{3}$$

$$16 + 4 - 4D + 2E + F = 0. \tag{4}$$



Solving (2), (3), and (4), for D, E, and F,

$$D = -2, E = -4, F = -20.$$

Whence the equation of the circle is

$$x^2 + y^2 - 2x - 4y - 20 = 0. \quad (5)$$

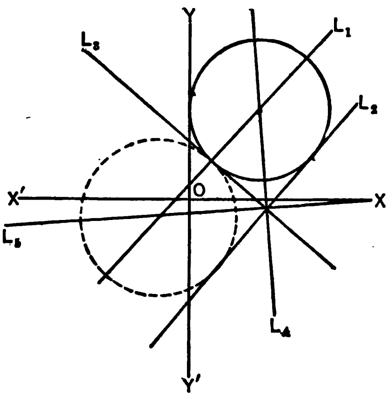
Changing to form (17), $(x - 1)^2 + (y - 2)^2 = 25$, from which it is seen that the center is (1, 2) and the radius is 5. The figure is as shown.

This problem may also be solved by finding the equations of the perpendicular bisectors of the lines joining two pairs of the points. The intersection of these bisectors will be the center, and the distance from this center to any one of the given points will be the radius of the required circle. Substitution in standard equation (17) will give the equation of the circle.

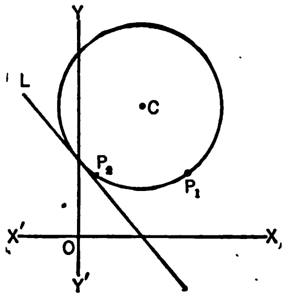
2. Find the equation of the circle whose center lies on the line $y - x = 1$, and which is tangent to each of the lines $4x - 3y = 15$ and $3x + 4y = 10$.

Represent the three lines in the order given by L_1, L_2, and L_3.

It is seen from the figure that there are two circles which fulfil the conditions mentioned, and from geometry it is known that the center of each lies on one of the bisectors of the angles between L_2 and L_3. Let the bisectors be represented by L_4 and L_5. The equation of that circle whose center C lies on the bisector L_4 will first be determined. By the method of Art. 27, the equation of L_4 is found to be $7x + y = 25$. The intersection of this line with L_1 determines the center, C (3, 4). The radius is the distance from either L_2 or L_3 to C, and by Art. 26 this is 3. Substituting the coördinates of the center and the radius in standard equation (17), the equation of the circle in the first quadrant is found to be $(x - 3)^2 + (y - 4)^2 = 9$. The equation of the other circle can be found in a similar manner to be $(x + 2)^2 + (y + 1)^2 = 16$.



3. Find the equation of the circle tangent to the line $4x + 3y = 15$ and passing through P_1 (7, 4) and P_2 (1, 4).



Let C (h, k) represent the center of the required circle and r the radius. Also let L represent the given line. The three given conditions lead to three equations in h, k, and r. Since L is tangent to the circle, the distance from line L to C is r. By formula of Art. 26, this distance is

$$\frac{4h + 3k - 15}{5} = r. \tag{1}$$

The points (7, 4) and (1, 4) are on the circle. Therefore their coördinates must satisfy the equation

$$(x - h)^2 + (y - k)^2 = r^2.$$

Whence

$$(7 - h)^2 + (4 - k)^2 = r^2 \tag{2}$$

$$(1 - h)^2 + (4 - k)^2 = r^2. \tag{3}$$

Subtracting (2) from (3), $\quad h = 4. \quad$ (4)

Substituting in (1), $\quad r = \dfrac{1 + 3k}{5}. \quad$ (5)

Combining (4) and (5) with either (2) or (3), $k = 8$ or $\frac{29}{8}$ and $r = 5$ or $\frac{25}{8}$. The equations of the required circles then are

$$(x - 4)^2 + (y - 8)^2 = 25,$$

and

$$(x - 4)^2 + (y - \tfrac{29}{8})^2 = \tfrac{625}{64}.$$

EXERCISES

1. Find the coördinates of the center and the radius of each of the following circles:

(a) $x^2 + y^2 - 4x + 8y + 4 = 0.$
(b) $3x^2 + 3y^2 - 6x + 12y = 1.$
(c) $x^2 + y^2 = 4x.$
(d) $2x^2 + 2y^2 + 4x + 8y = 0.$
(e) $x^2 + y^2 + 10ax - 24ay = 0.$
(f) $x^2 + 2(a + b)x + y^2 + 2(a - b)y = 4ab.$
(g) $2x^2 + 2y^2 = 3y.$
(h) $x^2 + 4x + y^2 - 6y + 13 = 0.$

2. Find the equation of a circle through the three points (3, 1), (6, 0), (−1, −7). *Ans.* $x^2 + y^2 - 6x + 8y = 0.$

3. Find the equation of a circle

(*a*) center at $(-1, 4)$, tangent to $5x + 12y + 9 = 0$.
(*b*) center on y-axis, passing through the points $(3, -1)$ and $(3, 7)$.
(*c*) center on x-axis, passing through $(0, 0)$ and $(1, 5)$.
(*d*) passing through $(5, -5)$, having the same center as $2x^2 + 2y^2 + 4x - 12y + 3 = 0$.
(*e*) having line joining $(-1, 6)$ and $(5, 2)$ as diameter.
(*f*) passing through $(1, 0)$ and $(6, 1)$ and having center on line $2x + y + 4 = 0$.
(*g*) radius 4, tangent to x-axis at $(3, 0)$ and lying above it.

4. Find the equation of that diameter of the circle $3x^2 + 3y^2 + 12x - 12y - 1 = 0$ which makes an angle of 45° with the x-axis.

5. A diameter of the circle $x^2 + y^2 + 4x + 6y = 3$ passes through $(1, -1)$. What is its equation and the slope of the chords it bisects?

6. Find the equation of that chord of the circle $x^2 + y^2 = 25$ which is bisected at $(2, 3)$.

7. Prove that a circle can be drawn through the four points $(0, 2)$, $(3, 3)$, $(6, 2)$, and $(-1, -5)$. Find its center and radius.

8. Find the equation of the circle

(*a*) radius 10, passing through $(-2, -2)$ and $(0, -4)$.
(*b*) in the first quadrant, of radius 3, and tangent to both axes.
(*c*) tangent to both axes, center on the line $y - 2x = 3$.
(*d*) passing through $(1, -3)$ and $(2, -2)$ and tangent to $3x - 4y = 15$.
(*e*) center on $2x + y = 4$ and tangent to $y - 3x = 6$ and $3x + y + 6 = 0$.
(*f*) tangent to both axes, distance from center to origin $= 4$, and lying in the fourth quadrant.

9. Find the equation of the circle inscribed in the triangle whose sides are the lines $y - 3 = 0$, $12x - 5y = 21$, and $12x + 5y + 21 = 0$. *Ans.* $x^2 + y^2 - 2y = 3$.

10. Find the equation of the circle circumscribed about the triangle whose sides are the lines $y + x = 0$, $3y + x = 0$, and $2y + x - 1 = 0$.

11. Find the equation of the circle tangent to the x-axis, through the point $(4, 1)$ and center on the line $y = 5x$.

12. Find the equation of the circle whose center is on the y-axis and which passes through the points of intersection of the two circles $x^2 + y^2 - 5x - 7y + 6 = 0$ and $x^2 + y^2 - 4x - 4y + 3 = 0$.

13. Find the equation of the common chord of the two circles $x^2 + y^2 + 6x - 4y + 3 = 0$ and $x^2 + y^2 - 2x + 4y - 5 = 0$, and prove that it is perpendicular to the line of centers.

14. Prove that the square of the length of the tangent from P_1 (x_1, y_1) to the circle $x^2 + y^2 + Dx + Ey + F = 0$ is $x_1^2 + y_1^2 + Dx_1 + Ey_1 + F$.

Hint. — Join the center with the point P_1 and with the point of contact. These lines with the tangent form a right triangle.

15. Prove that the point $(-1, -1)$ is on the radical axis of the two circles $x^2 + 6x + y^2 - 4y + 9 = 0$ and $x^2 + y^2 - 4x - 2y + 1 = 0$, and show that the tangents from this point to the two circles are equal.

16. Find the equation of the locus of the point which moves so that the lengths of the tangents from this point to the two circles $x^2 + y^2 + Dx + Ey + F = 0$ and $x^2 + y^2 + D_1x + E_1y + F_1 = 0$ are equal. Show that this locus is the radical axis of the two circles.

17. Given the three circles $x^2 + 6x + y^2 - 4y + 9 = 0$, $x^2 + y^2 - 4x - 2y + 1 = 0$, and $x^2 + y^2 - 3x - y + \frac{5}{2} = 0$. Taking the circles in pairs, find the equations of the radical axes and prove that they meet in a point.

18. Prove analytically that every angle inscribed in a semicircle is a right angle.

Hint. — Take the extremities of the diameter as $(-a, 0)$ and $(a, 0)$, thus making the equation of the circle $x^2 + y^2 = a^2$.

19. Prove analytically that if a perpendicular is drawn from a point on a circle to a diameter, the length of the perpendicular is a mean proportional between the segments it cuts off on the diameter.

20. Prove that the following loci are circles and find the radius and the coördinates of the center in each:

(*a*) A point moves so that the sum of the squares of its distances from $(3, 0)$ and $(-1, -4)$ is always 40.

(*b*) A point moves so that its distance from $(1, 3)$ is twice its distance from $(-2, -3)$.

(*c*) A point moves so that the square of its distance from $(2, 3)$ is equal to its distance from the line $4x - 3y - 15 = 0$.

21. Prove that the following loci are circles:

(*a*) A point moves so that the sum of the squares of its distances from two fixed points is constant.

Hint. — When no mention is made of axes or coördinates, it is always advisable to choose these in such a way as to make the work as simple as possible. Thus, in the above problem take the x-axis through the two points with the origin halfway between them.

(*b*) A point moves so that the sum of the squares of its distances from the four sides of a square is constant.

(*c*) A point moves so that the square of its distance from the base of an isosceles triangle is equal to the product of its distances from the other two sides.

(*d*) A point moves so that the square of its distance from a fixed point is proportional to its distance from a fixed line.

22. A point moves so that its distances from two fixed points are in a constant ratio K. Show that this is a circle excepting when $K = 1$, in which case it is a straight line.

CHAPTER VII

THE PARABOLA

47. Conic sections. — The three curves next considered belong to a general class called *conic sections*. This name arises from the fact that each of these curves can be obtained by passing a plane through a right circular cone.

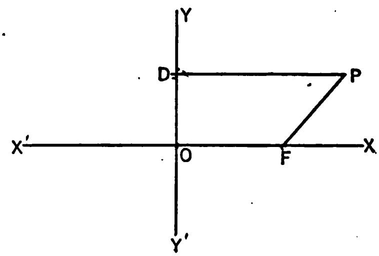
Many of the properties of these curves were known by the early Greek geometers among whom the principal investigators were Archimedes and Appolonius about 200 B.C. The former computed the area of a parabolic segment and of an ellipse. The latter discovered that all three curves can be cut from the same cone and investigated many problems peculiar to the hyperbola.

That the knowledge of conic sections could be made of great practical use in studying the laws of the universe was not learned until after the passage of many centuries. About 1600, Kepler in Germany discovered their importance in the study of the motion of the heavenly bodies, and, about the same time, Galileo in Italy discovered that the path of a projectile is a parabolic curve. The field of their usefulness has spread until a large group of problems in physics, mechanics, and architecture are now known to depend upon a knowledge of these curves for their solution.

Although these conic sections differ very much in appearance, it is found that they can all be generated by the same law, *viz.*, *A conic section is the locus traced by a point which moves so that its distance from a fixed point bears a constant ratio to its distance from a fixed straight line.*

The fixed line is called the **directrix**, the fixed point the **focus**, and the fixed ratio the **eccentricity**, represented by e.

Equation of a conic section. — Take the directrix as the y-axis and the perpendicular through the focus on the directrix as the x-axis. Let $P\ (x, y)$ be any point on the curve. Draw PD perpendicular to YY'. Call the distance $OF = 2p$.



By definition, $\dfrac{FP}{DP} = e.$

By formula (1), $FP = \sqrt{(x-2p)^2 + y^2}$, $DP = x$.

Therefore, $$\frac{\sqrt{(x-2p)^2+y^2}}{x} = e.$$

Clearing of fractions and collecting,

$$(1 - e^2)\,x^2 - 4px + 4p^2 + y^2 = 0. \qquad (20)$$

From this equation it is seen that the curve is symmetrical with respect to the x-axis which is the perpendicular from the focus on the directrix. For this reason the line is called the **principal axis** of the curve.

By letting $y = 0$, the intercepts on the principal axis can be found to be $\dfrac{2p}{1+e}$ and $\dfrac{2p}{1-e}$.

When $e = 1$, the curve is called a parabola. It cuts the principal axis in one finite point, halfway between the focus and the directrix.

When $e < 1$, the curve is called an ellipse. It cuts the principal axis in two points on the same side of the directrix as the focus.

When $e > 1$, the curve is called an hyperbola. It cuts the principal axis in two points on opposite sides of the directrix.

48. Equation of the parabola. — Since, in the parabola, $e = 1$, the definition of this curve can be stated:

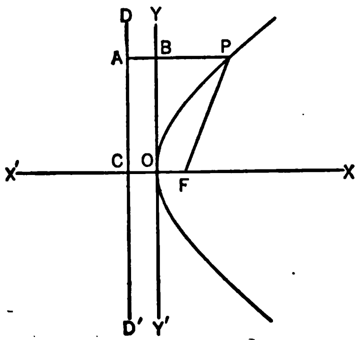
A parabola is a locus traced by a point equidistant from a fixed point called the focus and a fixed line called the directrix.

It was seen in Art. 47 that the parabola passes through a point halfway between the focus and directrix. This point is called the **vertex.** It is found that the simplest form of the equation is obtained when this point is taken as origin and the x-axis coincides with the principal axis.

First standard equation of the parabola. — *The equation of a parabola whose vertex is at the origin and whose axis is the x-axis is*

$$y^2 = 4\,px, \tag{21}$$

p being the distance from the vertex to the focus.



Proof. — Let DD' be the directrix and F the focus. Through F draw the x-axis perpendicular to the directrix, meeting it in C. At O halfway between C and F erect the y-axis. Let $P(x, y)$ be any point on the curve and draw AP perpendicular to the directrix, meeting the y-axis in B. Let CF be represented as before by $2\,p$. Then the coördinates of F are $(p, 0)$, also $CO = OF = p$.

By the definition given above,

$$FP = AP.$$

From formula (1),

$$FP = \sqrt{(x-p)^2 + y^2}.$$
$$AP = AB + BP = p + x.$$

Therefore, $\sqrt{(x-p)^2 + y^2} = p + x.$

Clearing of radicals and simplifying,

$$y^2 = 4\,px.$$

The equation shows, as has been previously discovered, that the curve is symmetrical with respect to its axis and passes through the origin, that is, through a point halfway between the directrix and the focus. It also shows that when p is positive, the curve extends indefinitely to the right, while no part lies to the left of the origin. When p is negative, the curve extends indefinitely to the left, while no part lies to the right of the origin.

A chord through the focus of any conic section is called a **focal chord.**

The **latus rectum** is that focal chord parallel to the directrix.

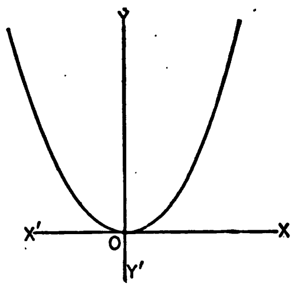
The equation of the latus rectum is $x = p$. Solving this simultaneously with the equation of the parabola $y^2 = 4\,px$, the ordinates of the intersections are $y = \pm 2\,p$. Whence *the length of the latus rectum is* $4\,p$.

It is helpful in sketching a parabola, to locate the vertex and focus, then erect the latus rectum equal to 4 times the distance from the vertex to the focus. The parabola passes through the extremities of this latus rectum and the vertex.

The second standard equation of a parabola. — *The equation of a parabola whose vertex is at the origin and whose axis is the y-axis is*

$$\boldsymbol{x^2 = 4\,py,} \qquad \textbf{(22)}$$

p being the distance from the vertex to the focus.



Proof. — Rotating the axes through $(-90°)$, equation (21) becomes

$$[x \sin(-90°) + y \cos(-90°)]^2 = 4\,p\,[x \cos(-90°) - y \sin(-90°)],$$

or

$$x^2 = 4\,py.$$

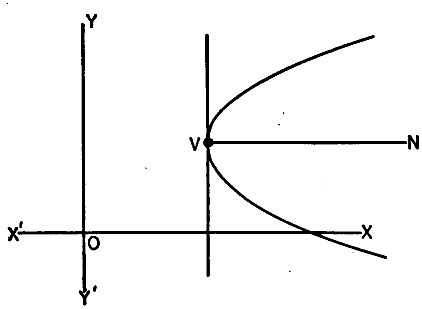
This equation may also be obtained directly from the figure by taking steps similar to those used in deriving equation (21).

The third standard equation of a parabola. — *The equation of the parabola whose vertex is at the point (h, k) and whose axis is parallel to the x-axis is*

$$(y - k)^2 = 4p(x - h), \tag{23}$$

p being the distance from the vertex to the focus.

Proof. — Let the figure be drawn as indicated with the vertex V having coördinates (h, k) and principal axes VN parallel to $X'X$. The equation of the parabola, considering V as the origin and VN as the x-axis, is $y^2 = 4px$. The equation with the origin at V is known, and the equation with the origin at O is required. The problem then is to translate the axes to a new origin. The coördinates of the new origin 0 with respect to the old axes through V are $(-h, -k)$. Hence the equation becomes, after translation of axes to 0,



$$(y - k)^2 = 4p(x - h).$$

The fourth standard equation of a parabola. — *The equation of a parabola whose vertex is at (h, k) and whose axis is parallel to the y-axis is*

$$(x - h)^2 = 4p(y - k), \tag{24}$$

p being the distance from the vertex to the focus.

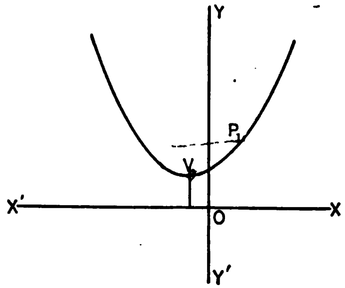
The proof is identical with that used in deriving (23).

ILLUSTRATIVE EXAMPLES

1. Find the equation of the parabola with axis parallel to the y-axis, vertex at $(-1, 2)$, and passing through the point P_1 $(1, 3)$.

The equation of a parabola whose vertex is at $(-1, 2)$ and whose axis is parallel to the y-axis is $(x + 1)^2 = 4p(y - 2)$ by equation (24). Since the point $(1, 3)$ is on this locus, its coördinates must satisfy the equation, whence

$$(1 + 1)^2 = 4p(3 - 2), \quad \text{or} \quad p = 1.$$



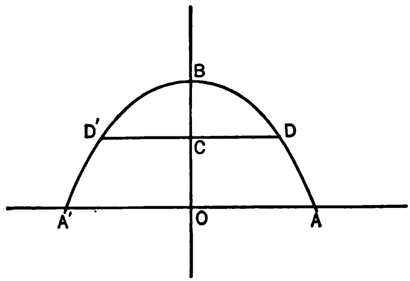
The equation of the parabola then is

$$(x + 1)^2 = 4(y - 2)$$

or $\quad x^2 + 2x - 4y + 9 = 0.$

2. An arch is in the form of a parabola with vertical axis. Its highest point is 18 feet above the base which is 36 feet wide. Find the length of the beam horizontally across the arch, 10 feet above the base.

Let $A'BA$ represent the given arch. If the origin is taken at the center of the base, the coördinates of the vertex are $(0, 18)$, and the equation of the parabola is $x^2 = 4p(y - 18)$, by standard equation (24). The point A $(18, 0)$ is on this parabola and its coördinates must satisfy the equation, whence $(18)^2 = 4p(0 - 18)$, or $p = -\frac{9}{2}$. The equation then becomes $x^2 = -18(y - 18)$.



Let $D'CD$ represent the position of a beam 10 feet above the base. Then the ordinate of D is 10. Substituting 10 for y in the equation of the parabola,

$$x^2 = -18(10 - 18) = 144.$$

Whence $\quad x = \pm 12 \quad \text{or} \quad D'D = 24 \text{ feet}.$

EXERCISES

1. Find the coördinates of the focus, the equation of the directrix, and the length of the latus rectum for each of the following parabolas and plot the curves:

(*a*) $y^2 = 8x$. (*c*) $y^2 = -4x$. (*e*) $x^2 = -6y$.
(*b*) $3x^2 = 5y$. (*d*) $x + 4y^2 = 0$. (*f*) $2y^2 = -5x$.

2. Find the equations of the parabolas satisfying the following conditions:

(*a*) vertex (0, 0), axis $y = 0$, a point on curve $(-1, 3)$.
(*b*) vertex $(-2, -2)$, focus $(-3, -2)$.
(*c*) focus (0, 0), vertex $(0, -3)$.
(*d*) directrix $y = -2$, focus (1, 4).
(*e*) vertex (0, 1), axis parallel to x-axis, and the point (1, 3) on curve.
(*f*) focus $(1, -2)$, directrix $3x - y + 6 = 0$.

Hint. — Use the definition of a parabola.

3. Find the equation of the line joining the vertex and the upper extremity of the latus rectum of the parabola $y^2 = -8x$.

4. The equation of a parabola is $y^2 = 8x$. With center at the origin, and diameter equal to three times the distance from the vertex to the focus, a circle is described. Prove that the common chord of circle and parabola cuts the x-axis halfway between the vertex and the focus.

5. Find the equation of the circle through the vertex and the ends of the latus rectum of $x^2 = 4y$.

6. Find the equations of the parabolas with the axes parallel to the y-axis and satisfying in addition the following conditions:

(*a*) vertex $(2, -5)$ and a point on curve $(6, -1)$.
(*b*) three points on curve (0, 3), (4, 3), and $(-2, 6)$.

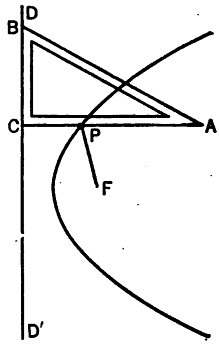
7. Find the equation of the focal chord of the parabola $y^2 = 6x$ through the point on the curve whose ordinate is 4.

8. A parabola has its vertex at the origin and axis along the y-axis. A focal chord has one extremity at $(3, -3)$. Find its equation and the coördinates of the other extremity.

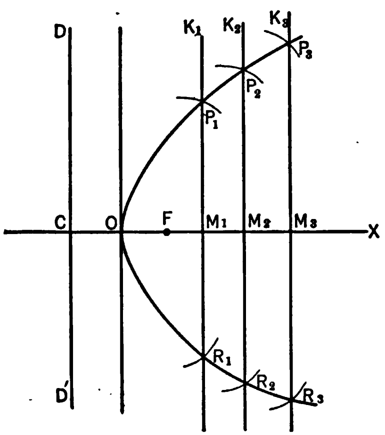
9. A trough whose cross section is a parabola with vertex downward is partly filled with liquid. The width of the trough one foot above the vertex is 4 feet and the width at the surface of the liquid is 8 feet. Find height of liquid. *Ans.* 4 feet.

10. An arch has the form of a parabola with vertical axis. The width of the base is 36 feet and the height above the base at a point 12 feet to the right of the center of the base is 10 feet. Find the height of the arch at its highest point. *Ans.* 18 feet.

49. Construction of the parabola. — Having given the directrix and the focus there are two principal methods of constructing the parabola mechanically.



First method. — Let DD' be the given directrix and F the focus. Place a right triangle ABC with one leg BC on the directrix, the other leg lying on the same side of the directrix as the focus. Fasten one end of a string of length CA at A and the other end at the focus. With a pencil point against the triangle at P, keep the string taut and move the triangle along the directrix. The pencil point will describe a parabola, since $CP = FP$, and therefore P is equidistant from the focus and the directrix.



Second method. — Locate the focus and directrix as in the first case. Draw OX through F perpendicular to the directrix, on it lay off a number of points, as M_1, M_2, M_3, etc., and erect ordinates M_1K_1, M_2K_2, M_3K_3 at these points. With F as a center and a radius equal to the distance from the directrix to the foot of any ordinate as CM_1, describe an arc cutting the ordinate in two points as P_1 and R_1. Similarly, locate the points P_2 and R_2, P_3 and R_3, etc. These points all lie on the parabola since they are equidistant from the focus and the directrix. Connect by a smooth curve and the figure is approximately a parabola.

50. General equation of a parabola, axis parallel to one of the coördinate axes. — When equation (23) is expanded, it takes the form

$$y^2 - 2\,ky - 4\,px + k^2 + 4\,ph = 0. \tag{1}$$

Similarly, equation (24) becomes

$$x^2 - 2\,hx - 4\,py + h^2 + 4\,pk = 0. \tag{2}$$

These results show that every equation of a parabola with axis parallel to a coördinate axis contains one and only one term which is the square of a variable and no xy term. It will be shown that every equation of the form

$$y^2 + Dx + Ey + F = 0 \tag{3}$$

or

$$x^2 + Dx + Ey + F = 0 \tag{4}$$

represents a parabola.

Completing the squares and collecting, equation (3) becomes

$$\left(y + \frac{E}{2}\right)^2 = 4\left(-\frac{D}{4}\right)\left(x - \frac{E^2 - 4\,F}{4\,D}\right), \tag{5}$$

which is in the form of equation (23) if D is not 0.

Similarly, (4) becomes

$$\left(x + \frac{D}{2}\right)^2 = 4\left(-\frac{E}{4}\right)\left(y - \frac{D^2 - 4\,F}{4\,E}\right), \tag{6}$$

which is in the form of equation (24) if E is not 0.

Comparing the general equations of the parabola (3) and (4) with the general equation of second degree $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ it is seen that:

The general equation of second degree represents a parabola with axis parallel to a coördinate axis if $B = 0$, and if there is only one second degree term, either x^2 or y^2, provided the first degree term in the other variable is present.

ILLUSTRATIVE EXAMPLE

Determine the vertex, focus, latus rectum, equation of the directrix, and of the axis for the parabola whose equation is $x^2 + 6x + 8y + 1 = 0$.

Completing the squares of the x-terms,

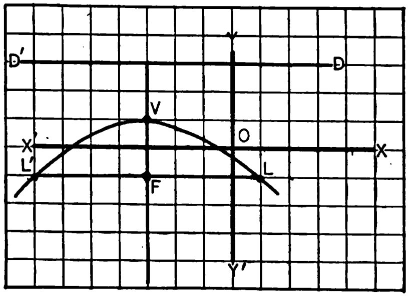
$$(x + 3)^2 = -8(y - 1).$$

This is in the form of the fourth standard equation of the parabola,

$$(x - h)^2 = 4p(y - k).$$

Whence the vertex is at $(-3, 1)$, p the distance from the vertex to the focus is -2, and the length of the latus rectum is 8.

The facts just determined are sufficient to roughly sketch the figure. Since the vertex bisects the distance from the focus to the directrix, that line can now be drawn and its equation is seen to be $y = 3$. The equation of the axis VF can likewise be read from the figure, and is $x = -3$.



EXERCISES

Determine the coördinates of the vertex and focus, length of latus rectum, and equation of the directrix and of the axis for the following parabolas. Also sketch the figures.

1. $x^2 + 4x - 6y - 8 = 0$.
2. $y^2 - 6y + 8x = 15$.
3. $3x^2 + 6x + 5y = 7$.
4. $4x^2 + 8x + 8y = 3$.
5. $y^2 - 5y = x - 7$.
6. $3y^2 - 6y = 4x$.

CHAPTER VIII

THE ELLIPSE

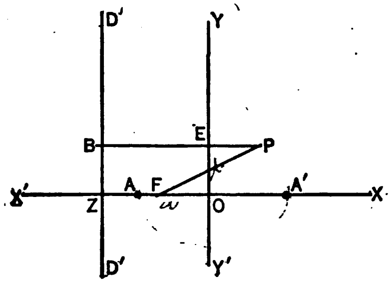
51. The ellipse has been defined as that conic section which is traced by a point which moves so that the ratio of its distance from a fixed point, called the focus, to its distance from a fixed line, called the directrix, is constant and less than 1.

It was shown in Art. 47 that the ellipse cuts the principal axis in two points, both on the same side of the directrix as the focus. The simplest form of the equation of an ellipse is obtained by taking the principal axis as the x-axis and the point halfway between the two intersections as origin. This point is called the **center** of the ellipse.

The first standard equation of the ellipse. — *The equation of an ellipse whose major axis is on the x-axis and whose center is at the origin is*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad (25)$$

in which a and b are the semi-major and semi-minor axes respectively.



Proof. Let the directrix of the ellipse be DD' and take the x-axis on the principal axis which is perpendicular to DD' through the focus F, meeting it at Z. Let A and A' represent the two points at which the curve cuts the principal axis. These two points are called the **vertices** of the ellipse.

At O midway between A and A' erect the y-axis. Call the distance $AO = OA' = a$. Take $P(x, y)$ any point on the ellipse and drop PB perpendicular to the directrix, cutting the y-axis at E.

From the definition of an ellipse,

$$\frac{FP}{BP} = e. \tag{1}$$

In order to compute the values of FP and BP, it is first necessary to find the distances from the directrix to the center and from the focus to the center. In finding these lengths, use is made of the fact that A and A' are on the ellipse. Applying the definition,

$$\frac{AF}{ZA} = e, \tag{2}$$

and

$$\frac{FA'}{ZA'} = e. \tag{3}$$

Clearing (2) and (3) of fractions and adding,

$$AF + FA' = e\,(ZA + ZA').$$

Substituting from the figure,

$$AA' = e\,[(ZO - a) + (ZO + a)].$$

Whence $\quad 2\,a = e\,(2\,ZO)$ and $ZO = a/e$.

The distance from the directrix of an ellipse to the center is a/e. (26)

Similarly, by subtracting (2) from (3),

$$FA' - AF = e\,(ZA' - ZA).$$

Whence $\quad (FO + a) - (a - FO) = e\,(AA')$ and $FO = ae$.

The distance from the focus of an ellipse to the center is ae. (27)

The coördinates of F are $(-ae, 0)$, whence

$$FP = \sqrt{(x + ae)^2 + y^2}, \text{ by formula (1).}$$
$$BP = BE + EP = a/e + x.$$

Substituting in equation (1),

$$\frac{\sqrt{(x + ae)^2 + y^2}}{a/e + x} = e.$$

Clearing of fractions and collecting,

$$x^2 (1 - e^2) + y^2 = a^2 (1 - e^2), \tag{4}$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 (1 - e^2)} = 1. \tag{5}$$

If $x = 0$, $y = \pm a \sqrt{1 - e^2}$, hence the ellipse cuts the y-axis in two points equidistant from the center. This distance will be represented by b. The equation of the ellipse then is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{6}$$

where

$$\mathbf{b^2 = a^2 (1 - e^2).} \tag{7}$$

This relation also shows that $\mathbf{ae = \sqrt{a^2 - b^2}}$. **(28)**

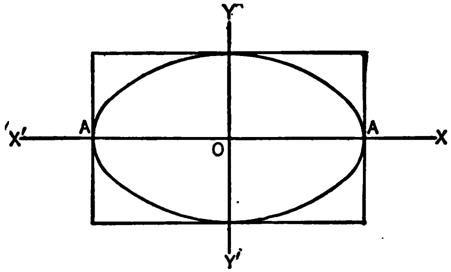
The portion of the principal axis cut off by the ellipse is called the **major axis.** It is represented by $2a$. The portion of the perpendicular to the principal axis through the center, cut off by the ellipse, is called the **minor axis.** It is represented by $2b$.

From the form of the equation, it is readily seen that the ellipse is symmetrical with respect to both axes.

When the equation of the ellipse is solved for y,

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2},$$

from which it is seen that y is imaginary for values of x numerically greater than a, and hence the curve lies entirely between the lines $x = -a$ and $x = +a$.

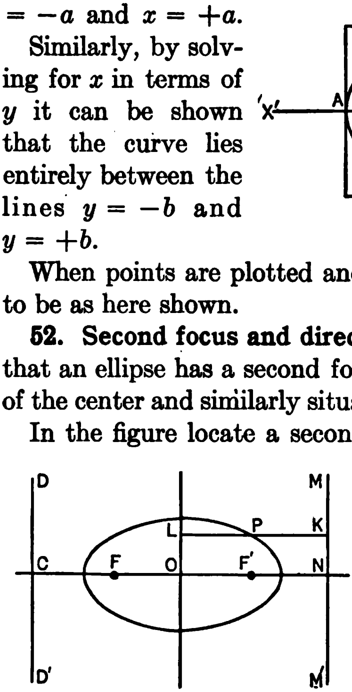


Similarly, by solving for x in terms of y it can be shown that the curve lies entirely between the lines $y = -b$ and $y = +b$.

When points are plotted and the curve drawn, it is found to be as here shown.

52. Second focus and directrix. — It will now be proved that an ellipse has a second focus and directrix on the right of the center and similarly situated with respect to the center.

In the figure locate a second focus F', making $OF' = FO = ae$. Also draw a second directrix MM' parallel to DD' meeting the principal axis at N and making $ON = CO = a/e$. It will now be shown that the ellipse which has F' for focus and MM' for directrix has the same equation and therefore is the same ellipse as the one having F as focus and DD' as directrix.



Let $P\,(x, y)$ represent any point on the ellipse whose focus is F' and whose directrix is MM'. Draw PK perpendicular to the directrix meeting it in K and meeting the y-axis in L. Then by the definition of an ellipse, $\dfrac{F'P}{PK} = e$,

but $\quad F'P = \sqrt{(x - ae)^2 + y^2}$, from formula (1),

and $\quad PK = LK - LP = a/e - x$,

whence
$$\frac{\sqrt{(x-ae)^2+y^2}}{a/e-x}=e.$$

Clearing and collecting,
$$x^2(1-e^2)+y^2=a^2(1-e^2),$$

which is the same as equation (4) of the previous article, in which F is the focus and DD' the directrix.

53. *The latus rectum of the ellipse is the chord through either focus parallel to the directrix. Its length is* $2\,b^2/a$.

Proof. — The equation of this chord is $x = \pm ae$.

Solving simultaneously with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $y = \pm b\sqrt{1-e^2} = \pm b^2/a$, since $b^2 = a^2(1-e^2)$. Therefore the latus rectum, which is twice the ordinate at the focus, is equal to $2\,b^2/a$.

54. The second standard equation of an ellipse. — *The equation of an ellipse whose major axis is on the y-axis and whose center is at the origin is*

$$\frac{y^2}{a^2}+\frac{x^2}{b^2}=1, \qquad (29)$$

where a and b are the semi-major and semi-minor axes, respectively.

Proof. — Rotating the axes through 90°, equation (25) becomes

$$\frac{(x\cos 90^\circ - y\sin 90^\circ)^2}{a^2}+\frac{(x\sin 90^\circ + y\cos 90^\circ)^2}{b^2}=1,$$

or
$$\frac{y^2}{a^2}+\frac{x^2}{b^2}=1.$$

The third standard equation of an ellipse. — *The equation of an ellipse whose major axis is parallel to the x-axis and whose center is at the point (h, k) is*

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1, \qquad (30)$$

where a and b are the semi-major and semi-minor axes respectively.

Proof. — The proof is identical to that given in deriving the third standard equation of the parabola.

The fourth standard equation of an ellipse. — *The equation of an ellipse whose major axis is parallel to the y-axis and whose center is at the point (h, k) is*

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1, \tag{31}$$

where a and b are the semi-major and semi-minor axes respectively.

Proof as above.

ILLUSTRATIVE EXAMPLES

1. An ellipse with semi-minor axis equal to 5 and passing through the point (6, 4) has its center at the origin and its major axis on the x-axis. Find the equation of the ellipse, the coördinates of the foci and the equations of the directrices.

Substituting the value of $b = 5$ in standard equation (25),

$$\frac{x^2}{a^2} + \frac{y^2}{25} = 1. \tag{1}$$

This is the equation of a family of ellipses all having 5 as semi-minor axis.

This ellipse must pass through the point (6, 4) whence the coördinates of this point satisfy equation (1).

Therefore, $$\frac{36}{a^2} + \frac{16}{25} = 1. \tag{2}$$

Solving, $$a^2 = 100.$$

Substituting back in equation (1),

$$\frac{x^2}{100} + \frac{y^2}{25} = 1.$$

Since $b^2 = a^2(1 - e^2)$, therefore $25 = 100(1 - e^2)$ and hence $e = \frac{\sqrt{3}}{2}$.

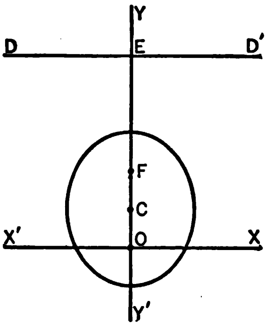
The distance from the center to the focus is $ae = 5\sqrt{3}$ and from the center to the directrix is $a/e = 20/\sqrt{3}$.

The coördinates of the foci are, therefore, $(\pm 5\sqrt{3}, 0)$ and the equations of the directrices $x = \pm 20/\sqrt{3}$.

2. Find the equation of the ellipse one of whose foci is at $(0, 2)$, the equation of whose corresponding directrix is $y = 5$ and whose eccentricity equals $\frac{1}{2}$.

The data given shows that the ellipse is in the fourth standard form

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1.$$



Locate the focus F at $(0, 2)$ and draw the directrix DD' 5 units above the origin meeting the y-axis at E. Then $CE = a/e$, $CF = ae$, whence by subtraction,

$$\frac{a}{e} - ae = FE = 3.$$

Substituting $e = \frac{1}{2}$, a is found to be 2.

$$b^2 = a^2(1 - e^2) = 4\left(1 - \tfrac{1}{4}\right) = 3.$$

$OC = OF - CF$. Since $OF = 2$ and $CF = ae = 1$, therefore $OC = 1$.

The coördinates of the center then are $(0, 1)$ and the equation is

$$\frac{(y-1)^2}{4} + \frac{x^2}{3} = 1.$$

EXERCISES

1. Determine the vertices, foci, equations of directrices, and length of latus rectum for each of the following ellipses. Plot each curve.

(*a*) $9x^2 + 25y^2 = 225$.
(*b*) $3x^2 + 4y^2 = 48$.
(*c*) $16y^2 + 25x^2 = 400$.
(*d*) $4x^2 + 9y^2 = 36$.
(*e*) $4x^2 + 3y^2 = 108$.
(*f*) $\dfrac{x^2}{36} + \dfrac{y^2}{64} = 1$.

2. Find the equations of the following ellipses which have their centers at $(0, 0)$, major axis along the x-axis. Construct the figures.

(*a*) Semi-major axis $= 6$, $e = \frac{1}{3}$.
(*b*) Distance between the foci $= 6$, $e = \frac{1}{2}$.
(*c*) Minor axis $= 12$, a focus at $(8, 0)$.
(*d*) Equation of a directrix is $x = 6$, $e = \frac{1}{3}$.
(*e*) A focus at $(3, 0)$, the equation of the corresponding directrix, $x = \frac{25}{3}$.

(*f*) Major axis = 16, and (4, 3) is a point on the curve.
(*g*) Minor axis = 4, and (3, 1) is a point on the curve.
(*h*) The two points (4, 2) and $(\sqrt{6}, 3)$ are on the curve.
(*i*) Latus rectum = 3, $e = \frac{1}{2}$.
(*j*) Latus rectum = 9, one vertex (8, 0).

3. Find the equations of the following ellipses, the coördinates of foci and vertices and length of latus rectum. Draw each curve.

(*a*) Center $(-1, -2)$, major axis = 6 and parallel to y-axis, minor axis = 4.
(*b*) Center $(-4, -2)$, major axis = 10 and parallel to x-axis, minor axis = 8.
(*c*) Center (0, 5), one vertex (0, 0), $e = \frac{4}{5}$.

4. Find the equations of the following ellipses:

(*a*) Center $(-1, -2)$, major axis = 12, latus rectum equal to one half of minor axis, principal axis parallel to x-axis.
(*b*) Major axis = 10, foci at $(-1, 3)$ and $(-1, -5)$.
(*c*) Minor axis = 6, foci $(-3, 4)$ and $(5, 4)$.
(*d*) Center at (2, 1), major axis = 8 and parallel to x-axis, and the center twice as far from the vertex as from the focus.

5. By translation of axes reduce each of the following equations to standard forms (25) or (29). Draw both sets of axes and the curve.

(*a*) $x^2 - 2x + 2y^2 - 4y + 1 = 0$.
(*b*) $x^2 - 6x + 4y^2 - 8y - 3 = 0$.

6. Prove that in the ellipse $\frac{x^2}{16} + \frac{y^2}{8} = 1$, the line joining the positive ends of the axes is parallel to the line joining the center to the upper end of the left hand latus rectum.

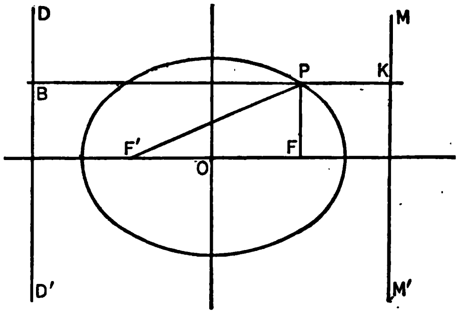
7. Find the equation of the circle whose diameter is the major-axis of the ellipse $9x^2 + 25y^2 = 225$ and whose center is at the center of the ellipse. Find the coördinates of the points where the right hand latus rectum produced, cuts the circle.

8. Find the equations of the lines through the left hand focus of $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and the extremities of the right hand latus rectum. Find the distances of these lines from the origin.

9. Find the equation of the locus of a point which moves so that the sum of the distances from the two points (0, 4) and (0, −4) is equal to 10. Prove that the locus is an ellipse.

55. Construction of an ellipse. — A proposition which readily leads to the construction of the ellipse is as follows:

The sum of the focal distances of any point on an ellipse is constant and equal to the major axis.



Proof. — Draw the ellipse with foci F and F' and directrices DD' and MM'. From P (x, y), any point on the ellipse, draw PK perpendicular to the directrices meeting them in B and K respectively.

From the definition of an ellipse,

$$F'P = e\,(BP) = e\left(\frac{a}{e} + x\right) = a + ex. \qquad (1)$$

Similarly,

$$FP = e\,(PK) = e\left(\frac{a}{e} - x\right) = a - ex. \qquad (2)$$

Adding (1) and (2),

$$FP + F'P = 2\,a = \text{major axis.}$$

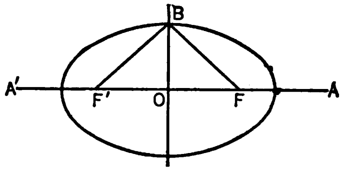
This fact leads to a second and important definition of an ellipse:

An ellipse is the locus of a point which moves so that the sum of its distances from two fixed points is constant.

From this definition, an ellipse can be constructed as follows, if the foci and the length of the major axis are given:

In a drawing board, fasten a tack at each focus F and F'. Tie about the tacks a string equal in length to the distance $FF' + 2\,a$ and with a pencil point hold the string taut while describing the curve. The locus will be an ellipse since the sum of the focal distances is always $2a$.

By use of this property, it can be shown that the foci are at a distance a from the extremities of the minor axis. Hence, to locate the foci, take an extremity of the minor axis B as center and with a radius equal to the semi-major axis describe an arc cutting the major axis in two points F and F'. These points are its foci.



56. General equation of an ellipse, axes parallel to coördinate axes. — When equations (30) and (31) are expanded, they become

$$b^2x^2 - 2b^2hx + a^2y^2 - 2a^2ky + b^2h^2 + a^2k^2 - a^2b^2 = 0$$

and

$$b^2y^2 - 2b^2ky + a^2x^2 - 2a^2hx + a^2h^2 + b^2k^2 - a^2b^2 = 0.$$

Either of these equations is of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$, in which A and C are positive and different.

It will be shown that every equation of the above type represents an ellipse.

Completing squares and collecting,

$$A(x + D/2A)^2 + C(y + E/2C)^2 = \frac{CD^2 + AE^2 - 4ACF}{4AC}.$$

After dividing by the second member, this becomes

$$\frac{(x + D/2A)^2}{\frac{CD^2 + AE^2 - 4ACF}{4A^2C}} + \frac{(y + E/2C)^2}{\frac{CD^2 + AE^2 - 4ACF}{4AC^2}} = 1,$$

which is of standard form (30) or (31).

Whether the major axis is parallel to the x-axis or the y-axis will depend upon whether the first or second denominator is the larger. If A and C are equal, the axes of the ellipse are equal and the figure is a circle. If the denominators are negative, the axes are imaginary and the ellipse impossible; if zero, it is a point-ellipse.

It is seen from the foregoing that:

The general equation of second degree, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, *represents an ellipse with axes parallel to the coördinate axes, if* $B = 0$, *and if* A *and* C *have like signs but different numerical values.*

ILLUSTRATIVE EXAMPLE

Determine for the ellipse $9x^2 + 25y^2 + 18x - 50y = 191$, center, foci, vertices, semi-axes, latus rectum, and equations of directrices.

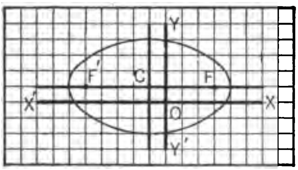
Completing the squares,

$$9(x+1)^2 + 25(y-1)^2 = 225.$$

Dividing by 225,

$$\frac{(x+1)^2}{25} + \frac{(y-1)^2}{9} = 1.$$

On comparing with standard equation (30), it is seen that the center is at $(-1, 1)$, the semi-major axis is 5 and the semi-minor axis 3. The principal axis is parallel to the x-axis. Sketching in a figure and using all the data obtained, it is seen that the vertices are at $(4, 1)$ and $(-6, 1)$.



Since $b^2 = a^2(1 - e^2)$ and $b = 3$ and $a = 5$, therefore $e = \frac{4}{5}$. The distance from the center to the focus is $ae = 4$. The coördinates of the foci then are $(3, 1)$ and $(-5, 1)$.

The latus rectum $= 2b^2/a = \frac{18}{5}$. The distance from the center to the directrix is $a/e = \frac{25}{4}$. Hence the equations of the directrices are $x = \frac{21}{4}$ and $x = -\frac{29}{4}$.

EXERCISES

1. Determine for the following ellipses, center, foci, vertices, semi-axes, latus rectum, and equations of directrices.

(*a*) $4x^2 + 16x + 3y^2 - 6y = 29$.

(*b*) $7y^2 + 14y + 16x^2 - 64x = 41$.

(*c*) $4x^2 - 8x + 8y^2 - 64y + 68 = 0.$

(*d*) $x^2 + 4y^2 + 6x - 8y = 87.$

(*e*) $3x^2 + 6x + 4y^2 + 24y = 69.$

(*f*) $9x^2 + 54x + 8y^2 - 16y = 199.$

2. Find the equation of an ellipse whose foci are at (3, 0) and (−3, 0) and the sum of whose focal radii is 10.

3. Prove that the point (4, 1) is on the ellipse $\frac{x^2}{18} + \frac{y^2}{9} = 1.$ Find the focal distances of the point and prove that their sum is equal to the major axis.

CHAPTER IX

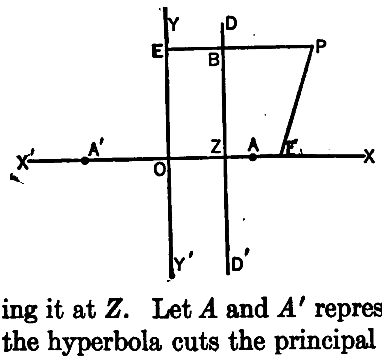
THE HYPERBOLA

57. The hyperbola is that conic section traced by a point which moves so that the ratio of its distance from a fixed point called the focus to its distance from a fixed line called the directrix is constant and greater than 1.

It was shown in Art. 47 that the hyperbola cuts the principal axis in two points on opposite sides of the directrix. The simplest form of the equation of an hyperbola is obtained by taking the principal axis as the x-axis and a point halfway between the two intersections as origin. This point is called the **center** of the hyperbola.

The first standard equation of an hyperbola. — *The equation of an hyperbola whose transverse axis is on the x-axis and whose center is at the origin is*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{32}$$



in which a and b are the semi-transverse and semi-conjugate axes respectively.

Proof. — Let the directrix of the hyperbola be DD' and take the x-axis on the principal axis which is perpendicular to DD' through the focus F, meeting it at Z. Let A and A' represent the two points at which the hyperbola cuts the principal axis.

These two points are called the **vertices** of the hyperbola.

At O, midway between A and A', erect the y-axis. Call the distance $A'O = OA = a$.

Take $P(x, y)$, any point on the hyperbola, and drop PB perpendicular to the directrix, cutting the y-axis at E.

From the definition of an hyperbola,

$$\frac{FP}{BP} = e. \tag{1}$$

The values of FP and BP are found in a manner almost identical to that used in the case of the ellipse, use being made of the fact that the points A and A' are on the hyperbola, and hence

$$\frac{AF}{ZA} = e \tag{2}$$

and

$$\frac{A'F}{A'Z} = e. \tag{3}$$

Whence $\quad A'F + AF = e\,(A'Z + ZA)$

or $\quad (a + OF) + (OF - a) = 2\,ae$ and $OF = ae$.

The distance from the center of an hyperbola to the focus is $\boldsymbol{ae}$. **(33)**

Similarly, $\quad A'F - AF = e\,(A'Z - ZA)$

or $\quad 2\,a = e\,[(a + OZ) - (a - OZ)]$ and $OZ = a/e$.

The distance from the center of an hyperbola to the directrix is $\boldsymbol{a/e}$. **(34)**

The coördinates of F, then, are $(ae, 0)$,

whence $\quad FP = \sqrt{(x - ae)^2 + y^2}$, by formula (1).

$\quad BP = EP - EB = x - a/e$.

Substituting in equation (1),

$$\frac{\sqrt{(x - ae)^2 + y^2}}{x - a/e} = e.$$

Clearing of fractions and collecting,

$$x^2(e^2-1) - y^2 = a^2(e^2-1).$$

If $x = 0$, $y = \pm a\sqrt{1-e^2}$, which since $e > 1$, is imaginary, and the curve does not cross the y-axis.

It is found convenient to make the substitution

$$b^2 = a^2(e^2-1).$$

The equation of the hyperbola then becomes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The portion of the principal axis which is cut off by the hyperbola is called the **transverse axis.** It is represented by $2a$.

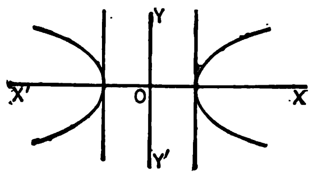
The segment on the perpendicular to the principal axis through the center such that its length is $2b = 2a\sqrt{e^2-1}$ is called the **conjugate axis.**

Since $b^2 = a^2(e^2-1)$, it is readily seen that

$$ae = \sqrt{a^2+b^2}. \qquad (35)$$

From the form of the equation, it is evident that the hyperbola is symmetrical with respect to both axes.

When the equation of the hyperbola is solved for y, $y = \pm\frac{b}{a}\sqrt{x^2-a^2}$, from which it is seen that y is imaginary for all values of x numerically less than a, and hence no part of the curve lies between the lines $x = -a$ and $x = a$. For all values of x numerically greater than a, y is real, showing that the curve extends indefinitely both right and left.



Similarly, by solving for x in terms of y, $x = \pm\frac{a}{b}\sqrt{b^2+y^2}$, from which it is seen that for every value of y, x is real and hence the curve extends indefinitely above and below the x-axis.

When points are plotted and the curve drawn it is found to be as shown.

It can be proved, as in the case of the ellipse, that the hyperbola has a second focus at $(-ae, 0)$ and a second directrix whose equation is $x = -a/e$.

58. *The* **latus rectum** *of the hyperbola is the chord through either focus parallel to the directrix. Its length is* $2\,b^2/a$.

Proof. — The equation of this chord is $x = \pm ae$.

Solving simultaneously with the equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $y = \pm b\sqrt{e^2 - 1} = \pm b^2/a$, since $b^2 = a^2(e^2 - 1)$.

Therefore the latus rectum, which is twice the ordinate at the focus, is equal to $2\,b^2/a$.

59. The second standard equation of an hyperbola. — *The equation of the hyperbola whose transverse axis is on the y-axis and whose center is at the origin is*

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, \tag{36}$$

where a and b are the semi-transverse and semi-conjugate axes, respectively.

The proof is left to the student. It is identical to that used in the case of the ellipse.

The third standard equation of an hyperbola. — *The equation of an hyperbola whose transverse axis is parallel to the x-axis and whose center is at the point (h, k) is*

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1, \tag{37}$$

where a and b are the semi-transverse and semi-conjugate axes, respectively. The proof is left to the student.

The fourth standard equation of an hyperbola. — *The equation of an hyperbola whose transverse axis is parallel to the y-axis and whose center is at the point (h, k) is*

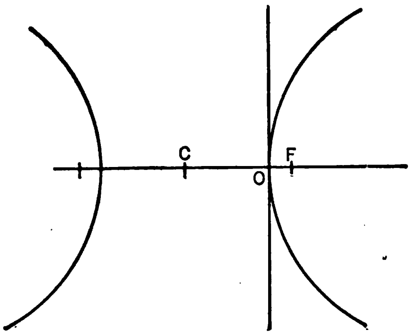
$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1, \tag{38}$$

where a and b are the semi-transverse and semi-conjugate axes, respectively. The proof is left to the student.

ILLUSTRATIVE EXAMPLE

An hyperbola in which the distance between the foci is 10 passes through the origin, has one focus at (1, 0), and its transverse axis on the x-axis. Find its equation.

Locate in a figure the center C, the focus F, and the vertex O. It is seen from the data given that the equation is in the form of the third standard equation,



$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1.$$

Here $CF = 5$ and $OF = 1$, therefore $CO = 4 = a$, and the coördinates of C are $(-4, 0)$.

Since $CF = \sqrt{a^2+b^2}$, then $b = 3$.

The equation then becomes

$$\frac{(x+4)^2}{16}-\frac{y^2}{9}=1.$$

EXERCISES

1. Determine lengths of axes and latus rectum, coördinates of vertices and foci and equations of directrices for each of the following hyperbolas. Plot each curve.

(a) $9x^2 - 25y^2 = -225$.
(b) $3x^2 - 4y^2 = 48$.
(c) $9y^2 - 4x^2 + 36 = 0$.
(d) $25x^2 - 16y^2 + 400 = 0$.
(e) $4x^2 - 3y^2 = 108$.
(f) $\frac{x^2}{36} - \frac{y^2}{64} = 1$.

2. Find the equations of the following hyperbolas having their centers at (0, 0) and their transverse axes along the x-axis. Construct the curves.

(a) Transverse axis = 4 and conjugate axis equal to one-half the distance between the foci.
(b) Transverse axis = 6 and (5, 3) is a point on the curve.
(c) Latus rectum = 10 and $e = \frac{3}{2}$.
(d) Transverse axis = 12 and a focus is at (8, 0).
(e) Distance between the foci = 8 and $e = \frac{4}{3}$.
(f) Latus rectum = 2 and $a = 2b$.
(g) $e = 2$ and distance from focus to nearest vertex = 1.

3. Find the equations of the following hyperbolas which have their centers at (0, 0) and their transverse axes along the y-axis. Construct the figures.

(*a*) Latus rectum $= 3$ and one vertex at (0, 2).
(*b*) Conjugate axis $= 8$ and (4, 6) is a point on the curve.
(*c*) $e = 2$ and the equation of a directrix is $y = 3$.
(*d*) One focus at (0, 6) and the equation of the corresponding directrix is $y = \frac{25}{6}$.
(*e*) The two points (3, 4) and (6, 7) are on the curve.

4. Find the equation of each of the following hyperbolas, determine the coördinates of foci and vertices and length of latus rectum:

(*a*) Center $(-1, 3)$, transverse axis $= 8$ and parallel to y-axis, conjugate axis $= 10$.
(*b*) Center $(-2, -3)$, transverse axis parallel to x-axis and $= 8$, conjugate axis $= 12$.
(*c*) Vertices are $(-1, -1)$ and $(-1, 7)$ and $e = 2$.

5. Find the equations of the following hyperbolas:

(*a*) Center at (2, 1), transverse axis $= 6$ and parallel to the x-axis and the center twice as far from the focus as from the vertex.
(*b*) $e = 2$, one focus at $(-1, -2)$ and the corresponding directrix $y = 4$.
(*c*) $2a = 6$ and foci at $(-2, -4)$ and $(-2, 6)$.
(*d*) Center $(-1, -3)$, transverse axis $= 8$ and parallel to the y-axis and the latus rectum equal to one-half of conjugate axis.
(*e*) Transverse axis $= 4$, one directrix is $x = 6$ and the corresponding focus $(3, -1)$.
(*f*) Vertices at (2, 5) and $(2, -1)$ and latus rectum $=$ transverse axis.

6. Prove that the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ and the hyperbola $\frac{x^2}{9} - \frac{y^2}{7} = 1$ have the same center and foci. Construct each curve.

7. By translation of axes reduce each of the following equations to standard forms. Draw both sets of axes and the curves.

(*a*) $9x^2 - 36x - 4y^2 - 24y = 36$.
(*b*) $x^2 + 6x - y^2 + 2y + 12 = 0$.

8. Find the equation of the locus of the point which moves so that the difference of its distances from the two points (6, 0) and $(-6, 0)$ is equal to 8. Prove that the locus is an hyperbola.

9. Prove that the foci of the hyperbolas $y^2/9 - x^2/16 = 1$ and $x^2/16 - y^2/9 = 1$ are equidistant from the center.

10. By rotation of axes, remove the xy-term from the equation $xy = 18$. Show that the curve is an hyperbola and construct both sets of axes and the curve.

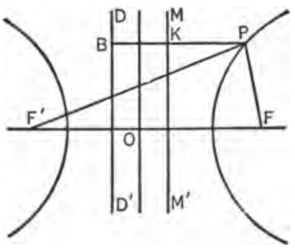
11. Prove that the latus rectum of an hyperbola is a third proportional to the transverse and conjugate axes.

12. Find the polar equation of the hyperbola $x^2/a^2 - y^2/b^2 = 1$.

13. Prove that the point (4, 1) is on the hyperbola $x^2/8 - y^2/1 = 1$ and that the difference of its focal distances is equal to the transverse axis.

60. Construction of an hyperbola. — A proposition which readily leads to the construction of the hyperbola is as follows:

The difference of the focal distances of any point on an hyperbola is constant and equal to the transverse axis.



Proof. — Draw the hyperbola with foci at F and F' and directrices DD' and MM'. From $P\ (x, y)$, any point on the hyperbola, draw PK perpendicular to the directrices and meeting them in B and K respectively.

From the definition of an hyperbola,

$$FP = e\,(KP) = e\,(x - a/e) = ex - a. \tag{1}$$

Similarly,

$$F'P = e\,(BP) = e\,(x + a/e) = ex + a. \tag{2}$$

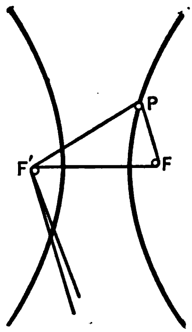
Subtracting (1) from (2),

$$F'P - FP = 2\,a = \text{transverse axis}.$$

This fact leads to a second and important definition of an hyperbola:

An hyperbola is the locus of a point which moves so that the difference of its distances from two fixed points is constant.

From this definition, an hyperbola can be constructed as follows, if the foci and length of the transverse axes are given.



In a drawing board fasten a tack at each focus F and F'. Let a pencil be tied to a string at P. Let one end of the string pass beneath F and then both ends over F' as shown. Adjust the string so that $F'P$ exceeds FP by $2a$. By holding the strings together below F' and pulling them in or letting them out, the point P will, if held firmly against the string, trace an hyperbola, for at each position $F'P - FP = 2a$.

By reversing the process, the other branch may be drawn.

61. General equation of an hyperbola, axes parallel to coördinate axes. — When equations (37) and (38) are expanded, they become

$$b^2x^2 - 2b^2hx + b^2h^2 - a^2y^2 + 2a^2ky - a^2k^2 - a^2b^2 = 0,$$
$$b^2y^2 - 2b^2ky + b^2k^2 - a^2x^2 + 2a^2hx - a^2h^2 - a^2b^2 = 0.$$

Either of these equations is of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

in which A and C have opposite signs.

It will now be shown that every equation of the above type represents an hyperbola.

Completing the squares and collecting,

$$A(x + D/2A)^2 + C(y + E/2C)^2 = \frac{CD^2 + AE^2 - 4ACF}{4AC}.$$

After dividing by the second member, this becomes

$$\frac{(x + D/2A)^2}{\frac{CD^2 + AE^2 - 4ACF}{4A^2C}} + \frac{(y + E/2C)^2}{\frac{CD^2 + AE^2 - 4ACF}{4AC^2}} = 1,$$

which is of standard form (37) or (38), since A and C have unlike signs and hence the denominators have unlike signs.

Whether the transverse axis is parallel to the x-axis or to the y-axis will depend upon whether the first or second denominator is positive.

It is seen from the foregoing that:

*The general equation of second degree, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, represents an hyperbola if $B = 0$ and if A and C have unlike signs.**

EXERCISES

1. Determine for each of the following hyperbolas, the center, semi-axes, foci, vertices, and latus rectum. Construct each curve.

(*a*) $9x^2 - 18x - 4y^2 + 16y - 43 = 0.$
(*b*) $4x^2 - 24x - 16y^2 - 64y + 36 = 0.$
(*c*) $x^2 - 6x - 9y^2 - 18y + 9 = 0.$
(*d*) $3y^2 + 6y - x^2 + 2x + 11 = 0.$
(*e*) $8x^2 - 8x - 28y^2 - 28y = 61.$
(*f*) $25y^2 - 4x^2 - 50y - 39 = 0.$

2. Find the equation of an hyperbola with $e = \sqrt{2}$, the line $x - y = 4$ as one directrix and the corresponding focus at $(-1, -1)$.

3. By rotation of axes, reduce the equation $xy + 50 = 0$ to one of the standard forms of the equation of an hyperbola. Draw both sets of axes and the curve.

4. Find the equation of an hyperbola whose foci are $(0, 8)$ and $(0, -8)$ and the difference of whose focal radii is 10.

5. Find the equation of the hyperbola whose center is at $(1, 1)$, whose transverse axis is parallel to the x-axis and which passes through $(6, 5)$ and $(-7, -7)$.

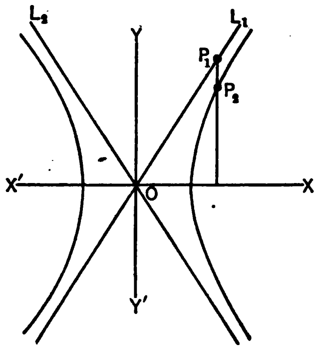
62. Asymptotes to the hyperbola. — The two lines represented by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

have a very important relation to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

* The general equation of second degree will under these same conditions sometimes take the form of the difference of two squares, in which case it will represent a pair of straight lines. This will be the case when $CD^2 + AE^2 - 4ACF$ in the equation above is zero.

It will now be shown that as a point on the hyperbola recedes indefinitely, the curve approaches coincidence with these lines and therefore these are the asymptotes to the curve.



Let L_1 and L_2 be the two lines represented by the equation $x^2/a^2 - y^2/b^2 = 0$, and take P_1 (x_1, y_1), any point on these lines, such that a perpendicular from it on the axis of x meets the hyperbola $x^2/a^2 - y^2/b^2 = 1$ in the point P_2 (x_1, y_2).

Then $$y_1 = \pm\frac{b}{a}x_1,$$

$$y_2 = \pm\frac{b}{a}\sqrt{x_1^2 - a^2}.$$

Subtracting,

$$y_1 - y_2 = \pm\frac{b}{a}\left(x_1 - \sqrt{x_1^2 - a^2}\right).$$

Rationalizing the numerator,

$$y_1 - y_2 = \pm\frac{b}{a}\,\frac{[x_1^2 - (x_1^2 - a^2)]}{x_1 + \sqrt{x_1^2 - a^2}} = \frac{\pm ab}{x_1 + \sqrt{x_1^2 - a^2}},$$

which approaches zero as x_1 recedes to infinity.

This shows that as the curve recedes to infinity, it approaches indefinitely close to the lines

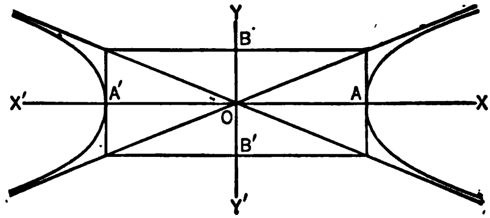
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0,$$

which are therefore the asymptotes to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

If the distances $A'O = OA = a$ and $B'O = OB = b$ are laid off on the x-axis and y-axis respectively, and parallels to the axes through A, B, A', and B' are drawn, the diagonals of the rectangle thus formed will be the asymptotes.

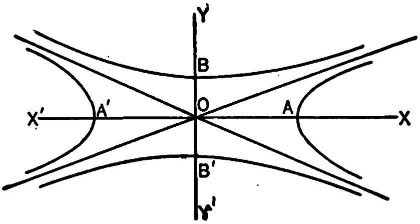
It is often found convenient in constructing an hyperbola to first construct this rectangle and the asymptotes and then draw in the hyperbola touching the rectangle at A and A' and approaching the asymptotes as it recedes to infinity.



63. Conjugate hyperbolas. — If in two hyperbolas the transverse axis in each is the conjugate axis in the other, the hyperbolas are said to be *conjugate*.

Thus, in the figure, if $A'A$ $(= 2a)$ is the transverse axis and $B'B$ $(= 2b)$ the conjugate axis, then the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$



If another hyperbola is constructed in which $B'B$ is the transverse axis and $A'A$ the conjugate axis, the equation is

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

In either case the equation can be written

$$\frac{(\text{Distance from conjugate axis})^2}{(\text{Semi-transverse axis})^2} - \frac{(\text{Distance from transverse axis})^2}{(\text{Semi-conjugate axis})^2} = 1.$$

If the eccentricity of the first hyperbola is represented by e_1 and of the conjugate hyperbola by e_2, then

$$e_1 = \frac{\sqrt{a^2 + b^2}}{a} \quad \text{and} \quad e_2 = \frac{\sqrt{b^2 + a^2}}{b},$$

hence,
$$ae_1 = be_2 = \sqrt{a^2 + b^2}.$$

This shows that the foci of the two hyperbolas are equidistant from the center and thus the foci of an hyperbola and its conjugate lie on a circle about the center with a diameter equal to the diagonal of the rectangle constructed as in the last article.

The asymptotes to the conjugate hyperbola $y^2/b^2 - x^2/a^2 = 1$ are found to be $y = \pm (b/a)\, x$, from which it is seen that the hyperbola and its conjugate have the same asymptotes.

64. Equilateral or rectangular hyperbola. — If $b = a$ in the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

it becomes
$$x^2 - y^2 = a^2.$$

The equation of the conjugate hyperbola is $y^2 - x^2 = a^2$. These are evidently equal hyperbolas.

The asymptotes of these hyperbolas are $y = \pm x$, two lines making angles of 45° and 135° with the x-axis and therefore at right angles with each other.

Since the semi-axes a and b are equal, these hyperbolas are sometimes called **equilateral**; since the asymptotes are at right angles to each other, they are sometimes called **rectangular hyperbolas.**

EXERCISES

1. Given the hyperbola $9\,x^2 - y^2 = 36$, find the equations of the asymptotes and of the conjugate hyperbola. Construct the hyperbolas and the asymptotes.

2. Find the equations of the asymptotes to the hyperbola $9x^2 - 16y^2 = 144$ and the tangent of the angle between them.

3. Write the equation of an hyperbola conjugate to the hyperbola $4x^2 - 9y^2 = 36$ and find the lengths of its axes and latus rectum, the coördinates of its foci and the equations of its directrices.

4. Prove that the distance from an asymptote to a focus is equal to the semi-conjugate axis.

5. Find the equation of an hyperbola whose foci are at the points $(5, 0)$ and $(-5, 0)$, the inclination of one of whose asymptotes is $30°$.

6. Find the equation of an hyperbola whose transverse axis is along the x-axis, which passes through the point $(5, 8)$ and whose asymptotes are given by the equation $y^2 = 4x^2$.

7. Write the equation of an hyperbola conjugate to the hyperbola $x^2 - 2x - 4y^2 - 8y = 7$.

8. Show by the method of Art. 62, that the equation of the asymptotes to the hyperbola $4x^2 - y^2 = 4$ is $4x^2 = y^2$.

9. Find the equation of the rectangular hyperbola $x^2 - y^2 = a^2$ referred to its asymptotes.

10. Find the equation of the asymptotes to the hyperbola $x^2 - y^2 = 9$. Prove that any line parallel to an asymptote meets the curve in only one finite point.

11. If e_1 and e_2 are the eccentricities of two conjugate hyperbolas,

prove that
$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1.$$

12. In an hyperbola, if the value of e is very little more than unity, how does the value of b compare with that of a? Discuss the slope of asymptotes and form of curve. As e increases, what is the effect on the slope of the asymptotes and the form of the curve?

13. Prove that the distance of any point on the rectangular hyperbola $x^2 - y^2 = a^2$ from the center is a mean proportional to its distances from the foci.

14. Prove that the product of the distances of any point on an hyperbola from its asymptotes is constant.

15. Find the coördinates of the foci of the hyperbola $x^2 - y^2 = 9$. By rotating the axes, find the coördinates of the foci and the equation of the hyperbola when its asymptotes are taken as axes.

16. Find the equation of an hyperbola whose transverse axis is along the x-axis, which passes through the point $(5, 2)$ and has the same asymptotes as $4x^2 - 9y^2 = 36$.

17. Prove that an asymptote and the perpendicular from the focus upon it meet upon the corresponding directrix.

18. Prove that the directrices of an hyperbola and the circle whose diameter is the line joining the foci intersect on the conjugate hyperbola.

19. Prove that the portion of the asymptotes intercepted between the directrices is equal to $2a$.

20. Through the point P_1 on the hyperbola $x^2/a^2 - y^2/b^2 = 1$, a straight line is drawn parallel to the transverse axis cutting the asymptotes. Prove that the product of the distances of P_1 from these points of intersection is equal to a^2.

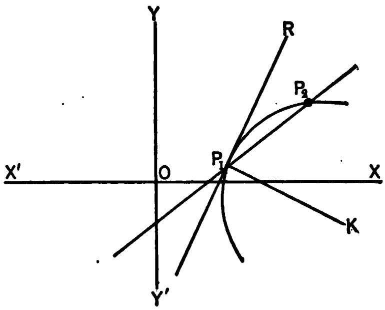
21. If the crack of a rifle and the thud of the ball on the target are heard at the same instant, prove that the locus of the hearer is an hyperbola.

CHAPTER X

TANGENTS AND NORMALS

65. A line which cuts a curve is called a **secant.** The line P_1P_2 in the figure is such a line.

If one of the points of intersection, as P_2, is made to move along the curve and approach the other point P_1, the line P_1P_2 will approach a limiting position P_1R. This line is called the **tangent** to the curve. The point P_1 is called the **point of contact** of the tangent line. The following definition may then be stated: *A tangent to a curve at a given point is the limiting position of the secant line connecting the given point with a second point on the curve, as this second point moves along the curve and approaches coincidence with the given point.*



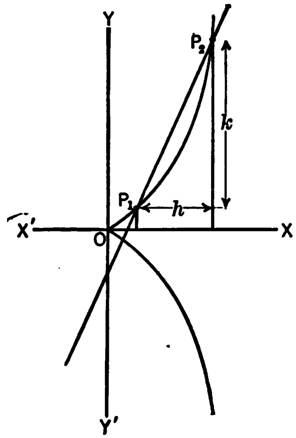
The line which is perpendicular to the tangent at the point of contact is called the ***normal*** *to the curve.* The line P_1K is the normal to the curve at P_1.

The equations of the tangent and normal to any curve at a given point on the curve. — A point on the tangent line is given. If the slope of the tangent can be determined, its equation can be readily found by substitution in the standard equation,

$$y - y_1 = m(x - x_1).$$

The method of finding this slope will now be illustrated. Let it be required to find the slope of the tangent to the curve $y^2 = x^3$ at the point $P_1\ (x_1, y_1)$ on the curve.

Let $P_2\ (x_1 + h, y_1 + k)$ represent a second point on the curve. Then by the formula



$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

the slope of P_1P_2 is found to be $\frac{k}{h}$.

If the point P_2 is now made to approach P_1, the values of h and k each approach 0, and the slope takes the indeterminate form of $\frac{0}{0}$. This difficulty arises from having failed to make use of the fact that the points P_1 and P_2 lie on the curve and thus their coördinates must satisfy the equation of the curve.

Substituting these in the equation $y^2 = x^3$,

$$y_1^2 = x_1^3, \tag{1}$$

$$(y_1 + k)^2 = (x_1 + h)^3. \tag{2}$$

Expanding and subtracting (1) from (2),

$$2\,y_1k + k^2 = 3\,x_1^2h + 3\,x_1h^2 + h^3, \tag{3}$$

or $$k\,(2\,y_1 + k) = h\,(3\,x_1^2 + 3\,x_1h + h^2). \tag{4}$$

Whence the slope of the secant,

$$\frac{k}{h} = \frac{3\,x_1^2 + 3\,x_1h + h^2}{2\,y_1 + k}. \tag{5}$$

If now h and k are made to approach 0, the slope of the tangent =

$$\underset{h=0,\,k=0}{\text{limit}} \left(\frac{k}{h}\right) = \underset{h=0,\,k=0}{\text{limit}} \left(\frac{3\,x_1^2 + 3\,x_1h + h^2}{2\,y_1 + k}\right) = \frac{3\,x_1^2}{2\,y_1}.$$

Consequently, the equation of the tangent is

$$y - y_1 = \frac{3\,x_1^2}{2\,y_1}(x - x_1),$$

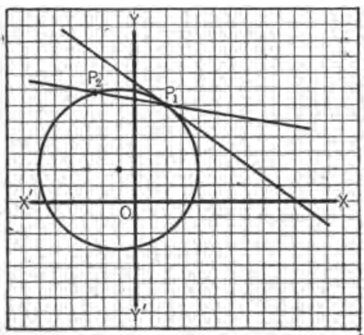
or
$$3\,x_1^2\,x - 2\,y_1 y = 3\,x_1^3 - 2\,y_1^2,$$

or
$$3\,x_1^2\,x - 2\,y_1 y = x_1^3 \text{ (since } y_1^2 = x_1^3).$$

Answers will, in general, be simplified by collecting all the variable terms in the first member of the equation and the constants in the second, and then reducing the second member to simpler form by making use of the fact that $P_1\,(x_1, y_1)$ is a point on the curve.

The steps taken may be summarized as follows: *To find the slope of the tangent to a given curve at a given point $P_1\,(x_1, y_1)$, choose a second point $P_2\,(x_1 + h, y_1 + k)$ on the curve. Substitute the coördinates of P_1 and P_2 in the given equation and subtract. Find the value of k/h, the slope of the secant. The limiting value of this slope as h and k approach zero is the slope of the tangent.*

Where the point of contact P_1 is given by numerical coördinates, the substitution of the coördinates of the second point P_2 gives sufficient data from which to determine the value of k/h.



This is illustrated in the following example. Find the equation of the tangent to the circle $x^2 + 2\,x + y^2 - 4\,y = 20$ at the point (2, 6).

Let P_1 (2, 6) and P_2 $(2 + h, 6 + k)$ be two points on the given circle.

The substitution of $P_2\,(2 + h, 6 + k)$ in the given equation gives

$$4 + 4\,h + h^2 + 4 + 2\,h + 36 + 12\,k + k^2 - 24 - 4\,k = 20;$$

from which

$$k^2 + 8k = -(h^2 + 6h) \quad \text{or} \quad k(k+8) = -h(h+6).$$

Whence the slope of the secant $= \frac{k}{h} = -\frac{h+6}{k+8}$. Letting h and k approach 0, the slope of the tangent is found to be $-\frac{3}{4}$. Therefore the equation of the tangent is

$$y - 6 = -\tfrac{3}{4}(x - 2).$$

or

$$4y + 3x = 30.$$

Since the normal to a curve is a line perpendicular to the tangent at the point of contact, therefore the slope of the normal is the negative reciprocal of the slope of the tangent at that point. The equation of the normal may then be determined by substitution in the equation

$$y - y_1 = m(x - x_1).$$

Thus, the slope of the normal to the circle given in the preceding example at the point $P_1(2, 6)$ is $+\frac{4}{3}$ and the equation of the normal is $y - 6 = \frac{4}{3}(x - 2)$ or $4x - 3y + 10 = 0$.

EXERCISES

1. Find the equation of the tangent to each of the following curves at the point (x_1, y_1):

(*a*) $xy = 4$. *Ans.* $x_1y + y_1x = 8$.
(*b*) $y^2 = 4px$. *Ans.* $yy_1 = 2p(x + x_1)$.
(*c*) $x^2 + y^2 = r^2$. *Ans.* $x_1x + y_1y = r^2$.
(*d*) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. *Ans.* $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$.
(*e*) $y^2 + ay + bx + c = 0$. *Ans.* $(2y_1 + a)y + bx + ay_1 + bx_1 + 2c = 0$.

2. Find the equations of the tangent and normal to each of the following curves at the indicated point. Draw the curve, tangent, and normal in each case.

(*a*) $y = x^3$ at $(1, 1)$.
(*b*) $y = \frac{x+3}{x-1}$ at $(2, 5)$.

(c) $y^2 = 2x^3$ at $(2, 4)$.
(d) $yx^2 = 1$ at $(-1, 1)$.
(e) $9x^2 + 25y^2 = 225$ at the positive extremity of the right hand latus rectum.
(f) $y = x^2 - 4x + 8$ at the point whose abscissa is 2.
(g) $y = x^2(x - 2)$ at the point $(3, 9)$.
(h) $y^3 = x^2$ at the point whose abscissa is 8.

66. The equation of the tangent to the curve represented by the general equation of second degree. — The most general equation of second degree is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \tag{1}$$

In order to find the tangent at the point $P_1(x_1, y_1)$ on this curve, the same steps are followed as in the preceding article. Let a second point on the curve be $P_2(x_1 + h, y_1 + k)$. Substitute the coördinates of P_1 and P_2 in equation (1).

Then $$Ax_1^2 + Bx_1y_1 + Cy_1^2 + Dx_1 + Ey_1 + F = 0, \tag{2}$$
and $$A(x_1 + h)^2 + B(x_1 + h)(y_1 + k) + C(y_1 + k)^2 + D(x_1 + h) + E(y_1 + k) + F = 0. \tag{3}$$

Expanding and subtracting (2) from (3),

$$2Ax_1h + Ah^2 + By_1h + Bx_1k + Bhk + 2Cy_1k + Ck^2 + Dh + Ek = 0, \tag{4}$$

which, on collecting, becomes

$$k(Bx_1 + 2Cy_1 + Ck + E) = -h(2Ax_1 + Ah + By_1 + Bk + D). \tag{5}$$

Then $$\frac{k}{h} = -\frac{2Ax_1 + Ah + By_1 + Bk + D}{Bx_1 + 2Cy_1 + Ck + E}. \tag{6}$$

Letting h and k approach 0

$$\text{the slope of the tangent} = -\frac{2Ax_1 + By_1 + D}{Bx_1 + 2Cy_1 + E}. \tag{7}$$

The equation of the tangent is therefore

$$y - y_1 = -\frac{2Ax_1 + By_1 + D}{Bx_1 + 2Cy_1 + E}(x - x_1). \tag{8}$$

Clearing this of fractions and placing the variable terms in the first member and the constant terms in the second,

$$\begin{aligned} 2Ax_1x + By_1x + Bx_1y + 2Cy_1y + Ey + Dx \\ = 2Ax_1^2 + 2Bx_1y_1 + 2Cy_1^2 + Dx_1 + Ey_1. \end{aligned} \tag{9}$$

From equation (2), the first three terms of the second member equal $-2Dx_1 - 2Ey_1 - 2F$.

Substituting this value in (9) and transposing,

$$\begin{aligned} 2Ax_1x + By_1x + Bx_1y + 2Cy_1y \\ + D(x + x_1) + E(y + y_1) + 2F = 0. \end{aligned} \tag{10}$$

Dividing by 2,

$$\begin{aligned} Ax_1x + \frac{B}{2}(x_1y + y_1x) + Cy_1y + \frac{D}{2}(x + x_1) \\ + \frac{E}{2}(y + y_1) + F = 0, \end{aligned} \tag{39}$$

which is the equation of the desired tangent at $P_1(x_1, y_1)$ on the curve $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. This result is very important and should be remembered.

A convenient statement is as follows: *The tangent to the curve represented by any equation of second degree is found by replacing x^2 by x_1x, y^2 by y_1y, xy by $\frac{x_1y + y_1x}{2}$, x by $\frac{x + x_1}{2}$, and y by $\frac{y + y_1}{2}$.*

ILLUSTRATIVE EXAMPLE

Find the equation of the tangent to $x^2 + 6xy + y^2 - 2x + 4y + 6 = 0$ at (x_1, y_1).

Applying the above rule, the tangent is

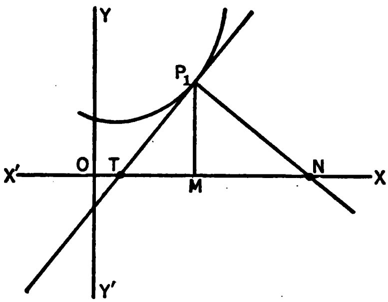
$$x_1x + 6\left(\frac{x_1y + y_1x}{2}\right) + y_1y - 2\left(\frac{x + x_1}{2}\right) + 4\left(\frac{y + y_1}{2}\right) + 6 = 0,$$

or

$$(x_1 + 3y_1 - 1)x + (3x_1 + y_1 + 2)y = x_1 - 2y_1 - 6.$$

A very convenient check on the correctness of the equation of the tangent at the point P_1 (x_1, y_1) is to drop the subscripts in the equation of the tangent and to notice that the result should be identical with the equation of the given curve.

67. Lengths of tangents and normals, subtangents and subnormals. — The tangent and normal lines are indefinite in extent, but it is customary to speak of that portion of the tangent between its intersection with the x-axis and the point of contact as the **tangent length** and that portion of the normal between the point of contact and its intersection with the x-axis as the **normal length**. In the figure, TP_1 is the tangent length and P_1N is the normal length.



The projection of the tangent length on the x-axis is called the **subtangent** and the projection of the normal length on the x-axis is called the **subnormal**. In the figure, TM is the subtangent and MN the subnormal.

There is little occasion to use the sign of the tangent and normal lengths and they are usually treated as positive, but in the case of the subtangent and subnormal the signs are important. The subtangent is always measured from the point where the tangent crosses the x-axis to the foot of the ordinate of the point of contact, and the subnormal from the foot of this ordinate to the intersection of the normal with the x-axis. These lengths are easily computed, for, from the figure,

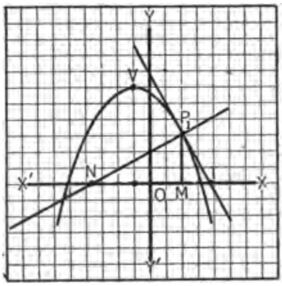
$$\text{subtangent} = TM = OM - OT,$$
$$\text{subnormal} = MN = ON - OM.$$

The abscissa OM of the point of contact is given, and OT and ON can be found since they are the intercepts of the tangent and normal respectively on the x-axis.

The tangent length is the hypotenuse of the right triangle of which the legs are the subtangent and the ordinate of the point of contact. The normal length is the hypotenuse of the right triangle of which the legs are the subnormal and the ordinate of the point of contact.

ILLUSTRATIVE EXAMPLE

Find the equations of the tangent and normal to $x^2 + 2x + 3y = 17$ at the point (2, 3), and determine the lengths of the subtangent, subnormal, tangent, and normal. Sketch the figure.



Rearranging terms and collecting, the equation $x^2 + 2x + 3y = 17$ becomes $(x + 1)^2 = -3(y - 6)$, which shows that the figure is as here sketched.

From the rule in paragraph 66, the equation of the tangent to $x^2 + 2x + 3y = 17$ at P_1 (2, 3) is found to be $y + 2x = 7$. The normal, which is perpendicular to the tangent at P_1 (2, 3), has for its equation $2y - x = 4$.

OT, the intercept of the tangent on the x-axis $= \frac{7}{2}$.

ON, the intercept of the normal on the x-axis $= -4$.

OM, the abscissa of $P_1 = 2$.

The subtangent $= TM = OM - OT = 2 - \frac{7}{2} = -\frac{3}{2}$.

The subnormal $= MN = ON - OM = -4 - 2 = -6$.

The tangent length $= TP_1 = \sqrt{\overline{TM}^2 + \overline{MP_1}^2} = \sqrt{\frac{9}{4} + 9} = \frac{3}{2}\sqrt{5}$.

The normal length $= P_1N = \sqrt{\overline{MP_1}^2 + \overline{MN}^2} = \sqrt{9 + 36} = 3\sqrt{5}$.

EXERCISES

1. Determine the equations of the tangent and normal, and lengths of subtangent and subnormal to the following curves at the point given. Draw the figure in each case.

(a) $y = 2x^2$ at $(1, 2)$.
(b) $x^2 + y^2 = 25$ at $(3, 4)$.
(c) $x^2 - 4x = 4y$ at $(4, 0)$.
(d) $xy = 4$ at point whose abscissa is 1.
(e) $2x^2 - y^2 = 14$ at $(3, -2)$.
(f) $\frac{x^2}{7} + \frac{y^2}{16} = 1$ at positive end of the upper latus rectum.
(g) $y^2 - 6y - 8x = 31$ at $(-3, 7)$.

2. Find the equations of the tangents to $y^2 = 4x - 4$ at the extremities of the latus rectum. Prove that they are perpendicular and meet on the directrix.

3. Find the equations of the tangent and normal to $x^2 + 4x + y^2 + 6y = 12$ at $(1, 1)$ and prove that the normal passes through the center.

4. Write the equations of the tangent and normal to each of the following conics at the point given. Draw each figure.

(a) $4x^2 - 16x + 9y^2 - 18y = 11$ at $(2, 3)$.
(b) $x^2 - 4x - 2y - 1 = 0$ at $(1, -2)$.
(c) $3x^2 + 10xy + 3y^2 = 3$ at $(-\frac{10}{3}, 1)$.
(d) $x^2 - 2xy + y^2 = 4x + 4y$ at $(4, 0)$.

5. Prove that the tangents at the extremities of a latus rectum of the curve $7x^2 + 16y^2 = 112$ meet on the corresponding directrix.

6. Prove that the tangents at the extremities of the latus rectum intersect on the directrix in the case of: (a) any parabola; (b) any ellipse; (c) any hyperbola.

7. How far from the vertex are the tangents at the extremities of the latus rectum of $x^2 + 4y + 4 = 0$?

8. Find the angle formed by the tangents at the extremities of a latus rectum of the hyperbola $9x^2 - 16y^2 = 144$.

9. Prove that the normal at one extremity of the latus rectum of a parabola is parallel to the tangent at the other extremity.

10. Given the ellipse $\frac{x^2}{9} + \frac{y^2}{25} = 1$. Find the equations of the tangents whose intercepts on the axes are numerically equal.

11. Prove that the tangents at the extremities of the latus rectum of a parabola are twice as far from the focus as from the vertex.

12. Prove that the tangent at any point of the parabola $y^2 = 4px$, the perpendicular from the focus upon it, and the tangent at the vertex meet in a point.

13. Prove that the subtangent of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{x_1^2 - a^2}{x_1}$ and that the subnormal is $(e^2 - 1)x_1$.

14. What is the point of contact of that tangent to the parabola $y^2 = 4x$, whose intercepts on the axes are numerically equal and (*a*) of same sign; (*b*) of opposite sign.

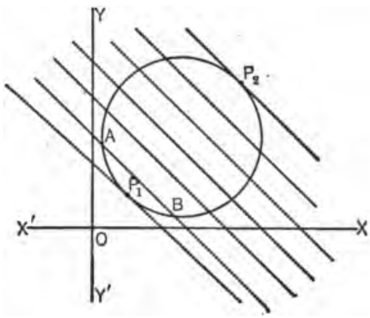
15. Prove that the perpendicular from the focus of an ellipse upon any tangent and the line joining the center to point of contact meet on the corresponding directrix.

68. The equation of the tangent when the slope is given. — The process used in finding the equation of tangents to curves of second degree, when the slope is given, will be illustrated by the following example.

Let it be required to find for the circle $x^2 - 6x + y^2 - 6y + 10 = 0$, the equations of the tangents of slope -1.

Let $y = -x + b$ represent any one of the system of parallel lines of slope -1.

It is evident from the following figure that some of these lines such as AB cut the curve in two distinct points, and that if this line is moved parallel to itself, the two distinct intersections will approach each other and eventually coincide as at P_1 and P_2. In this position, the line $y = -x + b$ is a tangent to the curve. The problem then is to so determine b that the line $y = -x + b$ shall meet the curve in two coincident points.



If the equation of the line

$$y = -x + b$$

and the equation of the circle

$$x^2 - 6x + y^2 - 6y + 10 = 0$$

are solved simultaneously, in order to determine the coordinates of the intersection, the result, after eliminating y, is

$$x^2 - 6x + (-x + b)^2 - 6(-x + b) + 10 = 0,$$

or when expanded,

$$2x^2 - 2bx + b^2 - 6b + 10 = 0,$$

from which the abscissas of the intersections of any line of the system $y = -x + b$ with the given circle may be determined.

It was shown in the theory of equations, that an equation in the form

$$Ax^2 + Bx + C = 0$$

has equal roots if $B^2 - 4AC = 0$.

Applying this principle here, the equation

$$2x^2 - 2bx + b^2 - 6b + 10 = 0$$

has equal roots if

$$4b^2 - 8(b^2 - 6b + 10) = 0,$$

or if

$$b^2 - 12b + 20 = 0.$$

This last equation is true if $b = 2$ or 10, whence a line of the system $y = -x + b$ meets the given circle in two coincident points if $b = 2$ or 10.

Therefore,

$$y = -x + 2$$

and

$$y = -x + 10$$

are the equations of the desired tangents.

EXERCISES

1. For each of the following curves, find the equations and points of contact of the tangents whose slopes are as given:

(a) $x^2 + y^2 = 25$, slope $= -\frac{4}{3}$.
(b) $y^2 = 4x + 4$, slope $= 1$.
(c) $xy = 4$, slope $= -4$.
(d) $x^2 - 4x - y^2 - 4y = 3$, slope $= 2$.

2. For each of the following curves, find the equation of the tangent with slope m:

(a) $y^2 = 4\,px$. *Ans.* $y = mx + \dfrac{p}{m}$.

(b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. *Ans.* $y = mx \pm \sqrt{a^2m^2 + b^2}$.

(c) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. *Ans.* $y = mx \pm \sqrt{a^2m^2 - b^2}$.

3. Find the equation of the tangent to $4\,x^2 + 25\,y^2 = 100$ parallel to $3\,x + 10\,y = 60$.

4. Find the equation of the tangent to $x^2 = 4\,y + 4$ perpendicular to $2\,y + x = 7$.

5. Find the equations of the tangents to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ which are parallel to the line joining the positive extremities of the axes.

6. Prove that the line $5\,x - 2\,y - 11 = 0$ is tangent to $5\,x^2 - 20\,x - 2\,y^2 + 4\,y + 15 = 0$.

7. Find the equations of the tangents to $9\,x^2 + 4\,y^2 + 6\,x + 4\,y = 0$ parallel to $3\,x + 2\,y = 7$, and the equations of the normals at the points of contact.

CHAPTER XI

POLES, POLARS, DIAMETERS, AND CONFOCAL CONICS

69. Harmonic division. — If a line AB is divided in-

ternally by the point P and externally by the point P_1 in such a way that

$$\frac{AP}{PB} = -\frac{AP_1}{P_1B}, \tag{1}$$

the line is said to be divided harmonically.

Theorem. — *If two points P and P_1 divide a line AB harmonically, then conversely, the points A and B divide the line PP_1 harmonically.*

Proof. — If the proportion (1) above is taken by alternation, it becomes

$$\frac{AP}{AP_1} = -\frac{PB}{P_1B} = \frac{PB}{BP_1}.$$

Reversing the members,

$$\frac{PB}{BP_1} = \frac{AP}{AP_1} = -\frac{PA}{AP_1}.$$

From the proportion $\frac{PB}{BP_1} = -\frac{PA}{AP_1}$, it is seen that the line PP_1 is divided harmonically by the points A and B.

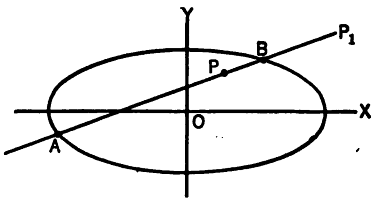
EXERCISES

1. Find the coördinates of the point P which together with P_1 (2, 3) divides harmonically the line joining $A(-1, 4)$ to $B(8, 1)$. *Ans.* $(-10, 7)$.

2. Show that the points A and B in example 1 divide harmonically the line joining P_1 and P.

70. Pole and polar. — If through a fixed point P_1, outside, inside, or on a conic, a secant is drawn to the conic meeting it in the points A and B, and if P is so chosen on the secant that the points P and P_1 divide the line AB harmonically, then the locus which contains all positions of P as the secant revolves about P_1 is called the **polar** of P_1 with regard to the conic, and the point P_1 is called the **pole** of that locus.

Equation of the polar for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. — Given the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the fixed point $P_1(x_1, y_1)$. Through P_1 draw any secant meeting the ellipse in the points A and B, and so locate $P(x, y)$ upon it that the line AB is divided harmonically by the points P and P_1. It is required to find the equation of the locus which contains P in all of its positions.



By the theorem in Art. 69, since AB is divided harmonically by P and P_1, then PP_1 is divided harmonically by A and B, and hence

$$\frac{PB}{BP_1}=-\frac{PA}{AP_1}.$$

Let the segments PB and BP_1 be in the ratio $r_1 : r_2$ (r_1 and r_2 will vary as the secant is revolved).

The coördinates of A and B can now be found by formulas (3), Art. 8. They are respectively

$$\left(\frac{r_1x_1 - r_2x}{r_1 - r_2},\ \frac{r_1y_1 - r_2y}{r_1 - r_2}\right) \text{ and } \left(\frac{r_1x_1 + r_2x}{r_1 + r_2},\ \frac{r_1y_1 + r_2y}{r_1 + r_2}\right).$$

The points A and B are on the ellipse and their coördinates satisfy the equation

$$b^2x^2 + a^2y^2 = a^2b^2. \tag{1}$$

Therefore,

$$b^2\left(\frac{r_1x_1 - r_2x}{r_1 - r_2}\right)^2 + a^2\left(\frac{r_1y_1 - r_2y}{r_1 - r_2}\right)^2 = a^2b^2, \tag{2}$$

and

$$b^2\left(\frac{r_1x_1 + r_2x}{r_1 + r_2}\right)^2 + a^2\left(\frac{r_1y_1 + r_2y}{r_1 + r_2}\right)^2 = a^2b^2. \tag{3}$$

Expanding equations (2) and (3) and clearing of fractions,

$$b^2(r_1^2x_1^2 - 2r_1r_2x_1x + r_2^2x^2) + a^2(r_1^2y_1^2 - 2r_1r_2y_1y + r_2^2y^2)$$
$$= a^2b^2(r_1^2 - 2r_1r_2 + r_2^2) \tag{4}$$

$$b^2(r_1^2x_1^2 + 2r_1r_2x_1x + r_2^2x^2) + a^2(r_1^2y_1^2 + 2r_1r_2y_1y + r_2^2y^2)$$
$$= a^2b^2(r_1^2 + 2r_1r_2 + r_2^2). \tag{5}$$

In addition to the variables x and y, equations (4) and (5) contain r_1 and r_2 which also vary as the secant is revolved. It is desired to find an equation, containing no variables other than x and y, which will be true for any position of P as the secant revolves. Therefore such equation must be independent of r_1 and r_2.

Subtracting (4) from (5),

$$b^2(4r_1r_2x_1x) + a^2(4r_1r_2y_1y) = a^2b^2(4r_1r_2). \tag{6}$$

Dividing by $4r_1r_2$,

$$b^2x_1x + a^2y_1y = a^2b^2, \tag{7}$$

or

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1, \tag{8}$$

which is the desired equation of the polar. Since it is of first degree, the polar is a straight line.

It will be noticed that equation (8) is of the same form as the equation of the tangent to the ellipse found in example 1 (*d*) of Art. 65. In the polar, however, P_1 may be inside, outside, or on the conic.

If P_1 is outside the ellipse, P must be inside the ellipse in order that the secant may be divided harmonically, therefore only that portion of line (8) which lies inside the conic fulfils this condition. If P_1 is inside the ellipse, P_1 and any point P on line (8) will divide AB harmonically, therefore the whole line fulfils the condition. If P_1 is on the ellipse, the polar line is the tangent, the point of contact is the pole, and no point other than the point of contact itself fulfils the condition.

By the same method, the equation of the polar of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is found to be

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1, \tag{9}$$

and the equation of the polar of the parabola $y^2 = 4\,px$ is

$$yy_1 = 2\,p\,(x + x_1). \tag{10}$$

The proof of the properties which follow is based upon the equation of the polar of the ellipse, but can be shown to hold for the hyperbola and parabola by using equations (9) and (10).

Important properties of poles and polars. —

1. *If the polar of the point P_1 passes through the point P_2, then the polar of the point P_2 passes through the point P_1.*

Proof. — The equations of the two polars for the ellipse are

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1 \tag{1}$$

and

$$\frac{x_2x}{a^2} + \frac{y_2y}{b^2} = 1. \tag{2}$$

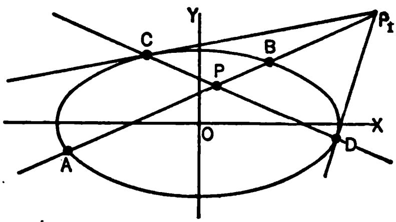
Since by hypothesis P_2 lies on the polar of P_1, the coördinates of P_2 satisfy (1), and, therefore,

$$\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} = 1. \tag{3}$$

If, however, the coördinates of P_1 are substituted in equation (2), the equation is identical to equation (3). Therefore the coördinates of P_1 satisfy equation (2) which shows that P_1 lies on the polar of P_2.

2. *If the pole P_1 is outside the conic, the polar is the chord joining the points of contact of the two tangents drawn from P_1 to the conic.*

It is evident from the figure that as the secant revolves about P_1 there will be two positions, P_1C and P_1D, in which it will be tangent to the conic. As the secant rotates toward the position P_1C, the points A and B approach each other and come into coincidence at C. But P lies between A and B, therefore P coincides with C and hence the point of contact C is on the polar. Similarly, the point D is on the polar, and the polar passes through the two points of contact of the tangents drawn from P_1.



EXERCISES

1. By the method of this article, find the equation of the polar of $P_1(x_1, y_1)$

(*a*) with respect to the hyperbola $x^2/a^2 - y^2/b^2 = 1$.
(*b*) with respect to the parabola $y^2 = 4px$.

2. Find the polar of the point

(*a*) $(-1, -3)$ with respect to the conic $x^2 + 4y^2 = 16$.
(*b*) $(2, 4)$ with respect to the conic $y^2 = x$.

3. Prove that the directrix is the polar of the focus

(*a*) for the parabola $y^2 = 4px$.
(*b*) for the ellipse $x^2/a^2 + y^2/b^2 = 1$.
(*c*) for the hyperbola $x^2/a^2 - y^2/b^2 = 1$.

4. Find the pole of $3x + 4y = 4$ with respect to the ellipse $6x^2 + 8y^2 = 16$.

Hint. — Let $P_1(x_1, y_1)$ represent the pole of $3x + 4y = 4$. The polar of (x_1, y_1) is $6xx_1 + 8yy_1 = 16$, but $3x + 4y = 4$ is also the polar of (x_1, y_1). Since these equations represent the same line, they will be identical if their second members are made equal.

5. Find the pole of $5x + 6y = 3$ with respect to the conic $15x^2 - 3y^2 = 9$.

6. Prove that the line joining any point $P_1(x_1, y_1)$ to the center of the circle $x^2 + y^2 = r^2$ is perpendicular to the polar of the point with respect to the circle.

7. Prove that the radius of a circle is a mean proportional between the distance from the center to the point $P_1(x_1, y_1)$ and the distance from the center to the polar of P_1.

8. Prove that the polar of $(-1, 2)$ with respect to the parabola $y^2 = 4x$ passes through $(2, 1)$. Verify the first property of this article by showing that the polar of $(2, 1)$ passes through $(-1, 2)$.

9. Find the polars of the vertices of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ with respect to its conjugate hyperbola.

10. Find the equations of the tangents to the circle $x^2 + y^2 = 25$ through the external point $(10, -5)$.

Hint. — Find the equation of the polar of the given point, then find the points of contact by use of the second property of this article.

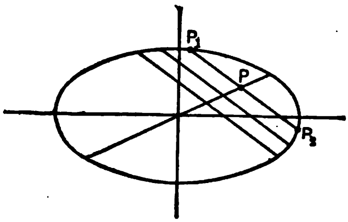
71. Diameters. — The locus of the middle points of any system of parallel chords of a conic is called a **diameter** of the conic.

The method of finding the equation of a diameter is illustrated by the following examples:

1. Find the equation of the diameter of the ellipse $x^2/a^2 + y^2/b^2 = 1$ which bisects all chords of slope m.

Let $y = mx + k$, in which m is fixed, represent a system of parallel chords. The constant k will have different values for different chords.

Let P_1 and P_2 be the points in which any one of these chords cuts the ellipse and let $P\ (x, y)$ be the center of this chord.



To find the abscissas of P_1 and P_2, y must be eliminated between the equations

$$x^2/a^2 + y^2/b^2 = 1 \qquad (1)$$

and

$$y = mx + k. \qquad (2)$$

The resulting equation is

$$(a^2m^2 + b^2)\, x^2 + 2\, a^2kmx + a^2k^2 - a^2b^2 = 0. \qquad (3)$$

Similarly, the ordinates of P_1 and P_2 may be found from the equation

$$(a^2m^2 + b^2)\, y^2 - 2\, b^2ky + b^2k^2 - a^2b^2m^2 = 0. \qquad (4)$$

It was found in the study of theory of equations that the sum of the roots of the quadratic equation $Ax^2 + Bx + C = 0$ is $-B/A$, therefore, from equation (3),

$$x_1 + x_2 = \frac{-2\, a^2km}{a^2m^2 + b^2}, \qquad (5)$$

and from equation (4),

$$y_1 + y_2 = \frac{2\, b^2k}{a^2m^2 + b^2}. \qquad (6)$$

But by formula (4), Art. 8,

$$x = \frac{x_1 + x_2}{2} \quad \text{and} \quad y = \frac{y_1 + y_2}{2}.$$

Therefore,

$$x = \frac{-a^2km}{a^2m^2 + b^2}, \qquad (7)$$

$$y = \frac{b^2k}{a^2m^2 + b^2}. \qquad (8)$$

In addition to the variables x and y, equations (7) and (8) contain k which also varies as P moves along the diameter. It is desired to find an equation, containing no variables other than x and y, which will be true for any position of P. Therefore, such equation must be independent of k.

Dividing equation (8) by equation (7), *the equation of the diameter of the ellipse bisecting chords of slope m is*

$$y = -\frac{b^2}{a^2 m}x. \tag{9}$$

2. Let the student show that *the equation of the diameter of the hyperbola which bisects chords of slope m is*

$$y = \frac{b^2}{a^2 m}x. \tag{10}$$

From the form of equations (9) and (10), it is evident that *every diameter of an ellipse or hyperbola passes through the center.*

3. Find the equation of the diameter of the parabola $y^2 = 4\,px$ which bisects chords of slope m.

Using a process identical with that used in example 1, the intersections P_1 and P_2 can be determined from the equations

$$m^2x^2 - (4\,p - 2\,mk)\,x + k^2 = 0 \tag{1}$$

and

$$my^2 - 4\,py + 4\,pk = 0. \tag{2}$$

Whence

$$x_1 + x_2 = \frac{4\,p - 2\,mk}{m^2} \tag{3}$$

and

$$y_1 + y_2 = \frac{4\,p}{m}. \tag{4}$$

Therefore,

$$x = \frac{2\,p - mk}{m^2}, \tag{5}$$

$$y = \frac{2\,p}{m}. \tag{6}$$

Following the plan of example 1, the next step would be to find from equations (5) and (6), an equation independent of k. (6) is such an equation and therefore *the equation of the diameter of the parabola which bisects chords of slope m is*

$$y = \frac{2p}{m}. \tag{7}$$

Its form shows that *the diameter is parallel to the axis of the parabola.*

Properties of diameters of central conics. —

1. *Any line through the center of an ellipse or hyperbola is a diameter bisecting some set of parallel chords.*

Proof. — Let $y = m_1x$ represent any line through the center of the ellipse $x^2/a^2 + y^2/b^2 = 1$. The equation of the diameter of this ellipse which bisects chords of slope m is

$$y = -\frac{b^2}{a^2m}x.$$

These lines are identical if

$$m_1 = -\frac{b^2}{a^2m}.$$

For a given ellipse, a and b are fixed, while m may have any value. Therefore, it is always possible to so choose m that m_1 shall equal $-\frac{b^2}{a^2m}$. Hence $y = m_1x$ always bisects some set of parallel chords.

A similar proof holds for the hyperbola.

2. *If one diameter bisects all chords parallel to a second diameter, then the second bisects all chords parallel to the first.*

Proof. — Let

$$y = m_1x \tag{1}$$

be a diameter of the ellipse $x^2/a^2 + y^2/b^2 = 1$ which bisects all chords parallel to the diameter $y = mx$.

The diameter which bisects chords of slope m has for its equation

$$y = -\frac{b^2}{a^2m}x. \tag{2}$$

Therefore,
$$m_1 = -\frac{b^2}{a^2m}. \tag{3}$$

The equation of the diameter which bisects all chords of slope m_1 is

$$y = -\frac{b^2}{a^2m_1}x, \tag{4}$$

from which it is seen that the slope of this diameter is $-b^2/a^2m_1$.

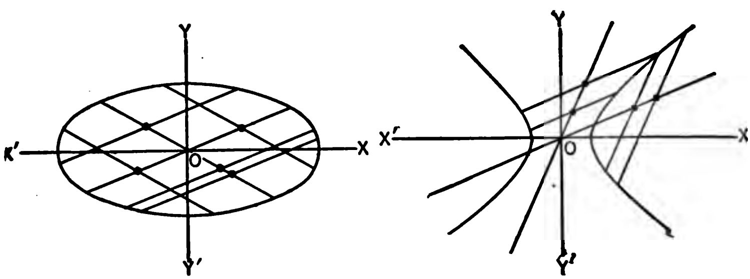
Substituting the value of m_1 from equation (3), this slope becomes

$$\left(-\frac{b^2}{a^2}\right)\left(-\frac{a^2m}{b^2}\right) = m,$$

and, therefore, $y = mx$ is the equation of the diameter bisecting chords of slope m_1.

A similar proof holds for the hyperbola.

72. Conjugate diameters. — If each of two diameters bisects all chords parallel to the other, the diameters are said to be **conjugate.**



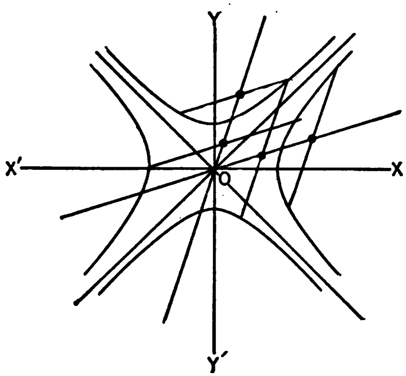
From equation (3) above it is seen that for the ellipse the relation which holds between the slopes of two conjugate diameters is

$$mm_1 = -\frac{b^2}{a^2}. \tag{1}$$

In the case of conjugate diameters of the hyperbola, the relation between the slopes is

$$mm_1 = \frac{b^2}{a^2}. \tag{2}$$

The equation of the hyperbola conjugate to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$. If the diameter bisecting chords of slope m for the second hyperbola is found by the process of Art. 71, it is seen to be $y = \frac{b^2x}{a^2m}$ as in equation (10) of that article.



It is thus seen that the same line is a diameter of each hyperbola, and bisects a system of chords in each.

EXERCISES

1. Find the equation of the diameter of the ellipse $4x^2 + 9y^2 = 36$, bisecting chords of slope 2.

2. Prove that the lines $2y + 3x = 0$ and $2y + x = 0$ are conjugate diameters of the hyperbola $3x^2 - 4y^2 = 12$.

3. Find the equation of the diameter of the parabola $y^2 = 8x$ bisecting chords parallel to the polar of (3, 4) with respect to the parabola.

4. Prove by means of equations (1) and (2), Art. 72, that a pair of conjugate diameters of the ellipse lie in different quadrants, and that a pair of conjugate diameters of the hyperbola lie in the same quadrant and on opposite sides of the asymptotes.

5. Lines are drawn joining the extremities of the major and minor axes of an ellipse. Prove that the diameters parallel to these are conjugate.

6. Find the equation of the diameter of the parabola $y^2 = 4x$ through the point (1, 2). Also find the equation of the chords which it bisects.

7. Given an extremity of a diameter of the ellipse $x^2 + 2y^2 = 24$ is (4, 2). Find the extremities of the conjugate diameter.

8. Find the equation of the chord of the hyperbola $x^2 - 2y^2 = 4$ through (3, 1) which is bisected by the diameter $2y = x$.

9. Find the equation of that chord of the ellipse $x^2 + 4y^2 = 16$ which is bisected at (1, 1).

10. Prove that the tangent at the extremity of a diameter of an ellipse is parallel to the conjugate diameter.

Hint. — Let $P_1(x_1, y_1)$ be the extremity of a diameter. Find the equation of the tangent and of the conjugate diameter.

11. Prove that the polar of any point $P_1(x_1, y_1)$ on a diameter of an ellipse is parallel to the conjugate diameter.

12. Write the equation of the diameter of the parabola $y^2 = 4px$ which bisects chords of slope m. Prove that the tangent at the extremity of this diameter is parallel to the chords.

13. In the rectangular hyperbola $x^2 - y^2 = a^2$, a diameter passes through the point $P_1(x_1, y_1)$ on the hyperbola. If P_2 is the point in which the conjugate diameter cuts the conjugate hyperbola, prove $OP_1 = OP_2$.

73. Confocal conics. — Consider the equation

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1, \tag{1}$$

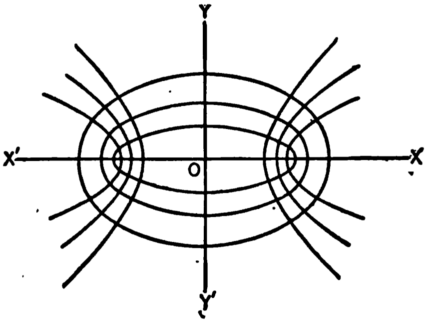
in which λ is an arbitrary constant and $a > b$.

1*st*. If λ is positive, or negative and $> -b^2$, equation (1) represents an ellipse.

From equation (28), Art. 51, the distance from the center to the focus is

$$\sqrt{(a^2 + \lambda) - (b^2 + \lambda)} = \sqrt{a^2 - b^2}.$$

Therefore the ellipses have the same foci for every value of λ.



If λ is positive, the ellipses lie without $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and if it is negative, they lie within.

2nd. If λ is negative and is such that $-b^2 > \lambda > -a^2$, equation (1) represents an hyperbola.

Let $\lambda = -\lambda_1$, then equation (1) may be written

$$\frac{x^2}{a^2-\lambda_1} - \frac{y^2}{\lambda_1 - b^2} = 1,$$

and the distance from the center to the focus is

$$\sqrt{(a^2-\lambda_1)+(\lambda_1-b^2)} = \sqrt{a^2-b^2}.$$

It is therefore seen that the equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda} = 1$$

represents a set of ellipses and hyperbolas all having the same foci.

The curves of the system represented by this equation are called **confocal conics.**

EXERCISES

1. Show that through the point (2, 1), two conics of the system $\frac{x^2}{7+\lambda}+\frac{y^2}{1+\lambda} = 1$ may be drawn. Find their equations and plot the curves. *Ans.* $\frac{x^2}{8}+\frac{y^2}{2} = 1$ and $x^2 - y^2 = 3$.

2. Find the equations of the tangents at (2, 1) on the conics found in exercise 1. Prove that these tangents are perpendicular.

CHAPTER XII

THE GENERAL EQUATION OF SECOND DEGREE

74. *The general equation of second degree*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad (1)$$

represents a conic section whose axes are inclined at an angle θ with the axes of coördinates, θ being the positive acute value determined from the equation

$$\tan 2\theta = \frac{B}{A - C}.$$

Proof. — It will first be shown that it is always possible to so rotate the axes as to cause the xy-term to vanish.

Making the substitutions from Art. 39, $x = x' \cos\theta - y' \sin\theta$, $y = x' \sin\theta + y' \cos\theta$, equation (1) becomes

$$\begin{aligned} &A(x'\cos\theta - y'\sin\theta)^2 + B(x'\cos\theta - y'\sin\theta) \\ &\quad (x'\sin\theta + y'\cos\theta) + C(x'\sin\theta + y'\cos\theta)^2 \\ &\quad + D(x'\cos\theta - y'\sin\theta) + E(x'\sin\theta + y'\cos\theta) \\ &\quad + F = 0, \end{aligned} \qquad (2)$$

which when expanded and collected is

$$\begin{aligned} &x'^2(A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta) \\ &\quad + x'y'(-2A\sin\theta\cos\theta - B\sin^2\theta + B\cos^2\theta \\ &\quad + 2C\sin\theta\cos\theta) + y'^2(A\sin^2\theta - B\sin\theta\cos\theta \\ &\quad + C\cos^2\theta) + x'(D\cos\theta + E\sin\theta) \\ &\quad + y'(E\cos\theta - D\sin\theta) + F = 0. \end{aligned} \qquad (3)$$

The $x'y'$ term will vanish if its coefficient is 0, that is, if

$$-2A\sin\theta\cos\theta - B\sin^2\theta + B\cos^2\theta + 2C\sin\theta\cos\theta = 0, \quad (4)$$

or if $$B(\cos^2\theta - \sin^2\theta) = (A - C)\,2\sin\theta\cos\theta, \tag{5}$$

or $$B\cos 2\theta = (A - C)\sin 2\theta, \tag{6}$$

or $$\tan 2\theta = \frac{B}{A - C}. \tag{7}$$

Since the tangent of an angle may have any value from $-\infty$ to $+\infty$, it is therefore always possible to rotate the axes through such an angle that the xy-term shall vanish.

Moreover, since any number may be the tangent of an angle in the first or second quadrant, there is always a value of $2\theta < 180°$ and a corresponding value of $\theta < 90°$ which satisfy equation (7). The positive acute value of θ will always be chosen.

The xy-term having been removed, the general equation becomes

$$A'x^2 + C'y^2 + D'x + E'y + F' = 0. \tag{8}$$

An equation of this form represents

(a) a circle if $A' = C'$ (this case never arises when $B \neq 0$ in the general equation).

(b) a parabola if A' and E' are present with C' absent or if C' and D' are present with A' absent.

(c) an ellipse if A' and C' are positive and unequal.

(d) an hyperbola if A' and C' have unlike signs (except where the equation can be resolved into first degree factors).

(e) a pair of straight lines. When A' and C' have unlike signs, the equation can frequently be factored and thus represents a pair of straight lines. When A' and D' each equal 0, or C' and E' each equal 0, the equation always represents a pair of parallel lines, distinct, coincident, or imaginary.

ILLUSTRATIVE EXAMPLE

By first removing the xy-term, determine the nature and position of the curve whose equation is $16x^2 - 24xy + 9y^2 - 85x - 30y + 175 = 0$ and plot the locus.

Rotate the axes through an angle θ, by substituting $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$, in the equation

$$16x^2 - 24xy + 9y^2 - 85x - 30y + 175 = 0. \qquad (1)$$

This gives

$$\begin{aligned}16(x'\cos\theta - y'\sin\theta)^2 &- 24(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta)\\ &+ 9(x'\sin\theta + y'\cos\theta)^2 - 85(x'\cos\theta - y'\sin\theta)\\ &- 30(x'\sin\theta + y'\cos\theta) + 175 = 0. \qquad (2)\end{aligned}$$

Collecting,

$$\begin{aligned}&x'^2(16\cos^2\theta - 24\sin\theta\cos\theta + 9\sin^2\theta)\\ &+ x'y'(-24\cos^2\theta + 24\sin^2\theta - 14\sin\theta\cos\theta)\\ &+ y'^2(16\sin^2\theta + 24\sin\theta\cos\theta + 9\cos^2\theta) - x'(85\cos\theta + 30\sin\theta)\\ &+ y'(85\sin\theta - 30\cos\theta) + 175 = 0. \qquad (3)\end{aligned}$$

The $x'y'$-term will vanish if

$$-24\cos^2\theta + 24\sin^2\theta - 14\sin\theta\cos\theta = 0,$$

or if $$24\cos 2\theta = -7\sin 2\theta,$$

or $$\tan 2\theta = -\tfrac{24}{7}.$$

If $\tan 2\theta = -\frac{24}{7}$, then $\sin 2\theta = \frac{24}{25}$ and $\cos 2\theta = -\frac{7}{25}$.

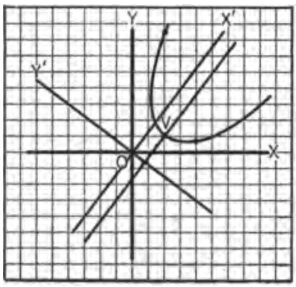
Whence $$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} = \sqrt{\frac{1+\frac{7}{25}}{2}} = \frac{4}{5}.$$

$$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}} = \sqrt{\frac{1-\frac{7}{25}}{2}} = \frac{3}{5}.$$

Substituting in equation (3) and clearing of fractions,

$$y'^2 + 2y' = 3x' - 7, \qquad (4)$$

or $$(y'+1)^2 = 3(x'-2). \qquad (5)$$



This is seen to be a parabola which when referred to the new axes has its vertex at $(2, -1)$, principal axis parallel to the x-axis, and distance from vertex to focus $\frac{3}{4}$. Constructing both sets of axes and the curve, the figure is as shown.

75. Test for distinguishing the type of conic when the equation contains the xy-term. — Comparing equations (3) and (8), Art. 74, it is seen that

$$
\begin{aligned}
A' &= A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta, && (9)\\
B' &= B\cos 2\theta - (A - C)\sin 2\theta, && (10)\\
C' &= A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta. && (11)
\end{aligned}
$$

Adding (9) and (11) and applying trigonometry,

$$A' + C' = A + C. \qquad (12)$$

Subtracting (9) and (11) and applying trigonometry,

$$A' - C' = (A - C)\cos 2\theta + B\sin 2\theta. \qquad (13)$$

Squaring (10) and (13) and adding,

$$(A' - C')^2 + B'^2 = (A - C)^2 + B^2. \qquad (14)$$

Squaring (12) and subtracting from (14),

$$B'^2 - 4A'C' = B^2 - 4AC. \qquad (15)$$

If θ is so chosen that $\tan 2\theta = \dfrac{B}{A - C}$, then $B' = 0$ and $-4A'C' = B^2 - 4AC$.

It was shown in the previous article that if the given conic is a parabola either A' or $C' = 0$. Therefore $B^2 - 4AC = 0$.

If the given conic is an ellipse, A' and C' have the same sign, therefore $B^2 - 4AC < 0$.

If the given conic is an hyperbola, A' and C' have opposite signs, therefore $B^2 - 4AC > 0$.

$A + C$ and $B^2 - 4AC$ can be shown to remain unchanged when the equation is translated to a new origin. Since these combinations of coefficients are unchanged by both rotation and translation, they are called **invariants**.

EXERCISES

1. Simplify the following equations by removing the xy-term. Plot both pairs of axes and the curve.

(a) $9x^2 - 24xy + 16y^2 - 80x - 60y + 100 = 0$.
(b) $x^2 + 4xy + 4y^2 + x + 2y - 2 = 0$.
(c) $7y^2 - 48xy - 7x^2 + 30x - 40y = 0$.

2. By rotation and translation, reduce the following equations to their simplest form. Plot the three sets of axes and the curve. Check the result by finding the nature of the conic by the method of this article.

(a) $5x^2 - 6xy + 5y^2 - 2\sqrt{2}(x + y) - 6 = 0$.
(b) $7x^2 - 48xy - 7y^2 + 70x + 10y + 100 = 0$.
(c) $2x^2 + 3xy - 2y^2 + 3x + y + 1 = 0$.
(d) $4x^2 - 4xy + y^2 + 8x - 4y = 5$.
(e) $16x^2 - 24xy + 9y^2 - 45x - 60y - 400 = 0$.

76. Conic through five points. — Since the general equation of second degree, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, has six constants any one of which can be divided out, it is seen that only five of these are independent. Therefore five conditions are sufficient to determine the equation of a conic. In particular, this fact makes it possible to write the equation of a conic through five given points.

ILLUSTRATIVE EXAMPLE

Write the equation of the conic through the 5 points (0, 0), (0, 1), (1, 2), (1, −2), (5, 0).

The general equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, after dividing by A, takes the form

$$x^2 + bxy + cy^2 + dx + ey + f = 0.$$

Substituting the coördinates of the given points, five equations result as follows:

$$\begin{aligned} f &= 0, \\ c + e + f &= 0, \\ 1 + 2b + 4c + d + 2e + f &= 0, \\ 1 - 2b + 4c + d - 2e + f &= 0, \\ 25 + 5d + f &= 0. \end{aligned}$$

Solving these equations simultaneously, $b = 1$, $c = 1$, $d = -5$, $e = -1$, and $f = 0$. Therefore the equation of the conic is

$$x^2 + xy + y^2 - 5x - y = 0.$$

If the equation of the conic through five given points is such that x^2 is not present, it would not be permissible to divide the general equation by A in the first step of the solution, and if such step is taken, the equations determined lead to contradiction. This difficulty can be avoided by dividing by some other constant. This is illustrated in Ex. 2 which follows.

EXERCISES

1. Find the equations of the conics through the following points:

(*a*) (0, 0), (0, 1), (1, 0), (1, 1), (−1, 2).

(*b*) (0, 1), (2, 1), (1, 0), (2, 0), (1, $\frac{3}{2}$).

(*c*) (−4, 0), (0, −2), (1, 0), (−5, −2), (1, 5).

2. Find the equation of the conic through (0, −2), (0, 3), (−2, 0), (1, 3), (1, −3).

Hint. — Divide the general equation by C.

3. Find the equation of the conic through the point (3, 2) and through the points of intersection of $x^2 + y^2 = 25$ and $xy = 12$. Plot all loci.

Hint. — Use Art. 16.

4. Find the equation of the conic through the point (−1, 1) and through the points of intersection of $4x^2 + 4xy + y^2 - 4x - 2y = 0$ and $x^2 + y^2 - 4x - 3y = 0$. Plot all loci.

5. What relation must hold between the coefficients of $x^2 + bxy + cy^2 + dx + ey + f = 0$ in order that its locus shall be tangent to the x-axis. Find the equation of a conic passing through (3, 2), (−1, 2), (3, 8), and tangent to the x-axis at (1, 0).

6. Find the equation of the conic through (−1, 0), (9, 0), (−1, 6), and tangent to the y-axis at (0, 3). Plot all loci.

Hint. — Use $ax^2 + bxy + y^2 + dx + ey + f = 0$ as the equation of the conic.

7. Find the equation of the parabola through the four points (0, 0), (1, 0), (0, 1), and (2, 1). Make use of the fact that in the parabola $B^2 = 4AC$.

CHAPTER XIII

TRANSCENDENTAL AND PARAMETRIC EQUATIONS

77. Loci of transcendental equations. — Thus far the discussions have been largely concerned with algebraic equations, that is, with equations involving variables raised to constant powers. In this chapter, attention will be given to other forms of equations such as $y = a^x$, $y = \sin x$, $y = \log x$, etc. Such equations are called **transcendental equations.**

Many of the steps taken in the discussion and plotting of such curves are the same as those used in algebraic equations, but there are some respects in which they require different treatment. A few examples will illustrate the method to be used.

78. The exponential curve $y = a^x$, where a is a positive constant. In the following discussion, a will be taken as greater than 1. A similar discussion would result from taking a less than 1, in which case the figure would be changed in position, but not in character.

1*st*. *Intercepts.* — Let $x = 0$, then $y = a^0 = 1$, whence the curve cuts the y-axis at the point (0, 1). Let $y = 0$, then $x = -\infty$.

2*nd*. *Symmetry.* — If y is replaced by $-y$, the equation is changed, therefore the curve is not symmetrical with respect to the x-axis. If x is replaced by $-x$, the equation is changed, therefore the curve is not symmetrical with respect to the y-axis. If x is replaced by $-x$ and y by $-y$, the equation is changed, therefore the curve is not symmetrical with respect to the origin.

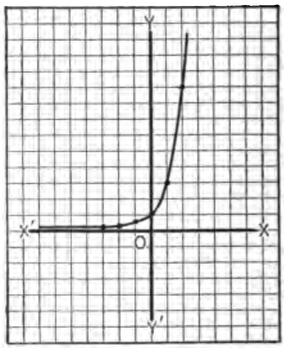
3rd. Extent. — It is seen that y is real for all values of x from $-\infty$ to $+\infty$, also that as x increases from 0 to $+\infty$, y increases from 1 to $+\infty$, and as x decreases from 0 to $-\infty$, y decreases from 1 to 0.

Solving for x in terms of y, $x = \log_a y$, from which it is seen that x is real for all positive values of y. All negative values of y are excluded since there are no logarithms of negative numbers.

4th. Asymptotes. — In the equation $y = a^x$, it is seen that no finite value of x will make y infinite, therefore there are no asymptotes parallel to the y-axis. In the equation $x = \log_a y$, the only finite value of y which makes x infinite is $y = 0$, therefore $y = 0$ is the only horizontal asymptote.

In plotting the curve, it is necessary to assign some value to a. The following table of values is computed for $a = 3$, from which the curve $y = 3^x$ is plotted.

x	y	x	y
0	1	0	1
1	3	-1	.33+
2	9	-2	.11+
3	27	-3	.03+



The exponential curve of most frequent occurrence is $y = e^x$, in which $e = 2.718+$. This number, 2.718+, is of great importance in all higher mathematics.

The student is asked to plot the three curves $y = 2^x$, $y = e^x$, and $y = 3^x$, using the same set of axes. Discuss the effect of an increase in a on the form of the curve. (Use $e = 2.7$.) Plot the locus $y = (\frac{1}{2})^x$.

79. Relation between natural and common logarithms. — A system of logarithms in which the base is 10 is called a system of **common logarithms.**

A system of logarithms in which the base is e is called a system of **natural logarithms.**

The common logarithm of a number N is the exponent of the power to which 10 must be raised to give N. If x = common logarithm of N, then

$$\log_{10} N = x \quad \text{or} \quad N = 10^x. \tag{1}$$

The natural logarithm of a number N is the exponent of the power to which e must be raised to give N. If y = natural logarithm of N, then

$$\log_e N = y \quad \text{or} \quad N = e^y. \tag{2}$$

Equating the values of N from equations (1) and (2),

$$e^y = 10^x. \tag{3}$$

Taking the common logarithm of each member of (3),

$$y \log_{10} e = x \log_{10} 10 \tag{4}$$

or

$$y = \frac{x}{\log_{10} e} = \frac{x}{.434} = (2.302)\, x.$$

Therefore the natural logarithm of a number is equal to the common logarithm of that number multiplied by 2.302.

For rough calculations of natural logarithms, a table of common logarithms may be used and each logarithm multiplied by **2.3.**

In all higher mathematics when no base is expressed, the base e is understood.

80. The logarithmic curve, $y = \log_e x$. —

1*st*. *Intercepts.* — When $x = 0$, $y = -\infty$; when $y = 0$, $x = 1$, therefore the curve cuts the x-axis at the point (1, 0) and has no finite intercept on the y-axis.

2nd. Symmetry. — As in the exponential equation, it can be shown that the curve is not symmetrical with respect to the x-axis, y-axis, or origin.

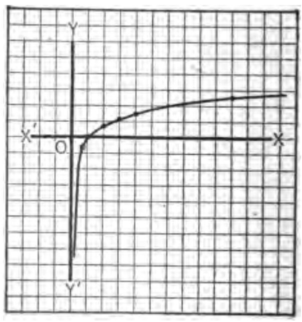
3rd. Extent. — Solving for y in terms of x, $y = \log_e x$, from which it is seen that y is real for all positive values of x, but that all negative values of x are excluded, since there are no logarithms of negative numbers.

Solving for x in terms of y, $x = e^y$, from which it is seen that x is real for all values of y from $-\infty$ to $+\infty$, also that as y increases from 0 to $+\infty$, x increases from 1 to $+\infty$, and as y decreases from 0 to $-\infty$, x decreases from 1 to 0.

4th. Asymptotes. — In the equation $y = \log_e x$, it is seen that $x = 0$ is the only finite value which makes y infinite, while in the equation $x = e^y$, there is no finite value of y which makes x infinite. $x = 0$ is then the only asymptote parallel to the axes.

Computing a table of values, and plotting the points, the figure is as shown.

x	$\log_{10} x$	$\log_e x$
0	$-\infty$	$-\infty$
.5	$-$ .301	$-$ .693
1	0	0
2	.301	.693
3	.477	1.098
4	.602	1.386
10	1	2.302
100	2	4.604



It will be observed that the logarithmic equation $y = \log_e x$ or $x = e^y$ is the same as the exponential equation $y = e^x$ with x and y interchanged. Hence the curve in this article might have been constructed in a manner identical with that used in Art. 78, except that values would have been assigned to y, and x computed.

These curves are of very frequent occurrence in expressing the laws of physics, and especially in problems of electromotive force.

EXERCISES

1. Construct the curves $y = e^x$ and $y = e^{-x}$ with the same axes and show that these curves are symmetrical with respect to the y-axis.

2. Plot the locus of $y = \log_2 x$ by changing to an exponential equation and assigning values to y to find the coördinates of points on the curve.

3. Discuss and plot the loci of the following equations:

(a) $y = e^{2x}$. (b) $y = 3\,e^{-2x}$.

(c) $y = e^{-\frac{x}{2}}$. (d) $y = 2\log_{10}(x + 1)$.

(e) $y = 2\log_e(x + 1)$. (f) $y = \frac{1}{2}e^{3x}$.

(g) $y = e^{-\frac{x}{3}}$. (h) $y = 2\log_{10}(1 + 3\,x)$.

(i) $y = \log_e(1 - 2\,x)$. (j) $y = \frac{1}{2}(e^x - e^{-x})$.

(k) $y = e^{-x^2}$. (l) $y = \log_e(1 + e^x)$.

(m) $y = \frac{1}{2}(e^x + e^{-x})$. *(n) $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$.

(o) $x = \frac{1}{2}(e^y + e^{-y})$. (p) $y = \log_e(1 - x^2)$.

(q) $x = \log_e(y + 2)$. (r) $y = \log_e \dfrac{e^x - 1}{e^x + 1}$.

81. The sine curve, $y = \sin x$. —

1*st*. *Intercepts.* — When $x = 0$, $y = 0$. When $y = 0$, $x = \sin^{-1} 0 = 0,\ \pi,\ 2\,\pi,\ 3\,\pi$, etc., $-\pi,\ -2\,\pi,\ -3\,\pi$, etc. Therefore the curve intersects the y-axis at the origin only, but intersects the x-axis at an infinite number of points at intervals of π.

2*nd*. *Symmetry.* — When y is replaced by $-y$, the equation is changed, therefore the curve is not symmetrical with respect to the x-axis. When x is replaced by $-x$, the equation is changed, therefore the curve is not symmetrical with respect to the y-axis. When x is replaced by $-x$ and y by $-y$,

* The locus of equation (n) is of frequent occurrence. It is called a *catenary* and has the form assumed by a heavy flexible cord suspended between two points.

the equation is unchanged, therefore the curve is symmetrical with respect to the origin.

3rd. Extent. — Solving for y in terms of x, $y = \sin x$, from which it is seen that y is real for every value of x from $-\infty$ to $+\infty$. Moreover, it is seen that as x increases from 0 to $\frac{1}{2}\pi$, y increases from 0 to 1; as x increases from $\frac{1}{2}\pi$ to π, y decreases from 1 to 0; as x increases from π to $\frac{3}{2}\pi$, y decreases from 0 to -1; as x increases from $\frac{3}{2}\pi$ to 2π, y increases from -1 to 0.

Since $\sin(x + 2n\pi) = \sin x$, where n is any integer, positive or negative, it follows that if the curve is plotted from $x = 0$ to $x = 2\pi$, the remainder of the curve can be obtained by moving the portion already plotted, right and left along the x-axis through successive multiples of 2π.

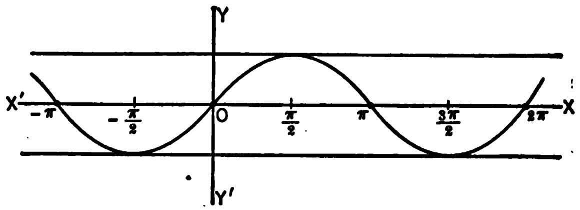
Solving for x in terms of y, $x = \sin^{-1} y$, from which it is seen that x is real for all values of y between -1 and $+1$, but that points whose ordinates are > 1 or < -1 are excluded.

4th. Asymptotes. — Since no finite value of either variable will make the other infinite, there are no asymptotes parallel to the axes.

In plotting curves of this type, it is customary to measure x in radian measure, using $\pi = 3.1416$ when laying off abscissas on the x-axis.

Plotting points in this way, the curve is found to be as follows:

x	y	x	y
0	0	$\pi = 3.14$	0
$\frac{\pi}{6} = .52$	.5	$\frac{7}{6}\pi = 3.66$	$-.5$
$\frac{\pi}{3} = 1.04$	.86	$\frac{4}{3}\pi = 4.18$	$-.86$
$\frac{\pi}{2} = 1.57$	1.00	$\frac{3}{2}\pi = 4.71$	-1.00
$\frac{2}{3}\pi = 2.09$	.86	$\frac{5}{3}\pi = 5.23$	$-.86$
$\frac{5}{6}\pi = 2.61$	.5	$\frac{11}{6}\pi = 5.75$	$-.5$
$\pi = 3.14$	0	$2\pi = 6.28$	0



82. Periodic functions. — The sine curve, discussed in the last article, is an illustration of a class called **periodic functions.** They are marked by the characteristic that when a definite constant is added to the variable, the function is unchanged. In the case of the sine function just discussed, this constant was $2n\pi$ where n was any integer. The least positive value of this constant is called the period of the function. In this case the period is 2π.

In plotting periodic functions, it is necessary to construct only that portion of the curve belonging to one period. The entire curve can then be sketched by use of the fact that its values are repeated in each successive period.

83. The curve $y = a \sin kx$. — Since $\sin kx = \sin (kx + 2\pi)$ $= \sin k\left(x + \frac{2\pi}{k}\right)$, therefore the period is $\frac{2\pi}{k}$.

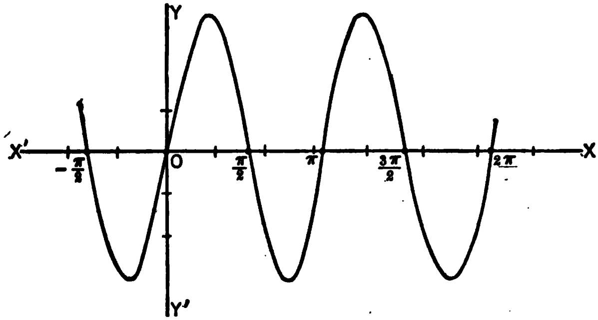
The values of $\sin kx$ vary between -1 and $+1$, hence the values of y vary between $-a$ and $+a$.

It is seen that the factor k divides the period and the factor a multiplies the function.

Letting $k = 2$ and $a = 3$, the following table is computed:

x	$\sin 2x$	y	x	$\sin 2x$	y
0	0	0	$\frac{1}{2}\pi$	0	0
$\frac{1}{8}\pi$	.707	2.121	$\frac{5}{8}\pi$	$-$ .707	-2.121
$\frac{1}{4}\pi$	1	3	$\frac{3}{4}\pi$	-1	-3
$\frac{3}{8}\pi$	.707	2.121	$\frac{7}{8}\pi$	$-$ .707	-2.121
$\frac{1}{2}\pi$	0	0	π	0	0

This table, together with a discussion similar to that used in the sine curve, shows the locus to be as follows:



Exercises. — Plot $y = \sin\frac{x}{2}$, $y = \sin x$, $y = \sin 2x$, using the same set of axes for the three curves.

Plot $y = \frac{1}{2}\sin x$, $y = \sin x$, $y = 2\sin x$, using the same set of axes for the three curves.

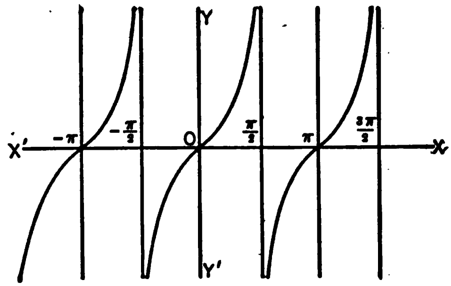
84. The tangent curve, $y = \tan x$. — Since $\tan x = \tan(\pi + x)$, therefore the period of this curve is π.

1*st. Intercepts.* — If $x = 0$, $y = 0$. If $y = 0$, $x = 0$, π, 2π, 3π, etc., $-\pi$, -2π, -3π, etc. Therefore the curve intersects the y-axis at the origin only, but intersects the x-axis at an infinite number of points at intervals of π.

2*nd. Symmetry.* — When y is replaced by $-y$, the equation is changed, therefore the curve is not symmetrical with respect to the x-axis. When x is replaced by $-x$, the equation is changed, therefore the curve is not symmetrical with respect to the y-axis. When x is replaced by $-x$ and y by $-y$, the equation is unchanged, therefore the curve is symmetrical with respect to the origin.

3*rd. Extent.* — Solving for y in terms of x, $y = \tan x$, from which it is seen that y is real for every value of x from $-\infty$ to

$+\infty$. Moreover, it is seen that as x increases from 0 to $\pi/2$, y increases from 0 to $+\infty$ and as x increases from $\pi/2$ to π, y increases from $-\infty$ to 0. Since the period of this curve is π, the values of the function are repeated beyond this point.



Solving for x in terms of y, $x = \tan^{-1} y$, from which it is seen that x is real for all values of y from $-\infty$ to $+\infty$.

4th. Asymptotes. — $y = \infty$ when $x = \pm\pi/2$, $\pm 3\pi/2$, $\pm 5\pi/2$, etc. These lines are asymptotes to the curve.

EXERCISES

1. Discuss and plot the loci of the following equations:

(*a*) $y = \cos x$. (*b*) $y = \cot x$. (*c*) $y = \sec x$.

(*d*) $y = \tan \dfrac{\pi x}{4}$. (*e*) $y = \sin \dfrac{\pi x}{2}$. (*f*) $y = \tan^2 x$.

(*g*) $y = \sin^2 x$. (*h*) $y^2 = \tan x$. (*i*) $y = \cot \dfrac{\pi x}{3}$.

2. (*a*) Plot $y = \cos^{-1} x$ or $x = \cos y$.

(*b*) In example (*a*), rotate the axes through $-\pi/2$ radians and show that the equation becomes $y = \cos x$. Plot the locus of $y = \cos x$ referred to the new axes and show that the same curve is obtained as in (*a*).

3. Prove that the sine curve differs from the cosine curve only in position by finding the equation of $y = \sin x$ when the axes are translated to new origin $(\pi/2, 0)$.

4. Plot the locus of $y = \csc x$. Draw ordinates at $x = \pi/8$ and $x = 3\pi/8$, and thus compute the cosecant of $\pi/8$ and $3\pi/8$. Check results from a table of natural functions.

5. Find the equation of $y = \sin(x - \pi/4)$ when the axes are translated to new origin at $(\pi/4, 0)$. Draw both sets of axes and the curve.

6. Find the period and the greatest value of the function in each of the following equations. Plot each locus.

(*a*) $y = 2 \sin 2x$. (*b*) $y = 4 \cos 3x$. (*c*) $y = 3 \tan 2x$.

(*d*) $y = 4 \sin \frac{x}{2}$. (*e*) $y = \cot \frac{\pi x}{3}$. (*f*) $y = \cos\left(x - \frac{\pi}{3}\right)$.

(*g*) $y = 5 \cos \frac{x}{4}$. (*h*) $y = \sec 2x$. (*i*) $y = \tan\left(x + \frac{\pi}{4}\right)$.

7. Discuss and plot the loci of the following equations:

(*a*) $y = \tan^{-1} x$. (*b*) $y = \sec^{-1} x$. (*c*) $y = \cot\left(x + \frac{\pi}{3}\right)$.

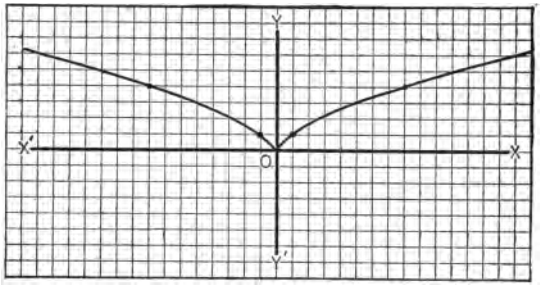
(*d*) $y = \log \sec x$. (*e*) $y = \log \csc x$. (*f*) $y = x \sin x$.

85. Loci of parametric equations. — It is sometimes convenient to express the coördinates x and y in terms of a third variable, thus $x = t^3$, $y = t^2$.

The third variable in such cases is called the parameter and the two equations are called the parametric equations of the curve. It is often possible to eliminate the parameter and thus derive the equation of the curve in rectangular coördinates. Thus, in the above example, from the first equation $t = \sqrt[3]{x}$. From the second equation, $t = \sqrt{y}$, whence $\sqrt{y} = \sqrt[3]{x}$ or $y^3 = x^2$. It is not always possible, however, to eliminate the parameter, and even when possible it is not always desirable.

To plot the graphs of such equations, values are assigned to the parameter and the corresponding values of x and y computed. The points on the curve are then located from the corresponding values of x and y. The parameter does not appear in plotting the graph. Thus in the preceding problem, if values are assigned to t, the following table of values of x and y is obtained:

t	x	y	t	x	y
0	0	0	0	0	0
1	1	1	−1	− 1	1
2	8	4	−2	− 8	4
3	27	9	−3	−27	9



The parameter usually represents time or some geometric magnitude, but may be any quantity whatever.

It is often possible to obtain equations of curves by means of a third variable which connects x and y, when it would be extremely difficult to derive the equation directly, without use of this variable. A number of important equations will be derived by this method in Art. 86.

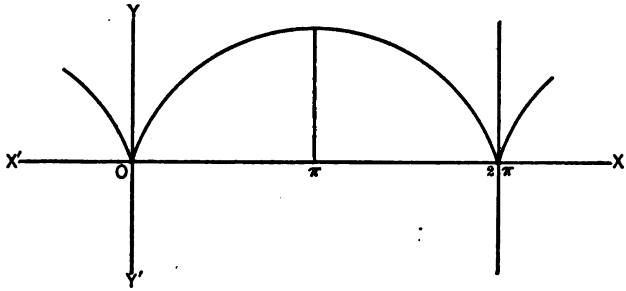
ILLUSTRATIVE EXAMPLES

1. Plot the locus of

$$x = a(\phi - \sin\phi), \quad y = a(1 - \cos\phi).$$

ϕ	Sin ϕ	Cos ϕ	x	y
0	0	1	0	0
$\frac{\pi}{6} = .52$	.5	.86	.02 a	.14 a
$\frac{\pi}{3} = 1.04$	.86	.5	.18 a	.5 a
$\frac{\pi}{2} = 1.57$	1	0	.57 a	a
$\frac{2}{3}\pi = 2.09$	.86	$-$ .5	1.23 a	1.5 a
$\frac{5}{6}\pi = 2.61$	.5	$-$ .86	2.11 a	1.86 a
$\pi = 3.14$	0	-1	3.14 a	2 a
$\frac{7}{6}\pi = 3.66$	$-$ .5	$-$ .86	4.16 a	1.86 a
$\frac{4}{3}\pi = 4.18$	$-$ .86	$-$ .5	5.04 a	1.5 a
$\frac{3}{2}\pi = 4.71$	-1	0	5.71 a	a
$\frac{5}{3}\pi = 5.23$	$-$ .86	.5	6.09 a	.5 a
$\frac{11}{6}\pi = 5.75$	$-$ .5	.86	6.25 a	.14 a
$2\pi = 6.28$	0	1	6.28 a	0

This is a periodic function of period 2π, since when the values of ϕ in the above table are increased by 2π, x is increased by $2\pi a$, and y remains unchanged. There are therefore an infinite number of arches like the one shown below.



2. If a projectile is thrown with initial velocity V ft. per second, at an angle ϕ with the horizon, it is found that the equation of its path is (resistance of air neglected), $x = Vt\cos\phi$, $y = Vt\sin\phi - 16t^2$, where t represents the time in seconds. If $V = 640$ ft. per sec. and $\phi = 30°$, plot the locus traced by the projectile and show that the equation of its path in rectangular coördinates represents a parabola.

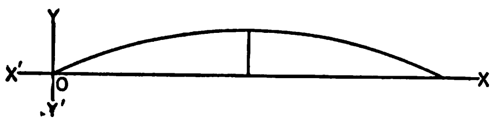
The parametric equations are

$$x = 320\sqrt{3}\,t. \tag{1}$$

$$y = 320t - 16t^2 = 16t(20 - t). \tag{2}$$

Compute a table of values and locate points as in the preceding problem.

t	x	y
0	0	0
5	2,768	1200
10	5,536	1600
15	8,304	1200
20	11,072	0



Since $y = 0$ when $t = 20$, the greatest value of x is at the end of 20 seconds and is equal to 11,072 ft. This is called the *range*. The greatest height is when $t = 10$, and is equal to 1600 ft.

In equation (1) above, $t = \dfrac{x}{320\sqrt{3}}$. Substituting this value in (2),

$$y = \frac{x}{\sqrt{3}} - \frac{x^2}{19{,}200},$$

or

$$(x - 3200\sqrt{3})^2 = -19{,}200\,(y - 1600),$$

a parabola with vertex at $(3200\sqrt{3}, 1600)$, and passing through the origin.

In finding the locus corresponding to a pair of parametric equations, it is often convenient to eliminate the parameter before plotting.

EXERCISES

1. Plot the loci of the following equations by assigning values to t and computing x and y:

(a) $x = 4t^2,\ y = 2t.$ (b) $x = t - 1,\ y = t^3.$
(c) $x = t - 1,\ y = 3t^2 - 6t.$ (d) $x = 6t^2,\ y = 1 + 3t.$
(e) $4x = t^3,\ 4y = t^2.$ (f) $x = 5\cos t,\ y = 5\sin t.$
(g) $x = 2\sin t,\ y = 3\cos t.$ (h) $x = \sec t,\ y = \tan t.$
(i) $x = \sin t,\ y = \sin 2t.$ (j) $x^2 = t,\ y = \log(t - 9).$
(k) $x = \sin\phi,\ y = \cos 2\phi.$ (l) $x = 1 + \cos\phi,\ y = 2\cos\frac{1}{2}\phi.$
(m) $x = a\phi,\ y = a(1 - \cos\phi).$ (n) $x = \log_e t,\ y = \frac{1}{2}t^2.$

2. Eliminate the parameter in Ex. 1 (c), (d), (f), (g), (h), (k), and (l). Name the curve and reduce to a standard form in each case.

3. A projectile leaves a gun with a velocity of 800 feet per second, the barrel of the gun being elevated at an angle $\tan^{-1}\frac{3}{4}$ with the horizontal. Find the equation of its path, using the equations in illustrative example 2. Eliminate the parameter and show that the curve is a parabola. Find the range and the highest point reached.

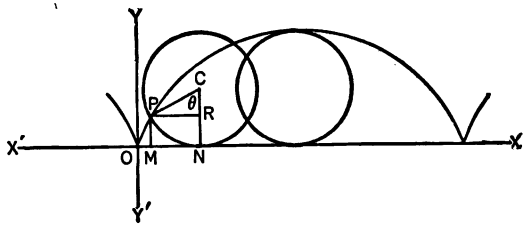
4. Prove that $2x = 3t + \dfrac{3}{t}$, $3y = 3t - \dfrac{3}{t}$, and $x = 3\sec\theta$, $y = 2\tan\theta$ represent the same curve. Plot the curve.

5. Plot the loci of the following equations:

(a) $x = a(\theta + \sin\theta),\quad y = a(1 - \cos\theta).$
(b) $x = a(2\cos t - \cos 2t),\quad y = a(2\sin t - \sin 2t).$
(c) $x = a\cos^3 t,\quad y = a\sin^3 t.$
(d) $x = \dfrac{3at}{1 + t^3},\quad y = \dfrac{3at^2}{1 + t^3}.$

86. Derivation of parametric equations. — Many equations of great importance in their applications, as well as in historical interest, are most readily derived in the form of parametric equations. In some cases this parameter can in the end be easily eliminated, while in others the parametric is the only practicable form of the result. A few illustrations will show the value of the parameter in finding the equations of curves often used in higher mathematics.

87. The Cycloid. — If a circle rolls along a straight line, the locus traced by a fixed point in its circumference is the cycloid.



The equation is derived as follows:

Let the given straight line be taken as the x-axis, and let $P\ (x,\ y)$ represent any point on the rolling circle of radius a. Take as origin the fixed point on the x-axis from which P started to move. From the x-axis erect perpendiculars NC to C, the center of the circle, and MP to P, any position of the generating point. Let θ represent the angle NCP. Then

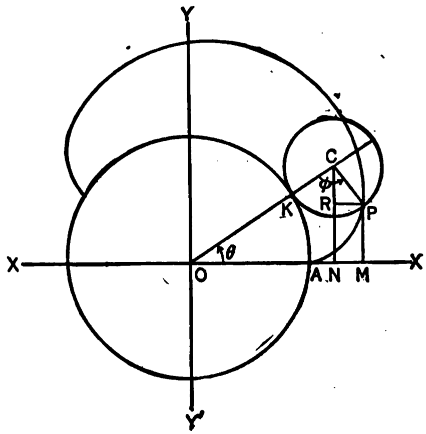
$$x = OM = ON - MN = \text{arc}\, PN - PR = a\theta - a\sin\theta,$$
$$y = MP = NC - RC = a - a\cos\theta.$$

Therefore, the parametric equations of the cycloid are

$$\boldsymbol{x = a\,(\theta - \sin\theta),}$$
$$\boldsymbol{y = a\,(1 - \cos\theta).}$$

88. The epicycloid. — If a circle rolls upon a fixed circle, a point on the circumference of the rolling circle generates a figure called the epicycloid.

Let a circle of radius b roll upon a fixed circle of radius a, and let P, a point on the rolling circle, generate the epicycloid. Take as origin the center of the fixed circle and as x-axis OA, A being the point at which P coincides with the fixed circle. From C the center of the rolling circle and from $P\ (x, y)$, any position of the generating point, draw NC and MP perpendicular to OX, also RP perpendicular to NC. Let angle $NOC = \theta$ and $OCP = \phi$. It can be seen from the figure that $NCP = \phi - (90° - \theta) = \phi + \theta - 90°$.



Therefore, $\quad \sin NCP = -\cos(\phi + \theta)$

and $\quad \cos NCP = \sin(\phi + \theta)$.

But by hypothesis, arc AK = arc PK or $a\theta = b\phi$, whence $\phi = \dfrac{a\theta}{b}$.

Therefore, $\quad \sin NCP = -\cos\left(\dfrac{a\theta}{b} + \theta\right) = -\cos\dfrac{a+b}{b}\theta$

and $\quad \cos NCP = \sin\left(\dfrac{a\theta}{b} + \theta\right) = \sin\dfrac{a+b}{b}\theta.$

Therefore,

$$x = OM = ON + NM = OC\cos\theta + CP\sin NCP$$
$$= (a+b)\cos\theta - b\cos\frac{a+b}{b}\theta,$$

$$y = MP = NC - RC = OC \sin\theta - CP \cos NCP$$
$$= (a+b)\sin\theta - b\sin\frac{a+b}{b}\theta.$$

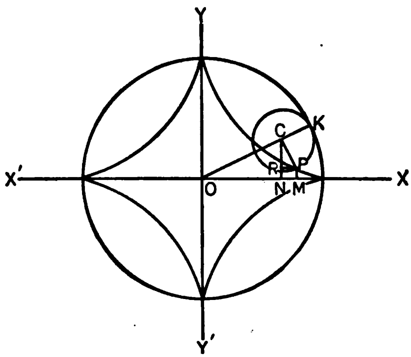
The equations of the epicycloid therefore are

$$x = (a+b)\cos\theta - b\cos\frac{a+b}{b}\theta,$$
$$y = (a+b)\sin\theta - b\sin\frac{a+b}{b}\theta.$$

When a and b are equal the curve is called a **cardioid.** Its equation is

$$x = 2a\cos\theta - a\cos 2\theta,$$
$$y = 2a\sin\theta - a\sin 2\theta.$$

89. The hypocycloid. — If a circle rolls within a fixed circle, a point on the circumference of the rolling circle generates a figure called the hypocycloid.



From the adjoining figure, the equation can be derived in a manner similar to that used in the preceding article, or the result may be obtained from the equations of the epicycloid by substituting $-b$ for b. In either case the results are

$$x = (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\right)\theta,$$
$$y = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\right)\theta.$$

90. Hypocycloid of four cusps. — A particular case of the hypocycloid in which $b = \frac{1}{4} a$ is of frequent occurrence. Making this substitution in the equation just obtained,

$$x = \frac{3}{4} a \cos \theta + \frac{a}{4} \cos 3\theta,$$

$$y = \frac{3}{4} a \sin \theta - \frac{a}{4} \sin 3\theta.$$

Changing $\sin 3\theta$ and $\cos 3\theta$ to terms of θ, these become

$$x = \frac{3}{4} a \cos \theta + \frac{a}{4} (4 \cos^3 \theta - 3 \cos \theta) = a \cos^3 \theta,$$

$$y = \frac{3}{4} a \sin \theta - \frac{a}{4} (3 \sin \theta - 4 \sin^3 \theta) = a \sin^3 \theta.$$

From the foregoing,

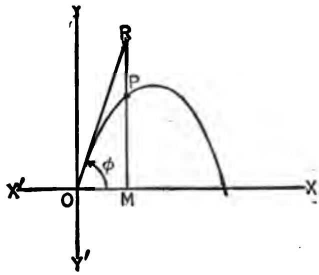
$$x^{\frac{2}{3}} = a^{\frac{2}{3}} \cos^2 \theta,$$

$$y^{\frac{2}{3}} = a^{\frac{2}{3}} \sin^2 \theta.$$

Adding, the result is

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

91. The path of a projectile. — The path traced by a body which is projected at a given angle and with a given velocity is a curve of much importance.



Let O, the initial position of the projectile, be taken as the origin and take OX and OY in horizontal and vertical positions, respectively. Let V_0 denote the initial velocity, ϕ the angle which the initial direction of the projectile makes with the horizontal, and t the time.

If there were no force acting other than that which originally projected the body, the path would be the straight line OR, the distance OR being V_0t.

The principal force tending to deflect the body from a straight line is the action of gravity which tends to pull it vertically downward. Let P represent the position of the body after t seconds, gravity alone being considered.

The ordinate of the point at which the body is found after t seconds is MP instead of MR, the difference being PR which is proved in mechanics to be $16\,t^2$. Then

$$x = OM = OR\cos\phi = V_0 t\cos\phi,$$
$$y = MP = MR - PR = V_0 t\sin\phi - 16\,t^2.$$

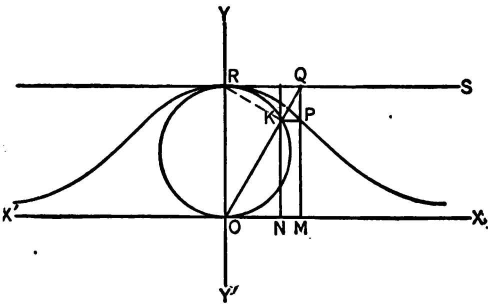
Eliminating t, the curve is found to be the parabola

$$y = x\tan\phi - \frac{16}{V_0^2\cos^2\phi}x^2.$$

In the preceding problem, no account has been taken of the resistance of the air or of other forces of which careful account is taken in figuring the paths of projectiles in military practice.

Exercise. — Find the path of a body which is projected with an initial velocity of 300 feet per second, in a direction inclined 45° with the horizontal.

92. The witch of Agnesi. — A circle of radius a lies between two parallel tangents OX and RS. A chord is drawn through the point of tangency O to the other tangent, meeting the circle at K and the tangent at Q.



Perpendiculars are dropped from Q to OX and from K to OX and MQ. The locus of the intersection P is the witch.

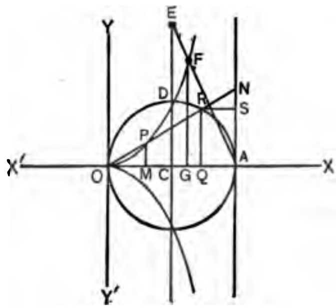
Let $\theta =$ the angle XOQ. Then

$$x = OM = RQ = 2\,a \cot\theta.$$
$$y = MP = NK = OK \sin\theta = 2\,a \sin^2\theta$$

(since the angle OKR is inscribed in a semicircle).

Eliminating θ, $y = \dfrac{8\,a^3}{x^2 + 4\,a^2}$.

93. The cissoid. — A circle of radius a passes through the origin and has its center on the line OA which is taken as the axis of x. A chord OR meets the tangent AN in the point N. If the point P on OR is so chosen that $OP = RN$, the point will describe the cissoid. Let $\theta =$ the angle AON and draw MP and QR perpendicular to OX, and RS perpendicular to AN. Then



$$x = OM = QA = OA - OQ = 2\,a - 2\,a\cos^2\theta = 2\,a\sin^2\theta.$$
$$y = MP = SN = AN - AS = 2\,a\tan\theta - 2\,a\sin\theta\cos\theta$$
$$= 2\,a\sin\theta\,(\sec\theta - \cos\theta).$$

The parametric equations of the cissoid are

$$\boldsymbol{x = 2\,a\sin^2\theta,}$$
$$\boldsymbol{y = 2\,a\sin\theta\,(\sec\theta - \cos\theta).}$$

Solving the first for $\sin\theta$, $\sin\theta = \sqrt{\dfrac{x}{2\,a}}$.

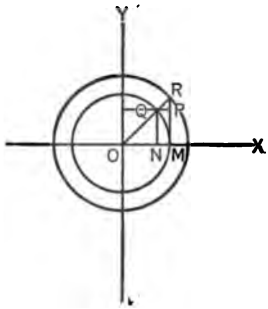
Substituting in the second, $y^2 = \dfrac{x^3}{2\,a - x}$.

Exercise. — Extend the vertical diameter CD to E making $CE = 2\,CD$, and join E to A. Let F be the point where

AE cuts the curve, and draw the ordinate *GF*. By means of the equation of the curve and similar triangles, prove $\overline{GF}^3 = 2\,\overline{OG}^3$.

This fact makes it possible to duplicate a given cube. For if a_1 represents the side of a given cube, and a fourth proportional a_2 is found to *OG*, *GF* and a_1, then a_2 will be the side of the cube of double volume.

The cissoid was invented by a Greek mathematician named Diocles about 150 B.C. His purpose was to solve the famous problem of duplicating the cube.



EXERCISES

1. Find the parametric equation of the circle with radius a and center at origin, in terms of θ, the angle between the x-axis and the radius to any point P.

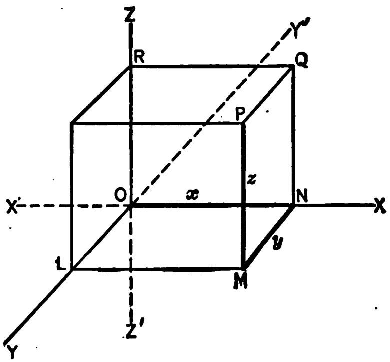
2. A radius is drawn from the center O of two concentric circles, cutting the inner circle at Q and the outer circle at R. Perpendiculars are dropped from R to OX and from Q to OY. Find the equation of the locus traced by the intersection P in terms of the angle $\theta = XOR$. Prove this locus is an ellipse.

Hint. — Draw the perpendicular NQ.

CHAPTER XIV

SOLID ANALYTIC GEOMETRY

94. Rectangular coördinates in space. — If a point is located in a plane, its distances from two fixed perpendicular lines in the plane are determined. If a point is located in space, its distances from three perpendicular planes are determined. In each case these distances are called the **coördinates** of the point.



Construct three mutually perpendicular planes, intersecting in the three perpendicular lines XX', YY', and ZZ'.

The three perpendicular planes are called the **coördinate planes;** the lines of intersection, the **coördinate axes;** and the common point of intersection of the coördinate planes, the **origin.**

Let P be any point in space. Through P draw planes parallel to the coördinate planes forming with them the rectangular parallelopiped $OLMN - P$. The edges, $ON = x$, $NM = y$, $MP = z$, are the **rectangular coördinates** of P. These edges measure the distances of P from the yz-, xz-, and xy-planes, respectively. It is often convenient in locating a point to draw only the lines ON, NM, and MP, taking $x = ON$ on the x-axis, $y = NM$ parallel to the y-axis and $z = MP$ parallel to the z-axis. Coördinates measured to the right, forward, and upward will be considered positive and those measured to the left, backward, and downward, negative. The eight equal parts into which space is divided by the coördinate planes are called **octants.** That octant in which all coördinates are positive is called the first octant. There is no established order in numbering the other octants.

The following suggestions are helpful in constructing figures on cross-section paper. Draw the x- and z-axes at right angles to each other and lay off units as indicated on such paper. Draw the y-axis making equal angles with the other two, and lay off units equal to one-half the diagonal of a square whose side is the unit on the x- and z-axes. This foreshortening of y-units aids in giving the figure the appearance of a solid.

EXERCISES

1. Plot the points $(0, 1, 2)$, $(2, 3, 4)$, $(-1, 4, -3)$, $(1, 0, -5)$, and $(2, -2, -3)$.

2. Where are the points for which $x = 0$? $y = 0$? $z = 0$? What are the equations of the coördinate planes? Where are the points for which both x and $y = 0$?

3. Where are the points for which $x = -1$? $y = 2$? $z = a$?

4. Write the coördinates of the points symmetrical to the following points with respect to each of the axes and with respect to the origin:

(*a*) $(-1, 2, 3)$. (*b*) (a, b, c). (*c*) $(-1, 0, 6)$.

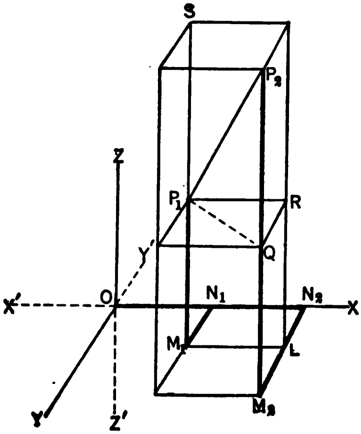
5. Find the coördinates of the feet of the perpendiculars drawn from the point $(2, -1, 3)$ to each of the coördinate planes.

6. From a point (x_1, y_1, z_1) perpendiculars are drawn to each of the coördinate planes. Find the feet of these perpendiculars.

95. Distance between two points.— *The distance between two points P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2) is given by the formula*

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}. \qquad (40)$$

Proof. — Let P_1 and P_2 represent any two given points, and let d represent the distance between them. Through P_1 and P_2 pass planes parallel to the coördinate planes forming the rectangular parallelopiped QS of which the required distance P_1P_2 is the diagonal.



Since P_1QP_2 is a right triangle,

$$P_1P_2 = \sqrt{\overline{P_2Q}^2 + \overline{QP_1}^2},$$

also since P_2QR is a right triangle,

$$\overline{P_2Q}^2 = \overline{P_2R}^2 + \overline{RQ}^2 = \overline{N_2N_1}^2 + \overline{LM_1}^2.$$

Substituting this value of $\overline{P_2Q}^2$,

$$P_1P_2 = \sqrt{\overline{N_2N_1}^2 + \overline{LM_1}^2 + \overline{QP_1}^2}.$$

In terms of the coördinates of P_1 and P_2,

$$P_1P_2 = d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

As a particular case, let ρ equal the distance from the origin to any point $P\ (x, y, z)$, then $\rho = \sqrt{x^2 + y^2 + z^2}$.

EXERCISES

1. Prove that the points (3, 3, 2), (6, −1, −3), (−2, 3, 7) are the vertices of an isosceles triangle.

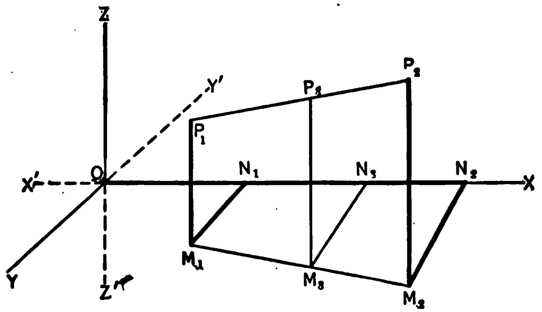
2. Prove that the points (5, 1, 5), (0, −4, 3), (7, −2, 0), and (−3, 3, 5) lie on a sphere whose center is (1, 2, −1).

3. Prove that the points (1, 2, 3), (−1, −2, 1), and (3, 6, 5) are on the same straight line.

4. Prove that the points (6, 7, 3), (3, 11, 1), and (0, 3, 4) are the vertices of a right triangle.

96. Point of Division. — *If the point P_3 divides the line joining the two points P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2) into segments such that the ratio $\frac{P_1P_3}{P_3P_2} = \frac{r_1}{r_2}$, the coördinates are given by the formulas*

$$x_3 = \frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \quad y_3 = \frac{r_1y_2 + r_2y_1}{r_1 + r_2}, \quad z_3 = \frac{r_1z_2 + r_2z_1}{r_1 + r_2}. \tag{41}$$



Proof. — Let P_1 and P_2 be the given points and let P_3 be the point which divides the line joining them in the ratio $r_1 : r_2$. Draw the lines M_1P_1, M_2P_2, and M_3P_3 perpendicular to the xy-plane. By plane geometry,

$$\frac{M_1M_3}{M_3M_2} = \frac{P_1P_3}{P_3P_2} = \frac{r_1}{r_2}.$$

The line M_1M_2 lies in the xy-plane and the x and y coördinates of M_3 which are also the x and y coördinates of P_3 are found, as in Art. 8, to be

$$x_3 = \frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \qquad y_3 = \frac{r_1y_2 + r_2y_1}{r_1 + r_2}.$$

By dropping perpendiculars from P_1, P_2, and P_3 on either of the other coördinate planes, the z-coördinate of P_3 may be found.

EXERCISES

1. Find the coördinates of the point which divides the line joining $(-1, 4, 3)$ and $(-5, -8, 7)$ in ratio $1 : 3$.

2. One extremity of a line is at $(-3, 2, 7)$ and the middle point is $(-1, 4, 2)$. What are the coördinates of the other extremity?

3. In what ratio does the point $(2, 3, 4)$ divide the line joining $(-1, 4, 5)$ and $(8, 1, 2)$?

4. Find the lengths of the medians of the triangle whose vertices are $(2, 5, 6)$, $(3, -7, 4)$, and $(-1, 1, 2)$.

5. In what ratio is the line joining $(5, -1, 4)$ and $(2, -4, -2)$ divided by the xy-plane? Find the coördinates of the point of intersection with this plane.

6. The line joining A $(1, 2, 2)$ and B $(-1, 3, 1)$ is produced to C so that $BC = 3\,AB$. Find the coördinates of C.

7. Two vertices of a triangle are $(2, 3, 0)$ and $(-2, -3, 4)$ and the center of gravity is $(0, 2, \frac{4}{3})$. Find the third vertex.

8. Prove that the lines joining the middle points of the opposite edges of the tetrahedron whose vertices are $(0, 0, 0)$, $(a, 0, 0)$, $(b, c, 0)$, and (d, e, f) meet in a point.

97. Orthogonal projections. — If through a point, a plane is passed perpendicular to a given line in space, the point in which the line pierces the plane is called the **projection of the point on the line.**

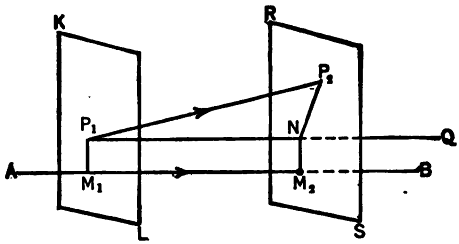
If through the extremities of a directed segment of a line, planes are drawn perpendicular to a given line in space, the portion of this line measured from the projection of the initial point of the segment to the projection of its terminal point is called the **projection of the segment on the line.**

Thus, in the figure of Art. 95, the projection of P_2P_1 on the x-axis is N_2N_1.

The angle between two lines which do not intersect is the angle between two intersecting lines respectively parallel to the two given lines.

Thus, in the figure of Art. 95, the angle between the line P_1P_2 and the axis OX is the angle RP_2P_1.

Theorem I.— The projection of the segment of a directed line upon another line in space is the product of the length of the segment by the cosine of the angle between the two lines.



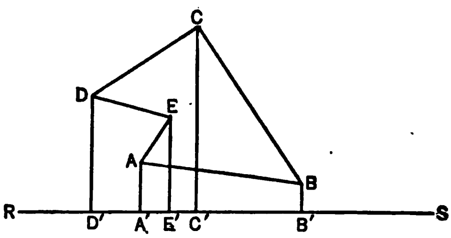
Proof.— Let P_1P_2 be a directed segment making an angle θ with AB, any other line in space. The planes KL and RS through P_1 and P_2 perpendicular to AB determine the projection M_1M_2. It is desired to prove

$$M_1M_2 = P_1P_2 \cos \theta.$$

Draw P_1Q parallel to AB, piercing the plane RS in N and join NP_2. By the definition above, the angle $QP_1P_2 = \theta$. The triangle P_1NP_2 is right angled at N and hence, by trigonometry,

$$M_1M_2 = P_1N = P_1P_2 \cos \theta.$$

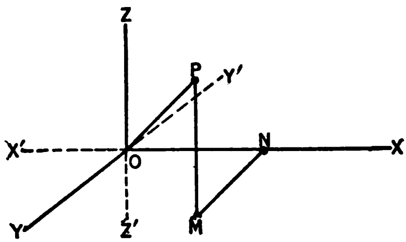
Theorem II.— The sum of the projections on any straight line, of the segments of the broken line joining the point A to the point B, is equal to the projection of the segment AB on that line.



Given the broken line $AEDCB$ and the straight line AB

joining the point A to the point B. Let the projections of the points A, B, C, D, and E on RS be A', B', C', D', and E'. It is evident that $A'E' + E'D' + D'C' + C'B' = A'B'$, that is, the sum of the projections on RS of the segments of the broken line $AEDCB$ is equal to the projection of the straight line AB.

98. Polar coördinates. — Using the same coördinate axes and origin as in the rectangular system, the line OP from the origin to the point P is called the *radius vector* and is represented by ρ. The angles which OP makes with the axes of x, y, and z are called the **direction angles** of the line OP and are represented by α, β, and γ, respectively. ρ, α, β, and γ are called the **polar coördinates** of the point P. The cosines of these angles are called the **direction cosines** of the line OP.



The direction cosines of a line are not independent but are connected by a very important relation, *viz.:*

The sum of the squares of the direction cosines of a line is unity, or

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \tag{42}$$

Proof. — From theorem I, Art. 97, it is evident that

$$x = \rho \cos \alpha,$$
$$y = \rho \cos \beta,$$
$$z = \rho \cos \gamma.$$

Also by Art. 95, $x^2 + y^2 + z^2 = \rho^2$.

Squaring and adding,

$$\rho^2 \cos^2 \alpha + \rho^2 \cos^2 \beta + \rho^2 \cos^2 \gamma = x^2 + y^2 + z^2 = \rho^2,$$

whence $$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

The direction cosines of the line joining the points P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2) are given by the equations:

$$\cos\alpha = \frac{x_2 - x_1}{d}, \quad \cos\beta = \frac{y_2 - y_1}{d}, \quad \cos\gamma = \frac{z_2 - z_1}{d}, \qquad (43)$$

in which d is the length P_1P_2.

This is evident from the figure of Art. 95.

ILLUSTRATIVE EXAMPLES

Find the direction cosines of a line if they are proportional to 1, -2, and 3.

It is given that $\dfrac{\cos\alpha}{1} = \dfrac{\cos\beta}{-2} = \dfrac{\cos\gamma}{3}.$

Then

$$\frac{\cos^2\alpha + \cos^2\beta + \cos^2\gamma}{1^2 + (-2)^2 + 3^2} = \frac{\cos^2\alpha}{1} = \frac{\cos^2\beta}{4} = \frac{\cos^2\gamma}{9}.$$

But the numerator of the first fraction equals 1, hence

$$\cos^2\alpha = \frac{1}{14},$$

$$\cos\alpha = \pm\frac{1}{\sqrt{14}},$$

$$\cos\beta = \pm\frac{2}{\sqrt{14}},$$

and

$$\cos\gamma = \pm\frac{3}{\sqrt{14}}.$$

EXERCISES

1. What are the projections of the point (3, 1, -6) on each of the axes?

2. A line makes an angle of 60° with the x-axis and of 30° with the y-axis. What angle does it make with the z-axis?

3. The direction cosines of a line are equal. Find their values.

4. The direction cosines of a line are proportional to 3, -1, and 2. Find their values.

5. Find the direction cosines of the line joining $(6, 3, -1)$ and $(-2, -1, 0)$, and the projection of the line upon each of the axes.

6. What are the direction cosines of a line parallel to the x-axis? to the y-axis? to the z-axis? of a line perpendicular to the x-axis?

7. A broken line joins $(3, 1, -2)$, $(3, 4, 6)$, $(-1, 2, 3)$, and $(2, -5, -7)$. Find the projections on the x-axis of the closing line and of each of the segments. Verify theorem II, Art. 97.

8. Find the polar and rectangular coördinates of a point, given $\rho = 4$, $\alpha = 120°$, and $\beta = 135°$. How many solutions?

9. Where do all the lines lie which pass through the origin and

(*a*) make an angle of 45° with the y-axis? with the z-axis?

(*b*) make an angle of 45° with both the y-axis and the z-axis? Is there any line making an angle of 45° with each of the coördinate axes?

10. What are the direction cosines of a line if $\alpha = \beta = 90°$? Where are all the points for which $\cos\gamma = 0$?

11. In which octant is a point found when

(*a*) $\cos\alpha > 0,\ \cos\beta < 0,\ \cos\gamma > 0$?

(*b*) $\cos\alpha < 0,\ \cos\beta < 0,\ \cos\gamma < 0$?

(*c*) $\cos\alpha > 0,\ \cos\beta > 0,\ \cos\gamma < 0$?

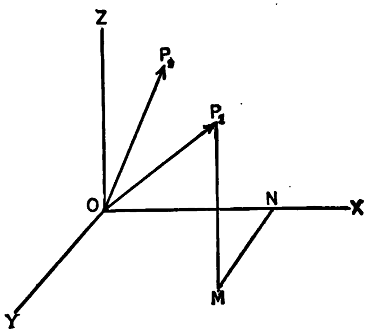
Name the octant by indicating the signs of the axes.

12. If the projections of P_1P_2 on the axes are respectively 4, 2, and -4 and the coördinates of P_1 are $(6, -3, 2)$, find the coördinates of P_2.

13. Given $\beta = 30°$, $y = 2$, $z = -1$. Find the coördinates of the point in both rectangular and polar coördinates. How many solutions?

14. Prove by means of direction cosines that the points $(1, 2, 3)$, $(-1, -2, 1)$, and $(3, 6, 5)$ are on the same straight line.

99. The angle between two directed lines. — *The cosine of the angle between two directed lines is equal to the sum of the products of the corresponding direction cosines.*



Let α_1, β_1, γ_1 and α_2, β_2, γ_2 represent the direction angles of two given lines and let OP_1 and OP_2 be two lines through the origin parallel to those lines. Also let θ equal the angle P_1OP_2. Construct ON, NM, and MP_1, the coördinates of P_1.

If the broken line $ONMP_1$ and the straight line OP_1 are projected on OP_2, from theorem II, Art. 97,

$$\text{Proj. } OP_1 = \text{Proj. } ON + \text{Proj. } NM + \text{Proj. } MP_1,$$

or by theorem I, Art. 97,

$$OP_1 \cos\theta = ON \cos\alpha_2 + NM \cos\beta_2 + MP_1 \cos\gamma_2.$$

But $\quad ON = OP_1 \cos\alpha_1,\ NM = OP_1 \cos\beta_1,$
$\quad MP_1 = OP_1 \cos\gamma_1.$

Substituting and dividing by OP_1,

$$\cos\theta = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2. \quad (44)$$

It is evident that if two lines are parallel and extend in the same direction, $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$; if parallel and extending in opposite directions, $\alpha_1 = \pi - \alpha_2$, $\beta_1 = \pi - \beta_2$, $\gamma_1 = \pi - \gamma_2$.

If two lines are perpendicular, $\cos\theta = 0$ and, therefore,

$$\cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2 = 0.$$

EXERCISES

1. Find the angle between two lines whose direction cosines are $\frac{2}{7}$, $-\frac{3}{7}$, $\frac{6}{7}$ and $\frac{2}{3}$, $\frac{1}{3}$, $\frac{2}{3}$, respectively?

2. Find the angle between two lines whose direction cosines are proportional to 1, 2, 5 and -1, 3, -2, respectively.

3. Prove that the lines whose direction cosines are $\frac{2}{3}$, $\frac{1}{3}$, $\frac{2}{3}$; $-\frac{1}{3}$, $-\frac{2}{3}$, $\frac{2}{3}$; and $\frac{2}{3}$, $-\frac{2}{3}$, $-\frac{1}{3}$ are mutually perpendicular.

4. Find the direction cosines of the line joining $(-1, 2, 4)$ and $(6, 5, -3)$.

5. Find the length of the projection of the line joining $(1, 1, 2)$ and $(2, -1, 4)$ upon the line joining $(2, 1, -2)$ and $(4, -5, 1)$.

100. The equation of a locus. — If a point moves in space according to some law, it traces a locus which will, in general, be a surface. To find the equation of this locus, the same steps will be followed as in plane analytic geometry, *viz.*, the discovery of some law which applies to the moving point in all of its positions and the translation of this law into an algebraic equation between the coördinates of the point.

Thus, to find the equation of a sphere of radius 5 and center at the origin, it is seen that if P (x, y, z) represents any point on the surface of the sphere, $OP = 5$. Whence $x^2 + y^2 + z^2 = 25$ is the equation of the surface of the sphere.

Again, the equation of a plane parallel to the yz-plane and three units to the right of it is $x = 3$, since every point in the given plane is at a distance 3 from the yz-plane.

EXERCISES

1. Find the equation of the plane which is

(*a*) parallel to the yz-plane and 4 units to the left of it.
(*b*) parallel to the xz-plane and 3 units in front of it.

2. Find the equation of the locus of a point which is equidistant from the points $(1, 0, -2)$ and $(2, -3, 0)$.

3. Find the equation of the locus of a point which is equidistant from the xy- and yz-planes.

4. Find the equation of the locus of a point

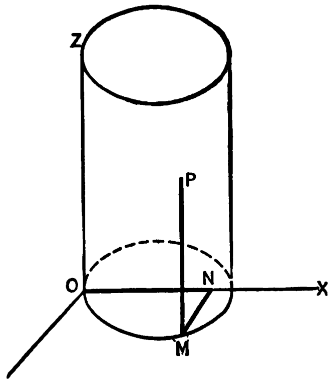
(*a*) whose distance from the x-axis is equal to 5.
(*b*) whose distance from the y-axis is equal to its distance from the xz-plane.
(*c*) whose distance from the x-axis is equal to its distance from the z-axis.

5. Find the equation of the locus of a point whose distance from the point $(3, 1, 0)$ is equal to its distance from the y-axis.

101. Cylindrical surface with elements parallel to one of the coördinate axes. — The method of finding the equation of such a surface is illustrated by the following example:

Find the equation of the cylindrical surface whose directing curve in the xy-plane is $x^2 - 4x + 4y^2 = 0$, and whose axis is parallel to the z-axis.

Let $P(x, y, z)$ be any point in this surface. The x and y coördinates of P on the surface are the same as those of M on the directing curve, and therefore the coördinates of P satisfy the equation $x^2 - 4x + 4y^2 = 0$. (Since z does not appear in this equation, it may have any value.) This equation therefore is the equation of the surface of the cylinder.



In general, *the equation of a cylindrical surface whose axis is parallel to one of the axes is the same as the equation of the generating curve in the plane of the other two axes.*

102. Spherical surface. — *The equation of a sphere whose center is at $C(h, k, l)$ and whose radius is r is*

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2. \qquad (45)$$

This equation results immediately from Art. 95.

EXERCISES

1. Describe the following surfaces: (*a*) $x^2 - z^2 = 4$. (*b*) $x^2 + y^2 = 4x$. (*c*) $y = \cos x$. (*d*) $4y^2 + z^2 + 8y = 0$.

2. Find center and radius of each of the following spheres:

(*a*) $x^2 - 2x + y^2 - 6y + z^2 + 2z - 5 = 0$.
(*b*) $4x^2 + 4y^2 + 4z^2 - 4x + 12y - 20z = 1$.

3. Find the equation of the sphere whose center is on the z-axis, whose radius is 7, and which passes through the point $(2, -3, 4)$.

4. The axis of a cylinder is parallel to the x-axis and its directing curve is a circle in the yz-plane with radius 5, with center on the z-axis and tangent to the y-axis. Find the equation of the cylinder.

5. Find the equation of the sphere whose diameter is the line joining (2, 4, −3) and (2, −2, 1).

6. Find the equation of the sphere whose center is at (−1, 3, −5) and which passes through the point (−3, 6, 1).

7. Find the equation of a sphere through the four points (0, 0, 0), (−3, 0, 3), (0, 3, 11), and (0, −8, 0).

8. Find the equation of the sphere of radius 7 whose center is in the yz-plane and which passes through the points (−2, −3, 3) and (−3, 6, −1).

9. Find the equation of the sphere with center at the origin and which is tangent to the sphere $x^2 - 12x + y^2 + 4y + z^2 - 6z + 24 = 0$.

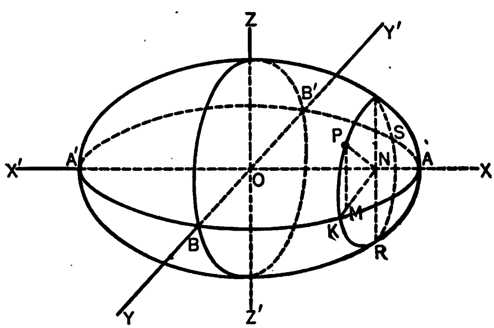
10. Find the equation of the sphere concentric with $x^2 - 2x + y^2 + 6y + z^2 - 8z + 1 = 0$ and passing through the point (5, 1, −3).

11. Prove that if a point moves so that the sum of the squares of its distances from (0, 0, 1) and (−1, 1, 0) is 7, its locus is a sphere. Find its center and radius.

12. Find the equation of the locus of a point which moves so that its distance from (−3, −6, 3) is twice its distance from the origin. Prove that the locus is a sphere and find its center and radius.

13. Find the equation of the locus of a point which moves so that its distance from the x-axis is equal to its distance from the point (1, 0, 2). Describe and construct the surface.

103. A surface of revolution is formed by revolving a plane curve about an axis in its plane. If the equation of a curve in one of the coördinate planes is given, and if the axis of revolution is one of the coördinate axes, the equation of the surface is readily found.



Thus, find the equation of the surface formed by revolving the ellipse $4x^2 + 9y^2 = 36$ about OX.

Let the ellipse $ABA'B'$ be revolved about OX. In order to avoid confusion of the coördinates of any point on the surface with the coördinates of the points on the generating curve, let $P(x', y', z')$ represent any point on the surface. Then $x' = ON$, $y' = NM$, $z' = MP$. Pass a plane through P perpendicular to OX. This section $PKRS$ is evidently a circle. In the triangle NMP,

$$\overline{NM}^2 + \overline{MP}^2 = \overline{NP}^2 \text{ or } y'^2 + z'^2 = \overline{NP}^2 = \overline{NK}^2 = y^2.$$

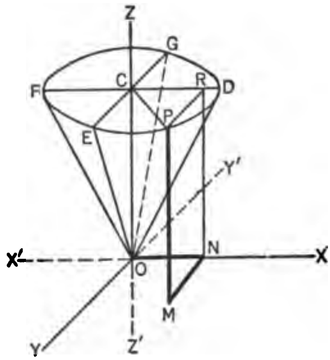
It is now required to express y^2 in terms of the coördinates of P. From the equation of the generating ellipse,

$$y^2 = \tfrac{4}{9}(9 - x^2) = \tfrac{4}{9}(9 - x'^2).$$

Hence
$$y'^2 + z'^2 = \tfrac{4}{9}(9 - x'^2)$$

or, dropping primes and simplifying,

$$\frac{y^2}{4} + \frac{z^2}{4} + \frac{x^2}{9} = 1.$$



Again, find the equation of the conical surface formed by revolving the line $z = 2x$ about OZ.

Let $P(x', y', z')$ be any point on the surface of the cone. Then $x' = ON$, $y' = NM$, $z' = MP$. Pass planes $DEFG$ and $MNRP$ through P perpendicular to OZ and OX respectively. Then $CR = ON = x'$, $RP = NM = y'$.

But $\overline{CR}^2 + \overline{RP}^2 = \overline{CP}^2$, or $x'^2 + y'^2 = \overline{CP}^2 = \overline{CD}^2 = x^2$.

From the equation of the generating line, $x = \tfrac{1}{2}z = \tfrac{1}{2}z'$; hence

$$x'^2 + y'^2 = \tfrac{1}{4}z'^2$$

or, dropping primes and simplifying,

$$4x^2 + 4y^2 = z^2.$$

EXERCISES

1. Find the equations of the surfaces of revolution generated by revolving

(*a*) $y = x$ about the x-axis.
(*b*) $4x^2 + y^2 = 16$ about the y-axis.
(*c*) $x^2 = 4z$ about the z-axis.
(*d*) $x^2 - z^2 = 4$ about the z-axis.
(*e*) $y - x = 1$ about the y-axis.

2. Find the equation of the surface generated by revolving the ellipse $x^2/a^2 + y^2/b^2 = 1$

(*a*) about its major axis. This surface is called a *prolate spheroid.*
(*b*) about its minor axis. This surface is called an *oblate spheroid.*

3. Find the equation of the surface generated by revolving about the x-axis, the line $z = 4$ in the xz-plane.

4. Show that the surface obtained by revolving the parabola $y^2 = 4x$ about the x-axis is the same as that obtained by revolving the parabola $z^2 = 4x$ about the x-axis.

5. Find the equation of the surface generated by revolving the circle $x^2 + y^2 = a^2$ around the y-axis. What curve in the xz-plane would generate the same surface when revolved about the x-axis?

6. A circle in the xz-plane of radius 4, with center on the z-axis at a distance 7 from the origin, is revolved about the z-axis. Find the equation of the surface generated.

104. Equations of a curve. — Two surfaces intersect in a curve. The equations of two surfaces when considered simultaneously define the curve of intersection, since the coördinates of any point on this curve of intersection satisfy both equations.

If the equations of the two surfaces are combined so as to obtain a third equation, this equation represents another surface through the curve of intersection, and this together with either of the given equations defines the curve of intersection. It is thus seen that the same curve may be represented in an infinite number of ways. In particular, if one of the variables is eliminated between the equations of the surfaces which define a curve, the resulting equation, containing but two variables, is a cylinder with elements parallel to one of the axes, and passing through the given curve. This is called a **projecting cylinder.**

105. The locus of an equation. — It is evident from Art. 100 that *every equation in one variable represents a plane or series of planes parallel to one of the coördinate planes.*

Also from Art. 101, it is evident that *every equation in two variables represents a cylindrical surface, the equation of whose directing curve in one of the coördinate planes is the same as the given equation and whose elements are perpendicular to the plane of this curve.*

In determining the loci of most other equations, a discussion somewhat similar to that used in plane analytic geometry is helpful. The principal points in such a discussion are:

1*st*. Symmetry.

2*nd*. Intercepts on the coördinate axes.

3*rd*. Intersections on the coördinate planes.

4*th*. Intersections on planes parallel to the coördinate planes.

Symmetry. — A locus is symmetrical with respect to

(*a*) one of the coördinate planes, if the variable corresponding to the axis perpendicular to that plane can be changed in sign without changing the equation.

(*b*) one of the coördinate axes, if the variables corresponding to the other two axes can be changed in sign without changing the equation.

(*c*) the origin, if all three variables can be changed in sign without changing the equation.

The proof is similar to that in Art. 13.

Intercepts on the coördinate axes. — These are found by setting two of the variables equal to zero and solving for the third.

Intersections of a surface with the coördinate planes. — These intersections are found by treating the equations of the coördinate planes $x = 0$, $y = 0$, and $z = 0$ simultaneously with the equation of the given surface. These curves are called the *traces*.

Intersections of a surface by planes parallel to the coördinate planes. — Represent these planes by $x = k$, $y = k_1$, $z = k_2$. These taken simultaneously with the equation of the given surface determine the curves of intersection. By giving k, k_1, k_2 different values, the general form and limits of the surface are determined.

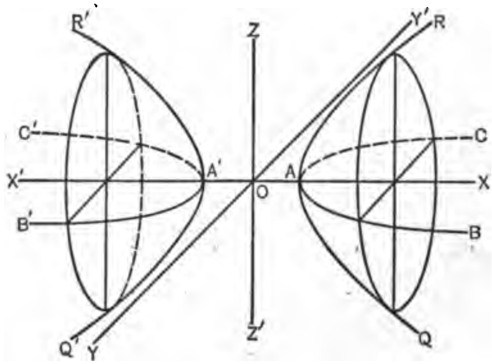
Frequently it is sufficient to discuss the set of planes parallel to but one coördinate plane.

ILLUSTRATIVE EXAMPLE

Discuss and construct the locus of $x^2 - y^2 - z^2 = 4$.

1st. This surface is evidently symmetrical with respect to the coördinate planes, the coördinate axes and the origin.

2nd. Intercepts on x-axis are ± 2. There is no intercept on y-axis or z-axis.



3rd. Let $x = 0$, then $y^2 + z^2 = -4$. Therefore the surface does not intersect the yz-plane. Let $y = 0$, then $x^2 - z^2 = 4$. Therefore, the trace is an hyperbola $QARQ'A'R'$ in the xz-plane. Let $z = 0$, then $x^2 - y^2 = 4$. Therefore, the trace is an hyperbola $BACB'A'C'$ in the xy-plane.

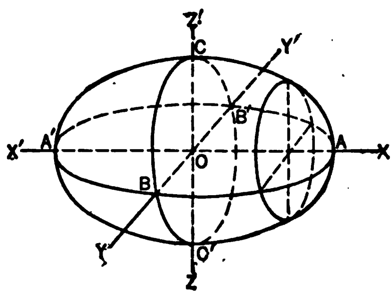
4th. To find the intersection of the plane $x = k$ with the surface, substitute $x = k$ in the equation of the surface. The result is $y^2 + z^2 = k^2 - 4$, which is a cylinder whose trace in the yz-plane is the circle

$y^2 + z^2 = k^2 - 4$ and whose elements are parallel to the axis of x. The curve of intersection of the surface by the plane is the same as the curve of intersection of the cylinder by the plane, that is, a circle of radius $\sqrt{k^2 - 4}$. If $-2 < k < 2$, the radius is imaginary and there is no intersection and if $k > 2$ or < -2, the intersection is a circle whose radius increases without limit as k increases without limit.

Without considering planes parallel to the other coördinate planes, the surface can be sketched as above.

106. Quadric surfaces. — The general equation of second degree in three variables is $Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Kz + L = 0$. The surface represented by this equation is called a quadric surface. Some special forms of this equation which are of frequent occurrence will be discussed here.

107. The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.



1st. This surface is symmetrical with respect to the origin, the coördinate planes, and the coördinate axes.

2nd. The intercepts on the axes are $x = \pm a$, $y = \pm b$, $z = \pm c$.

3rd. The traces in the coördinate planes are the ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$, and $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

4th. The intersection with the plane $x = k$ is the ellipse

$$\frac{y}{b^2} + \frac{z^2}{c^2} = \frac{a^2 - k^2}{a^2}, \quad x = k,$$

or

$$\frac{y^2}{\frac{b^2}{a^2}(a^2 - k^2)} + \frac{z^2}{\frac{c^2}{a^2}(a^2 - k^2)} = 1, \quad x = k.$$

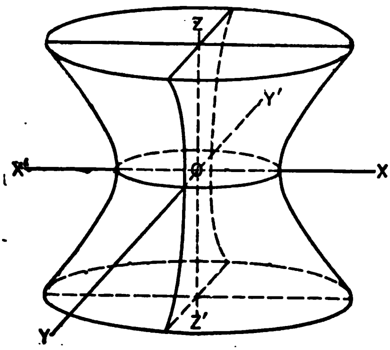
It is seen that as k increases numerically, the semi-axes $\frac{b}{a}\sqrt{a^2 - k^2}$ and $\frac{c}{a}\sqrt{a^2 - k^2}$ decrease from b and c respectively when $k = 0$ to 0 when $k = \pm a$. If $k > a$ or $< -a$ the ellipse is imaginary. The ellipsoid then lies between the planes $x = \pm a$.

A similar discussion shows that the sections made by planes parallel to the other coördinate planes are also ellipses, and that the figure lies between the planes $z = \pm c$ and $y = \pm b$.

108. The hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

1st. The surface is symmetrical with respect to the origin, the coördinate planes, and the coördinate axes.



2nd. The intercepts on the axes are $x = \pm a$, $y = \pm b$. There is no intercept on the z-axis.

3rd. The traces in the coördinate planes are the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ and the hyperbola $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.

4th. The intersection with the plane $z = k$ is the ellipse

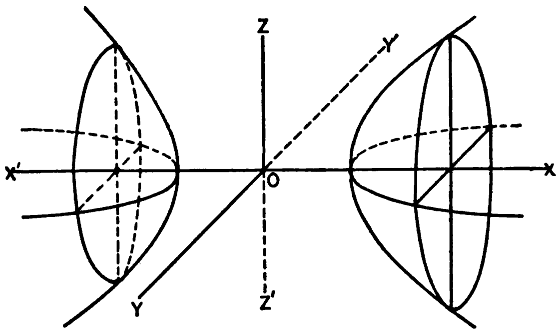
$$\frac{x^2}{\frac{a^2}{c^2}(c^2 + k^2)} + \frac{y^2}{\frac{b^2}{c^2}(c^2 + k^2)} = 1, \quad z = k.$$

It is seen that as k increases numerically from 0 to ∞, the semi-axes of the ellipse increase without limit, and therefore the surface extends indefinitely in the direction of the z-axis.

The intersections in planes parallel to the other coördinate planes are hyperbolas.

109. The hyperboloid of two sheets

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$



1*st*. The surface is symmetrical with respect to the origin, the coördinate planes, and the coördinate axes.

2*nd*. The intercepts on the x-axis are $\pm a$. There are no intercepts on the other axes.

3*rd*. The traces in the coördinate planes are the hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$. There is no trace in the yz-plane.

4*th*. The intersection with the plane $x = k$ is the ellipse

$$\frac{y^2}{\frac{b^2}{a^2}(k^2 - a^2)} + \frac{z^2}{\frac{c^2}{a^2}(k^2 - a^2)} = 1, \quad x = k.$$

If $-a < k < a$, the ellipse is imaginary. If k increases numerically from a to ∞, the semi-axes increase indefinitely.

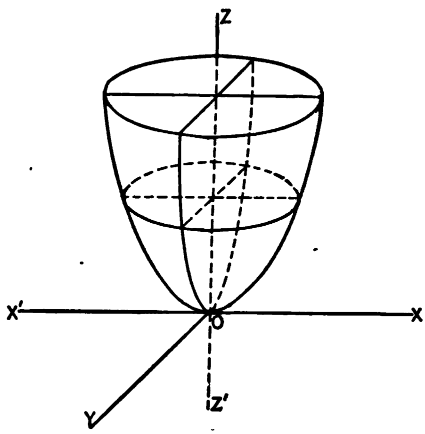
The intersections in planes parallel to the other coördinate planes are hyperbolas.

110. The elliptic paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz.$

1*st*. The surface is symmetrical with respect to the yz-plane and to the xz-plane, but not to the xy-plane. It is symmetrical with respect to the z-axis only. It is not symmetrical with respect to the origin.

2*nd*. The surface intersects the three axes at the origin only.

3*rd*. The traces in the coördinate planes are the point ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$, and the parabolas $\frac{x^2}{a^2} = cz$ and $\frac{y^2}{b^2} = cz$.



4*th*. The intersection with the plane $z = k$ (where k has the same sign as c) is the ellipse

$$\frac{x^2}{a^2kc} + \frac{y^2}{b^2kc} = 1, \quad z = k.$$

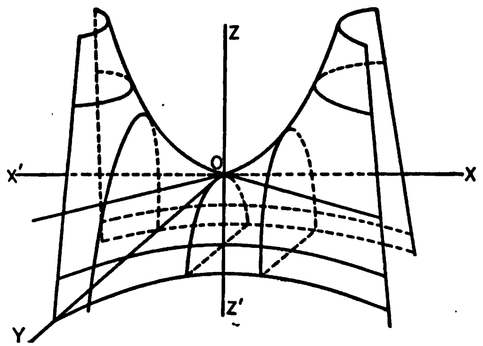
As k increases from 0 to ∞, the semi-axes increase from 0 to ∞ and therefore the surface extends indefinitely in the direction of the z-axis, lying entirely above the xy-plane when c is positive and entirely below if c is negative.

The intersections in the planes parallel to the other coördinate planes $x = k_1$ and $y = k_2$ are parabolas whose vertices recede from the xy-plane as k_1 and k_2 increase in numerical value.

111. The hyperbolic paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = cz$.

The following discussion considers c positive.

1st. The surface is symmetrical with respect to the yz-plane and to the xz-plane, but not to the xy-plane. It is symmetrical with respect to the z-axis only. It is not symmetrical with respect to the origin.



2nd. The surface intersects the three axes at the origin only.

3rd. The traces in the coördinate planes are the intersecting straight lines $x^2/a^2 - y^2/b^2 = 0$ and the two parabolas $x^2/a^2 = cz$ and $y^2/b^2 = -cz$.

4th. The intersection in the plane $z = k$ is the hyperbola

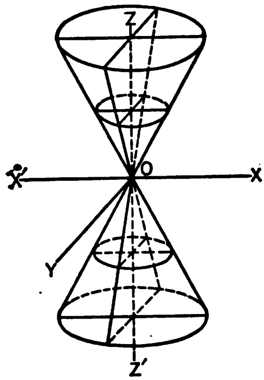
$$\frac{x^2}{a^2ck} - \frac{y^2}{b^2ck} = 1, \quad z = k.$$

If k is positive, the hyperbola in the plane $z = k$ has its principal axis parallel to the x-axis and its vertices on the parabola $\frac{x^2}{a^2} = cz$. These vertices recede indefinitely as k increases from 0 to $+\infty$. If k is negative, the hyperbola in the plane $z = k$ has its principal axis parallel to the y-axis and its vertices on the parabola $\frac{y^2}{b^2} = -cz$.

In a similar manner, the intersection of the surface by the plane $x = k_1$ is the parabola $\frac{y^2}{b^2} = -cz + \frac{k_1^2}{a^2}$, $x = k_1$, which has its vertex on the parabola $\frac{x^2}{a^2} = cz$. The intersection of the surface by the plane $y = k_2$ is the parabola $\frac{x^2}{a^2} = c + \frac{k_2^2}{b^2}$, $y = k_2$, which has its vertex on the parabola $\frac{y^2}{b^2} = -cz$. The surface extends indefinitely along all axes.

112. The cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$.

1st. The surface is symmetrical with respect to the origin, the coördinate planes, and the coördinate axes.



2nd. The surface intersects the three axes at the origin only.

3rd. The trace in the xy-plane is the point ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$, and in the xz-plane the intersecting straight lines $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 0$, and in the yz-plane the intersecting straight lines $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$.

4th. The intersection with the plane $z = k$ is the ellipse

$$\frac{x^2}{\frac{a^2k^2}{c^2}} + \frac{y^2}{\frac{b^2k^2}{c^2}} = 1, \quad z = k.$$

As k increases numerically, the semi-axes of the ellipse increase and the surface extends indefinitely in the direction of the z-axis.

The intersections in the planes $x = k_1$ and $y = k_2$ are hyperbolas with vertices on the straight lines $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 0$ and $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$, respectively.

EXERCISES

1. Discuss and construct the surfaces represented by the following equations:

(a) $4x^2 + 9y^2 + z^2 = 36.$
(b) $4x^2 - 9y^2 + z^2 = 36.$
(c) $4x^2 + z^2 = 4y.$
(d) $y^2 - x^2 - z^2 = 4.$
(e) $4x^2 - z^2 = 4y.$
(f) $9y^2 - 4x^2 - z^2 = 36.$
(g) $4x^2 - 9y^2 - 4z^2 = 0.$
(h) $y^2/b^2 - x^2/a^2 - z^2/c^2 = 1.$

2. Find the equation of the locus of a point which moves so that the sum of the squares of its distances from the x- and z-axes equals 4. Discuss and construct the locus.

3. A point moves so that the sum of its distances from two fixed points is constant. Prove that the locus is an ellipsoid.

Hint. — Take the straight line through the two points as x-axis and the point halfway between as origin.

4. A point moves so that the difference of its distances from two fixed points is constant. Prove that the locus is an hyperboloid.

5. Find the equation of the locus of a point which moves so that its distance from the xy-plane increased by 1 is equal to $1/\sqrt{2}$ times its distance from the point $(0, 0, -4)$.

6. Prove that the sections of the paraboloid $x^2/a^2 + y^2/b^2 = cz$ by planes parallel to the yz-plane are equal parabolas; also those parallel to the xz-plane.

7. Discuss and construct the locus of $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} - \dfrac{z^2}{c^2} = 1.$ Show that sections parallel to the xy-plane are circles. What curve revolved about the z-axis would generate this surface?

8. A point moves so that the sum of the squares of its distances from two perpendicular lines is constant. Prove that the locus is an ellipsoid.

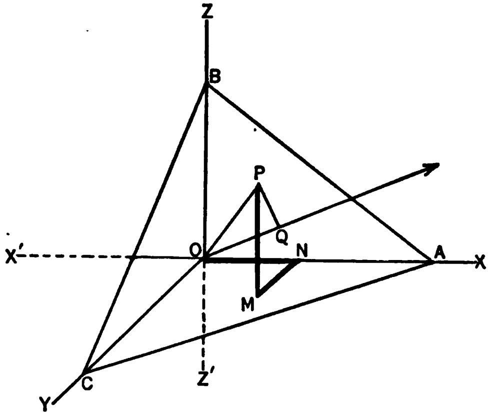
113. The normal form of the equation of a plane. — A plane is determined in position if the length of a perpendicular from the origin upon the plane and the direction angles of this perpendicular are known. This perpendicular from the origin to the plane is called the **normal to the plane**.

The normal form of the equation of a plane is

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p, \tag{46}$$

where p is the perpendicular distance from the origin to the plane, and α, β, and γ the direction angles of that perpendicular.

Proof. — Let ABC be any plane and let the perpendicular from the origin upon it be the line OQ which makes the angles α, β, and γ with the axes of x, y, and z, respectively. The direction OQ from the origin to the plane is always taken as positive, also α, β, and γ are considered positive angles.



Let $P\ (x, y, z)$ represent any point in the plane and draw its coördinates $ON = x$, $NM = y$, and $MP = z$.

Project $ONMP$ and OP on OQ. By theorem II, Art. 97, projection ON + projection NM + projection MP = projection OP. By theorem I, Art. 97, this becomes

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p.$$

This equation is seen to be of first degree.

114. *The general equation of first degree*

$$Ax + By + Cz + D = 0$$

represents a plane.

Proof. — Consider the equations

$$Ax + By + Cz + D = 0 \tag{1}$$

and $$x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0. \tag{2}$$

Equation (2) represents a plane. Equation (1) also represents a plane if it differs from equation (2) only by a constant multiplier as K. If, then,

$$KA = \cos\alpha, \quad KB = \cos\beta, \quad KC = \cos\gamma, \quad \text{and} \quad KD = -p,$$

it is desired to show that K can be determined.

By Art. 98, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ and, therefore,

$$K^2A^2 + K^2B^2 + K^2C^2 = 1 \text{ or } K = \frac{1}{\pm\sqrt{A^2 + B^2 + C^2}}.$$

This shows that equation (2) represents a plane in which

$$\cos\alpha = \frac{A}{\pm\sqrt{A^2 + B^2 + C^2}}, \quad \cos\gamma = \frac{C}{\pm\sqrt{A^2 + B^2 + C^2}},$$

$$\cos\beta = \frac{B}{\pm\sqrt{A^2 + B^2 + C^2}}, \quad p = \frac{-D}{\pm\sqrt{A^2 + B^2 + C^2}}.$$

Since p is always positive, the sign of the radical will be opposite to that of the constant term.

115. Plane determined by three conditions. — Of the four coefficients in the equation

$$Ax + By + Cz + D = 0 \tag{1}$$

only three are independent, and therefore three conditions are sufficient to determine three of them in terms of a fourth. After substituting these values, the equation can be divided by the fourth coefficient.

116. The equation of a plane in terms of its intercepts. — *The equation of a plane in terms of a, b, and c, the intercepts on the axes, is*

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \tag{47}$$

Let the intercepts of a plane on the axes of x, y, and z be a, b, and c, respectively. The three points $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$ on the plane are therefore known and the method suggested in Art. 115 applies.

Substituting the coördinates of these three points in equation (1) of that article,

$$Aa + D = 0, \quad Bb + D = 0, \quad Cc + D = 0.$$

Whence

$$A = -\frac{D}{a}, \quad B = -\frac{D}{b}, \quad C = \frac{D}{c}.$$

Substituting in (1),

$$-\frac{Dx}{a} - \frac{Dy}{b} - \frac{Dz}{c} + D = 0.$$

Dividing by D and transposing,

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

117. The angle between two planes. — *The cosine of the angle between two planes whose equations are of the form* $Ax + By + Cz + D = 0$ *and* $A_1x + B_1y + C_1z + D_1 = 0$ *is given by the equation*

$$\cos\theta = \frac{AA_1 + BB_1 + CC_1}{\sqrt{A^2 + B^2 + C^2}\sqrt{A_1^2 + B_1^2 + C_1^2}}. \tag{48}$$

The angle between two planes is evidently the same as the angle between their normals. Substituting in the formula of Art. 99, the values of $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ found in Art. 114, the above formula results immediately.

If two planes are parallel, $\frac{A}{A_1} = \frac{B}{B_1} = \frac{C}{C_1}$; *and if they are perpendicular,* $AA_1 + BB_1 + CC_1 = 0$. The proof is left to the student.

118. The distance from a plane to a point. — By finding the equation of a plane parallel to the given plane and passing through the given point, and computing the difference of the distances from the origin to the planes, it is found that *the distance from the plane* $Ax + By + Cz + D = 0$ *to the point* $P_1\ (x_1, y_1, z_1)$ *is*

$$\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}} \tag{49}$$

in which the sign of the radical is opposite to the sign of D.

EXERCISES

1. Reduce the following equations to intercept and normal forms:

(*a*) $7x - 2y - 2z + 14 = 0$.
(*b*) $2x + 6y - 3z - 42 = 0$.

2. Find the equations of the planes which satisfy the following conditions:

(*a*) passing through the points $(1, 1, 0)$, $(-2, 1, 2)$, and $(4, 0, 1)$.
(*b*) parallel to the plane $7x + 2y + 10z + 25 = 0$, and passing through $(3, 1, -2)$.
(*c*) perpendicular to the plane $3x + 2y - z + 11 = 0$ and passing through the points $(1, 0, 1)$ and $(-1, 1, 1)$.
(*d*) x-intercept $= 5$, y-intercept $= 3$, and z-intercept $= -7$.
(*e*) distance from origin to plane $= 5$, $\cos\alpha = \frac{2}{3}$, and $\cos\beta = -\frac{1}{3}$.
(*f*) passing through the point $(1, 5, 6)$ and perpendicular to each of the planes $4x - 5y + 2z = 5$ and $x - y + z = 3$.
(*g*) passing through the points $(1, -2, 3)$ and $(5, 0, 3)$ and at a distance of 3 from the origin.
(*h*) at a distance of 2 from the origin, the normal making equal angles with the axes.
(*i*) perpendicular to the line joining the points $(4, 3, 1)$ and $(1, 3, 5)$ at its middle point.
(*j*) containing the z-axis and the point (x_1, y_1, z_1).
(*k*) passing through the line of intersection of the planes $4x + y + 2z = 3$ and $2x + y + z = 1$, and perpendicular to the plane $3x + 4y - 2z = 7$.

Hint. — The equation of the plane through the line of intersection of the two given planes is $4x + y + 2z - 3 + \lambda(2x + y + z - 1) = 0$.

(*l*) perpendicular to the line through the points $(4, 3, 1)$ and $(2, 4, -1)$, and five units from the origin.

3. Prove that the planes $x + y + z - 1 = 0$, $2x + y - z = 0$, $x + 6y + 4z + 1 = 0$, and $5x + y - 4 = 0$ meet in a point.

4. Prove that the four points (8, 15, 4), (2, 1, 0), (0, 3, 2), and (2, 3, 1) lie in a plane.

5. Find the distance from the origin to the plane through (0, −3, 2), (2, 1, 2), and (5, 3, 0). In which octant does the foot of the normal lie?

6. Find the angles between the following planes:

(a) $4x - 7y + 4z = 5$ and $3x + 4y = 17$.
(b) $3x - 2y + 6z = 7$ and $4x - 3y + 12z = 0$.

7. Find the equations of the planes bisecting the angles between the planes $4x - 7y + 4z + 15 = 0$ and $2x - y - 2z - 5 = 0$.

8. So determine K that the plane $3x + Ky + 12z = 26$ shall be

(a) two units from the origin.
(b) perpendicular to the plane $x + 9y - z = 5$.

119. The general equations of a straight line. — Two planes intersect in a straight line. It has been shown that the locus of an equation of first degree is a plane and that the curve of intersection of two surfaces is defined by considering their equations simultaneously, hence:

The locus of two equations of first degree

$$Ax + By + Cz + D = 0, \quad A_1x + B_1y + C_1z + D_1 = 0$$

is a straight line.

120. The equations of a straight line through a given point and in a given direction. — *The equations of a straight line passing through the given point $P_1(x_1, y_1, z_1)$ and having direction angles α, β, and γ are*

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma}. \tag{50}$$

Proof. — Let $P(x, y, z)$ be any other point on the line, then by Art. 98,

$$\cos\alpha = \frac{x - x_1}{d}, \quad \cos\beta = \frac{y - y_1}{d}, \quad \cos\gamma = \frac{z - z_1}{d}.$$

Solving each of these equations for d and equating,

$$\frac{x - x_1}{\cos \alpha} = \frac{y - y_1}{\cos \beta} = \frac{z - z_1}{\cos \gamma}.$$

If instead of $\cos \alpha$, $\cos \beta$, and $\cos \gamma$, numbers a, b, and c proportional to them are given, it is readily seen that the equation will take the form

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}. \tag{51}$$

121. The equations of a straight line through two given points. — *The equations of a straight line through the two points $P_1\ (x_1, y_1, z_1)$ and $P_2\ (x_2, y_2, z_2)$ are*

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}. \tag{52}$$

Proof. — Substituting $\cos \alpha = \dfrac{x_2 - x_1}{d}$, $\cos \beta = \dfrac{y_2 - y_1}{d}$, $\cos \gamma = \dfrac{z_2 - z_1}{d}$, in equation (50), and dividing by d, the result is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

122. The projection form of the equations of a straight line. — A plane through a line perpendicular to one of the coördinate planes is called a *projecting plane of the line.*

If between two equations in the form $Ax + By + Cz + D = 0$ and $A_1x + B_1y + C_1z + D_1 = 0$, one of the variables is eliminated, an equation in two variables results. This from Art. 101 is a cylindrical surface with elements parallel to that axis which corresponds to the variable eliminated, and with its trace in the plane of the other two axes. The equation is of first degree and the cylindrical surface is therefore a plane. This plane is the projecting plane of the line. Two such planes will determine the line. By eliminating x, y, and z in turn, the projecting planes of the line $x + 2y + 3z = 6$, $2x - y - 3z = 5$ are found to be $5y + 9z = 7$, $5x - 3z = 16$, and $3x + y = 11$.

123. Direction angles of a line. — If the equations of a line are given in the form $Ax + By + Cz + D = 0$, $A_1x + B_1y + C_1z + D_1 = 0$, the direction cosines of the line may be found by a process illustrated in the following example.

Find the direction cosines of the line $x + 3y - 2z = 2$, $3x - 2y - 4z = 5$. Having determined two of the projecting planes to be $x - 8y = 1$ and $11y - 2z = 1$, the values of y may be equated, giving

$$\frac{x-1}{8} = \frac{y}{1} = \frac{2z+1}{11} \quad \text{or} \quad \frac{x-1}{8} = \frac{y}{1} = \frac{z+\frac{1}{2}}{\frac{11}{2}}.$$

This is in the form of equation (51) and therefore the direction cosines are proportional to 8, 1, and $\frac{11}{2}$. Dividing by $\sqrt{8^2 + 1^2 + (\frac{11}{2})^2} = \frac{1}{2}\sqrt{381}$, the direction cosines are found to be $\frac{16}{\sqrt{381}}$, $\frac{2}{\sqrt{381}}$, $\frac{11}{\sqrt{381}}$.

EXERCISES

1. Find the equations of the lines through the following pairs of points: (*a*) (0, 0, 0) and (1, 2, 2). (*b*) (1, 4, 0) and (3, −2, 3). (*c*) (1, 5, 1) and (−6, 1, 5).

2. Find the coördinates of the points in which each of the above lines cuts the coördinate planes.

3. Find the equations of the lines determined by the following conditions:

(*a*) passing through the point (3, 0, 1) and having $\cos\alpha = \frac{2}{3\sqrt{5}}$ and $\cos\beta = \frac{5}{3\sqrt{5}}$.

(*b*) passing through the point (5, −3, 1) and perpendicular to the plane $3x - 6y + 3z - 7 = 0$.

(*c*) passing through the point (3, 0, −1) and parallel to the line $\frac{x-5}{3} = \frac{y+11}{4} = \frac{z}{12}$.

(*d*) passing through the origin and perpendicular to the lines $\frac{x-4}{3} = \frac{y+6}{1} = \frac{z}{11}$ and $x + 3y - z - 3 = 0,\ 3x + 5y + z - 1 = 0$.

4. Prove that the following pairs of lines are perpendicular:

(*a*) $\frac{x-1}{3} = \frac{y}{1} = \frac{z+2}{5}$ and $\frac{x+1}{1} = \frac{y-3}{2} = \frac{z+5}{-1}$.

(*b*) $2x - y + z - 2 = 0,\ 4x + y - 4z - 4 = 0$ and $x + 7y - z + 8 = 0,\ x + 3y + 3z + 4 = 0$.

5. Find the angle between the following pairs of lines:

(*a*) $\frac{x}{2} = \frac{y+1}{1} = \frac{z+2}{-2}$ and $\frac{x}{0} = \frac{y}{1} = \frac{z-7}{-1}$.

(*b*) $x - 4y - 3z - 4 = 0,\ 2x - 2y + 3z + 1 = 0$ and $4x + 4y + 3z - 4 = 0,\ 4x + y - 6z - 1 = 0$.

6. Prove that the lines $5x + 8y - z + 20 = 0,\ 5x - 8y + 3z - 32 = 0$ and $4x + y + z - 2 = 0,\ 4x + 2y + 3z - 1 = 0$ meet in a point and find the angle between them.

7. Prove that the points (1, 0, −3), (4, 1, −1), and (7, 2, 1) lie on a line.

8. Prove that the line $2x + 6y + z - 2 = 0,\ 2x - 3y - 2z - 2 = 0$ is parallel to the plane $2x - 3y - 2z = 1$.

Hint. — Prove that the line is perpendicular to the normal to the plane.

9. Prove that the three planes $4x + y - z + 3 = 0$, $12x - y - z - 5 = 0$, and $4x - 3y + z - 11 = 0$ meet in a common line. Find its equation and direction cosines.

10. Find K such that the lines $x - 3y + 3 = 0,\ x + y - z - 1 = 0$ and $(7 + K)x - 7y + 7z - 28 - K = 0,\ 6x + 7z - 6 = 0$ are perpendicular.

11. Find K such that $(4, 15, K)$, $(1, K, 2)$, and $(-2, -1, -3)$ are collinear.

12. Prove that the line $2x + 6y + z - 2 = 0,\ 2x - 3y - 2z - 2 = 0$ is perpendicular to the plane $3x - 2y + 6z = 1$.

13. Find the projecting planes of the line $3x + 2y + z = 1,\ x - 4y - 2z = 3$.

14. Find the equations of the planes satisfying the following conditions:

(*a*) determined by the parallel lines $\frac{x-1}{1} = \frac{y+1}{-2} = \frac{z}{2}$ and $\frac{x+1}{-2} = \frac{y-3}{4} = \frac{z+1}{-4}$.

(*b*) determined by the intersecting lines $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z}{1}$ and $\frac{x+2}{3} = \frac{y+4}{2} = \frac{z+1}{1}$.

(*c*) containing the points (1, −1, 2) and the line $\frac{x-1}{2} = \frac{y+2}{4} = \frac{z-7}{2}$.

ANSWERS

Art. 10. Pages 21 and 22.

2. $y + \sqrt{3}\,x = 3 + 2\sqrt{3}$.

4. $4y - 2x = 5$.

5. $x - y = 7$.

6. $x + 3y = 14$.

7. $x + y = 7$.

9. $2x - y = 5$.

11. (*b*) $x^2 - 6x + y^2 - 4y = 12$.

12. $x^2 - 4x + y^2 - 6y + 4 = 0$.

13. $x^2 - 2x + y^2 - 4y = 20$.

15. $x^2 - 2x + y^2 - 6y = 15$.

17. $x^2 - 8x + y^2 + 8y + 22 = 0$.

18. $x^2 - 4x + y^2 - 8y + 10 = 0$.

Art. 13. Page 40.

3. (*a*) $x^2 = 6y - 9$.

(*c*) $y^2 + 8y + 4x + 20 = 0$.

(*d*) $y^2 - 2y - 4x + 13 = 0$.

(*e*) $3x^2 + 3y^2 - 18x - 14y + 10 = 0$.

(*f*) $9x^2 + 25y^2 = 225$.

(*g*) $16y^2 - 9x^2 = 144$.

Art. 14. Page 41.

1. $(6, 8)$ and $(-\frac{14}{5}, -\frac{48}{5})$.

2. $(1, \pm 2)$.

5. $(4, 3)$, $(-4, -3)$, $(3, 4)$, $(-3, -4)$.

7. $(2, 3)$ and $(-\frac{86}{7}, \frac{171}{7})$.

8. $(2, 2)$ and $(-\frac{8}{3}, \frac{46}{9})$.

10. $(0, 0)$, $\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{4}\right)$ and $\left(-\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{4}\right)$.

11. $(a, 2a)$ and $(9a, -6a)$.

17. $\frac{75}{4}$.

19. $\frac{1}{2}, -1, \infty$.

21. $\frac{5}{4}, \frac{\sqrt{85}}{2}, \frac{5\sqrt{13}}{4}$.

Art. 19. Pages 49 and 50.

1. (e) $x + 3y = 2$. (f) $3x + y = 12$. (g) $x + y = 6$.
8. $x - y = 8$, $x + 2y = 17$, and $5x + y + 14 = 0$.
9. $x + y = 5$ and $y - x = 1$.
10. $x - y = 6$, $5x - 2y = 31$, and $x + 2y = 7$.
11. $y - x = 2$ and $x + y = 6$.
12. (a) $2y = x$.
(b) $2x + y = 0$.
(c) $y(3 - \sqrt{3}) + x(3\sqrt{3} + 1) = 0$.

Art. 21. Pages 51 and 52.

1. (a) $3x - y = 7$.
5. $3x + 7y = 2$, $x + y + 2 = 0$, and $y = -1$.
6. $2x + y + 5 = 0$.
7. $x - 6y = 9$ and $3x - 5y = 1$.
9. $y = 1$, $12x + y = 21$, and $3x + 2y = 7$.
11. $2x - 3y = 12$, $3x - y = 4$, and $x + 2y + 1 = 0$.
12. $x = 4$.

Art. 24. Page 56.

3. $y + x\sqrt{3} = 10$.
4. (a) $x + y = 5\sqrt{2}$. (b) $x + y + 5\sqrt{2} = 0$.
5. (b) $y + x\sqrt{3} = 3$.
7. $x + y\sqrt{3} = 3$.

Art. 25. Page 58.

2. $x - 2y = 0$, $p = 0$.
3. $x + y + 9 = 0$, $p = -\frac{9}{2}\sqrt{2}$.

Art. 26. Page 61.

1. $-\frac{12}{5}$. **2.** 2. **3.** $4, \frac{24}{5}, \frac{24}{5}$.

Art. 27. Page 63.

1. $2x = 3$ and $y = 0$. **2.** $7x + y = 6$ and $7y - x = 6$.

Miscellaneous Exercises. Pages 63 and 64.

3. (*a*) $4x - 3y = 0$.
   (*b*) $5x + 12y + 13 = 0$.
4. (*a*) $5y - 12x = 52$.
   (*b*) $12x - 5y = 26$.
5. $y - 2x = 5\sqrt{5}$.
7. 14.
8. $\dfrac{30}{\sqrt{26}}$, $3\sqrt{2}$, and 5.
9. (4, 2).
10. (*a*) $x + y = 0$, $7x - y = 24$, and $y = -3$.
    (*b*) $17x - 17y = 4$, $7x + 17y = 0$, and $6x = 1$.
    (*c*) $x + y = 0$, $3x - 9y + 16 = 0$, and $2x - y + 4 = 0$.
13. $11x - 68y = 456$.
14. (*a*) $3\sqrt{13}$. (*b*) $3\sqrt{5}$.

Art. 28. Page 67.

5. $3x + y = -5$, $3y - x = 5$, and $x + 2y = 5$.
6. 90°, $\tan^{-1}\frac{1}{3}$, and $\tan^{-1} 3$.
8. $y - 3 = \dfrac{1+\sqrt{3}}{1-\sqrt{3}}(x-2)$ and $y - 3 = \dfrac{1-\sqrt{3}}{\sqrt{3}+1}(x-2)$.
10. $y - 1 = \dfrac{1+\sqrt{3}}{\sqrt{3}-1}(x-2)$, $y - 1 = \dfrac{\sqrt{3}-1}{1+\sqrt{3}}(x-2)$,
    $y + 3 = \dfrac{1+\sqrt{3}}{\sqrt{3}-1}(x+2)$, $y + 3 = \dfrac{\sqrt{3}-1}{1+\sqrt{3}}(x+2)$.

Art. 29. Page 70.

3. (*a*) $y = 2x + 5\sqrt{5}$.
   (*c*) $y = 2x + 12$.
   (*d*) $y = 2x - 3\sqrt{5}$.
4. (*a*) $y - 3x = 10$.
   (*b*) $2x - 5y = 2$.
   (*c*) $y - 2x = 6$ and $x - 4y = 4$.
   (*d*) $x + y + 6 = 0$.
   (*e*) $x + 2y + 8 = 0$.
5. (*a*) $x + 2y = 4\sqrt{5}$.
   (*b*) $4x + 3y = 20$.
   (*c*) $4y - 3x = 20$ and $4y + 3x = 20$.
   (*d*) $x + y = 4\sqrt{2}$.

6. $x + y = 4$.
7. $y + 2x = 6$ and $y + 8x = 12$.
8. $2x + y + 10 = 0$.
9. $x + y = \pm 2\sqrt{2}$.

Miscellaneous Examples on Chapter III. Pages 70, 71 and 72.

1. $3x + 2y = 16$ and $y = 2x + 1$.
2. $x - 5y = 9$ and $y + 2 = 0$.
4. The equations of two sides of one triangle are $y - 2 = -\dfrac{1}{\sqrt{3}}(x - 3)$ and $y + 4 = \dfrac{1}{\sqrt{3}}(x - 3)$, and of the other, $y - 2 = \dfrac{1}{\sqrt{3}}(x - 3)$ and $y + 4 = -\dfrac{1}{\sqrt{3}}(x - 3)$.
8. $x - 3y + 10 = 0$ and $3x + y = 0$; $(\frac{3}{10}, -\frac{9}{10})$ and $(\frac{29}{10}, \frac{13}{10})$.
9. (*b*) $5x + 25y = 143$.
(*c*) $103x + 44y = 91$.
(*d*) $3x + y = 7$.
11. $4y - 7x + 16 = 0$, $29y + 28x + 26 = 0$ and $7x - y = 17$.
15. (*a*) $C = -3A$.
(*d*) $C^2 = 25A^2 + 25B^2$.
(*f*) $3A + 5B + C = 0$.
18. $3x - 2y = 5$.

Art. 39. Page 93.

4. (*c*) $\sqrt{2}x^2 - x = 0$.
(*d*) $-5x^2 + 20y^2 + 8\sqrt{5}x - 16\sqrt{5}y = 25$.
5. $y\sqrt{2} = 1$.

Art. 41. Pages 96 and 97.

2. (*a*) $2x^2 = 9$.
(*c*) $4x^2 - y^2 = 4$.
3. (*a*) $x^2 + 3y^2 + 4x\sqrt{2} - 6y\sqrt{2} = 2$.
(*b*) $x^2 - y^2 - 2x - 2y = 4$.
4. $(1, -1)$, $2x - y = 0$, and $x + 2y = 0$.

Art. 42. Page 99.

4. (*d*) $xy = 2$.
(*e*) $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.
(*g*) $x^2 + y^2 = a\left(\sqrt{x^2 + y^2} - x\right)$.
(*i*) $x(x^2 + y^2) = 2ay^2$.
5. (*a*) $\rho^2 = 40$.
(*b*) $\rho^2 \cos 2\theta = 16$.

Art. 43. Page 101.

2. (*a*) $x^2 - 4x + y^2 - 2y = 20$.
(*c*) $x^2 + 2x + y^2 - 6y + 2 = 0$.
(*d*) $x^2 - 10x + y^2 - 12y + 25 = 0$.

Art. 46. Pages 106 to 109.

3. (*a*) $x^2 + 2x + y^2 - 8y + 1 = 0$.
(*b*) $x^2 + y^2 - 6y = 16$.
(*d*) $x^2 + 2x + y^2 - 6y = 90$.
(*f*) $3x^2 - 44x + 3y^2 + 112y + 41 = 0$.
(*g*) $x^2 - 6x + y^2 - 8y + 9 = 0$.
4. $x - y + 4 = 0$.
5. $2x - 3y = 5$; $-\frac{3}{2}$.
6. $2x + 3y = 13$.
8. (*a*) $x^2 - 12x + y^2 - 8y = 48$ and $x^2 + 16x + y^2 + 20y + 64 = 0$.
(*c*) $x^2 + 6x + y^2 + 6y + 9 = 0$ and $x^2 + 2x + y^2 - 2y + 1 = 0$.
(*d*) $x^2 + 4x + y^2 - 2y = 20$.
(*e*) $5x^2 + 5y^2 + 20x - 80y + 308 = 0$ and $5x^2 - 20x + 5y^2 = 52$.
(*f*) $x^2 + y^2 + 4(y - x)\sqrt{2} + 8 = 0$.
10. $x^2 - 6x + y^2 - 8y = 0$.
11. $x^2 - 2x + y^2 - 10y + 1 = 0$ and $x^2 - 34x + y^2 - 170y + 289 = 0$.
12. $x^2 + y^2 + 8y = 9$.

Art. 48. Page 116.

2. (*a*) $y^2 = -9x$.
(*b*) $y^2 + 4y + 4x + 12 = 0$.
(*c*) $x^2 = 12y + 36$.
(*d*) $x^2 - 2x - 12y + 13 = 0$.
(*e*) $(y - 1)^2 = 4x$.
(*f*) $x^2 + 9y^2 + 6xy - 56x + 52y + 14 = 0$.
3. $y = -2x$.
5. $x^2 + y^2 - 5y = 0$.
6. (*a*) $x^2 - 4x = 4y + 16$.
(*b*) $x^2 - 4x = 4y - 12$.
7. $7y = 24x - 36$.
8. $4y + 3x + 3 = 0$; $(-\frac{3}{4}, -\frac{9}{16})$.

Art. 50. Page 119.

1. $(-2, -2)$; $(-2, -\frac{1}{2})$; 6; $y = -\frac{7}{2}$; $x + 2 = 0$.
2. $(3, 3)$; $(1, 3)$; 8; $x = 5$; $y = 3$.
3. $(-1, 2)$; $(-1, \frac{19}{12})$; $\frac{5}{3}$; $12y = 29$; $x = -1$.

4. $(-1, \frac{7}{8})$; $(-1, \frac{3}{8})$; 2; $8y = 11$; $x = -1$.
5. $(\frac{3}{4}, \frac{5}{2})$; $(1, \frac{5}{2})$; 1; $2x = 1$; $2y = 5$.
6. $(-\frac{3}{4}, 1)$; $(-\frac{5}{12}, 1)$; $\frac{4}{3}$; $12x + 13 = 0$; $y = 1$.

Art. 54. Pages 126 and 127.

1. (*a*) $(\pm 5, 0)$; $(\pm 4, 0)$; $4x = \pm 25$; $\frac{32}{5}$.
   (*b*) $(\pm 4, 0)$; $(\pm 2, 0)$; $x = \pm 8$; 6.
   (*c*) $(0, \pm 5)$; $(0, \pm 3)$; $3y = \pm 25$; $\frac{32}{5}$.
2. (*a*) $\frac{x^2}{36} + \frac{y^2}{32} = 1$. (*b*) $\frac{x^2}{36} + \frac{y^2}{27} = 1$. (*c*) $\frac{x^2}{100} + \frac{y^2}{36} = 1$.
   (*d*) $\frac{x^2}{4} + \frac{9y^2}{32} = 1$. (*e*) $\frac{x^2}{25} + \frac{y^2}{16} = 1$. (*f*) $\frac{x^2}{64} + \frac{y^2}{12} = 1$.
   (*g*) $\frac{x^2}{12} + \frac{y^2}{4} = 1$. (*h*) $\frac{x^2}{24} + \frac{y^2}{12} = 1$. (*i*) $\frac{x^2}{4} + \frac{y^2}{3} = 1$.
3. (*a*) $\frac{(x+1)^2}{4} + \frac{(y+2)^2}{9} = 1$. (*b*) $\frac{(x+4)^2}{25} + \frac{(y+2)^2}{16} = 1$.
   (*c*) $\frac{x^2}{9} + \frac{(y-5)^2}{25} = 1$.
4. (*a*) $\frac{(x+1)^2}{36} + \frac{(y+2)^2}{9} = 1$. (*b*) $\frac{(x+1)^2}{9} + \frac{(y+1)^2}{25} = 1$.
   (*c*) $\frac{(x-1)^2}{25} + \frac{(y-4)^2}{9} = 1$. (*d*) $\frac{(x-2)^2}{16} + \frac{(y-1)^2}{12} = 1$.
7. $x^2 + y^2 = 25$; $(4, \pm 3)$ and $(-4, \pm 3)$.
8. $15y - 8x = 24$ and $15y + 8x + 24 = 0$, $\pm\frac{15}{17}$.
9. $25x^2 + 9y^2 = 225$.

Art. 56. Pages 130 and 131.

1. (*a*) $(-2, 1)$; $(-2, 3)$ and $(-2, -1)$; $(-2, 5)$ and $(-2, -3)$; 4 and $2\sqrt{3}$; 6; $y = 9$ and $y = -7$.
   (*b*) $(2, -1)$; $(2, 2)$ and $(2, -4)$; $(2, 3)$ and $(2, -5)$; 4 and $\sqrt{7}$; $\frac{7}{2}$; $3y = 13$ and $3y = -19$.
   (*c*) $(1, 4)$; $(1 \pm 2\sqrt{2}, 4)$; $(5, 4)$ and $(-3, 4)$; 4 and $2\sqrt{2}$; 4; $x = 1 \pm 4\sqrt{2}$.
   (*d*) $(-3, 1)$; $(-3 \pm 5\sqrt{3}, 1)$; $(7, 1)$ and $(-13, 1)$; 10 and 5; 5; $3x = -9 \pm 20\sqrt{3}$.
   (*e*) $(-1, -3)$; $(2, -3)$ and $(-4, -3)$; $(5, -3)$ and $(-7, -3)$; 6 and $3\sqrt{3}$; 9; $x = 11$ and $x = -13$.
   (*f*) $(-3, 1)$; $(-3, 3)$ and $(-3, -1)$; $(-3, 7)$ and $(-3, -5)$; 6 and $4\sqrt{2}$; $\frac{32}{3}$; $y = 19$ and $y = -17$.
2. $16x^2 + 25y^2 = 400$.

Art. 59. Pages 136 and 137.

1. (*a*) 6 and 10; $\frac{50}{3}$; $(0, \pm 3)$; $(0, \pm\sqrt{34})$; $y = \pm\frac{9}{\sqrt{34}}$.
   (*b*) 8 and $4\sqrt{3}$; 6; $(\pm 4, 0)$; $(\pm 2\sqrt{7}, 0)$; $x = \pm\frac{8}{\sqrt{7}}$.

2. (*a*) $\frac{x^2}{4} - \frac{3y^2}{4} = 1$; (*b*) $\frac{x^2}{9} - \frac{16y^2}{81} = 1$;
   (*c*) $\frac{x^2}{16} - \frac{y^2}{20} = 1$; (*d*) $\frac{x^2}{36} - \frac{y^2}{28} = 1$;
   (*e*) $\frac{x^2}{9} - \frac{y^2}{7} = 1$; (*f*) $\frac{x^2}{16} - \frac{y^2}{4} = 1$;
   (*g*) $\frac{x^2}{1} - \frac{y^2}{3} = 1$.

3. (*a*) $\frac{y^2}{4} - \frac{x^2}{3} = 1$; (*b*) $\frac{y^2}{18} - \frac{x^2}{16} = 1$; (*c*) $\frac{y^2}{36} - \frac{x^2}{108} = 1$;
   (*d*) $\frac{y^2}{25} - \frac{x^2}{11} = 1$; (*e*) $\frac{y^2}{5} - \frac{11x^2}{45} = 1$.

4. (*a*) $\frac{(y-3)^2}{16} - \frac{(x+1)^2}{25} = 1$; $(-1, 3 \pm \sqrt{41})$; $(-1, 7)$ and $(-1 -1)$; $\frac{25}{2}$.
   (*b*) $\frac{(x+2)^2}{16} - \frac{(y+3)^2}{36} = 1$; $(-2 \pm 2\sqrt{13}, -3)$; $(2, -3)$ and $(-6, -3)$; 18.
   (*c*) $\frac{(y-3)^2}{16} - \frac{(x+1)^2}{48} = 1$; $(-1, 11)$ and $(-1, -5)$; 24.

5. (*a*) $\frac{(x-2)^2}{9} - \frac{(y-1)^2}{27} = 1$; (*b*) $\frac{(y-6)^2}{16} - \frac{(x+1)^2}{48} = 1$;
   (*c*) $\frac{(y-1)^2}{9} - \frac{(x+2)^2}{16} = 1$; (*d*) $\frac{(y+3)^2}{16} - \frac{(x+1)^2}{4} = 1$;
   (*e*) $\frac{(x-7)^2}{4} - \frac{(y+1)^2}{12} = 1$; (*f*) $\frac{(y-2)^2}{9} - \frac{(x-2)^2}{9} = 1$.

8. $\frac{x^2}{16} - \frac{y^2}{20} = 1$.

12. $\frac{\rho^2 \cos^2\theta}{a^2} - \frac{\rho^2 \sin^2\theta}{b^2} = 1$.

Art. 61. Page 140.

1. (*a*) $(1, 2)$; 2 and 3; $(1 \pm \sqrt{13}, 2)$; $(3, 2)$ and $(-1, 2)$; 9.
   (*b*) $(3, -2)$; 2 and 4; $(3, -2 \pm 2\sqrt{5})$; $(3, -4)$ and $(3, 0)$; 16.
   (*c*) $(3, -1)$; 1 and 3; $(3, -1 \pm \sqrt{10})$; $(3, 0)$ and $(3, -2)$; 18.
   (*d*) $(1, -1)$; 3 and $\sqrt{3}$; $(1 \pm 2\sqrt{3}, -1)$; $(4, -1)$ and $(-2, -1)$; 2.

(*e*) $(\frac{1}{2}, -\frac{1}{2})$; $\sqrt{7}$ and $\sqrt{2}$; $(\frac{7}{2}, -\frac{1}{2})$ and $(-\frac{5}{2}, -\frac{1}{2})$; $(\frac{1}{2} \pm \sqrt{7}, -\frac{1}{2})$; $\frac{4}{\sqrt{7}}$.

(*f*) $(0, 1)$; $\frac{8}{5}$ and 4; $(0, 1 \pm \frac{4}{5}\sqrt{29})$; $(0, \frac{13}{5})$ and $(0, -\frac{3}{5})$; 20.

2. $xy + 5x - 3y = 7$.

3. $y^2 - x^2 = 100$.

4. $\frac{y^2}{25} - \frac{x^2}{39} = 1$.

5. $\frac{(x-1)^2}{12} - \frac{13(y-1)^2}{192} = 1$.

Art. 64. Pages 143 and 144.

1. $y = \pm 3x$; $\frac{y^2}{36} - \frac{x^2}{4} = 1$. **2.** $4y = \pm 3x$; $\frac{24}{7}$.

3. $9y^2 - 4x^2 = 36$; 4 and 6; 9; $(0, \pm\sqrt{13})$; $y = \pm\frac{4}{\sqrt{13}}$.

5. $4x^2 - 12y^2 = 75$. **6.** $4x^2 - y^2 = 36$.

7. $4y^2 + 8y - x^2 + 2x - 1 = 0$. **16.** $4x^2 - 9y^2 = 64$.

Art. 65. Pages 149 and 150.

2. (*a*) $3x - y = 2$; $3y + x = 4$.

(*b*) $y + 4x = 13$; $x - 4y + 18 = 0$.

(*c*) $3x - y = 2$; $x + 3y = 14$.

(*d*) $y - 2x = 3$; $x + 2y = 1$.

(*e*) $4x + 5y = 25$; $25x - 20y = 64$.

(*f*) $y = 4$; $x = 2$.

(*g*) $15x - y = 36$; $x + 15y = 138$.

(*h*) $x - 3y + 4 = 0$; $3x + y = 28$.

Art. 67. Pages 154 and 155.

1. (*a*) $4x - y = 2$; $4y + x = 9$; $\frac{1}{2}$; 8.

(*b*) $3x + 4y = 25$; $4x - 3y = 0$; $-\frac{16}{3}$; -3.

(*c*) $x - y = 4$; $x + y = 4$; 0; 0.

(*d*) $4x + y = 8$; $x - 4y + 15 = 0$; -1; -16.

(*e*) $3x + y = 7$; $x - 3y = 9$; $\frac{2}{3}$; 6.

(*f*) $4x + 3y = 16$; $16y - 12x = 27$; $-\frac{9}{4}$; -4.

(*g*) $y - x = 10$; $y + x = 4$; 7; 7.

2. $y = x$; $y = -x$.

3. $3x + 4y = 7$; $4x - 3y = 1$.

4. (a) $y = 3;\ x = 2.$
(b) $x + y + 1 = 0;\ x - y = 3.$
(c) $15x + 41y = -9;\ 123x - 45y + 455 = 0.$
(d) $x - 3y = 4;\ 3x + y = 12.$

7. $\dfrac{1}{\sqrt{2}}.$ 8. $\tan^{-1}\left(-\frac{40}{9}\right).$

10. $x + y = \pm\sqrt{34};\ x - y = \pm\sqrt{34}.$

14. (a) $(1, -2)$. (b) $(1, 2)$.

Art. 68. Pages 156 and 157.

1. (a) $4x + 3y = \pm 25;\ (\pm 4, \pm 3).$
(b) $y = x + 2;\ (0, 2).$
(c) $4x + y = \pm 8;\ (\pm 1, \pm 4).$
(d) $y = 2x - 3;\ y = 2x - 9;\ (0, -3);\ (4, -1).$

3. $3x + 10y = \pm 25.$

4. $2x - y = 5.$

5. $bx + ay = \pm ab\sqrt{2}.$

7. $3x + 2y = 0;\ 3x + 2y + 4 = 0;\ 2x = 3y;\ 6x - 9y = 5.$

Art. 70. Pages 162 and 163.

2. (a) $x + 12y + 16 = 0.$ 4. $(2, 2)$. 5. $(1, -6)$.

9. $x = \pm a.$ 10. $y + 5 = 0;\ 4x + 3y = 25.$

Art. 72. Pages 168 and 169.

1. $9y + 2x = 0.$ 3. $y = 4.$ 6. $y = 2;\ y - x = k.$

7. $(\pm 2\sqrt{2}, \mp 2\sqrt{2}).$ 8. $x - y = 2.$ 9. $4y + x = 5.$

Art. 75. Page 175.

1. (a) $y^2 - 4x + 4 = 0.$ (b) $5x^2 + \sqrt{5}x = 2.$
(c) $x^2 + 2y - y^2 = 0.$

2. (a) $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1.$ (b) $\dfrac{x^2}{4} - \dfrac{y^2}{4} = 1.$ (c) $x^2 - y^2 = 0.$
(d) $5y^2 = 9.$ (e) $y^2 = 3x.$

Art. 76. Page 176.

1. (a) $x^2 - y^2 - x + y = 0.$
(b) $x^2 + xy + 2y^2 - 3x - 4y + 2 = 0.$
(c) $7x^2 - 7xy + 3y^2 + 21x - 8y = 28.$

2. $xy + y^2 - 3x - y = 6.$

3. $x^2 - 2xy + y^2 = 1$.
4. $3x^2 + 4xy + y = 0$.
5. $4x^2 - 8x + y^2 - 10y + 4 = 0$.
6. $y^2 - 6y - x^2 + 8x + 9 = 0$.
7. $4x^2 - 4xy + y^2 - 4x - y = 0$.

Art. 96. Page 201.

1. $(-2, 1, 4)$.
2. $(1, 6, -3)$.
3. $1 : 2$.
5. $2 : 1$; $(3, -3, 0)$.
6. $(-7, 6, -2)$.
7. $(0, 6, 0)$.

Art. 98. Pages 204 and 205.

3. $\pm\frac{1}{3}\sqrt{3}$.
4. $\pm\dfrac{3}{\sqrt{14}}, \mp\dfrac{1}{\sqrt{14}}, \pm\dfrac{2}{\sqrt{14}}$.
5. $\pm\frac{8}{9}, \pm\frac{4}{9}, \mp\frac{1}{9}$; $-8, -4, 1$.
8. $(4, 120°, 135°, 60°)$ or $(4, 120°, 135°, 120°)$; $(-2, -2\sqrt{2}, 2)$ or $(-2, -2\sqrt{2}, -2)$.
12. $(10, -1, -2)$.

Art. 99. Page 206.

1. $\cos^{-1}\frac{13}{21}$.
2. $\cos^{-1} \pm\dfrac{5}{2\sqrt{105}}$.
4. $\pm\dfrac{7}{\sqrt{107}}, \pm\dfrac{3}{\sqrt{107}}, \mp\dfrac{7}{\sqrt{107}}$.
5. $\frac{20}{7}$.

Art. 100. Page 207.

2. $x - 3y + 2z = 4$.
5. $y^2 - 2y - 6x + 10 = 0$.

Art. 102. Pages 208 and 209.

2. (a) $(1, 3, -1)$; 4. (b) $(\frac{1}{2}, -\frac{3}{2}, \frac{5}{2})$; 3.
3. $x^2 + y^2 + z^2 + 4z = 45$ and $x^2 + y^2 + z^2 - 20z + 51 = 0$.
4. $y^2 + z^2 - 10z = 0$.
5. $x^2 - 4x + y^2 - 2y + z^2 + 2z = 7$.
6. $x^2 + 2x + y^2 - 6y + z^2 + 10z = 14$.
7. $x^2 - 8x + y^2 + 8y + z^2 - 14z = 0$.
8. $x^2 + y^2 + z^2 + 6z = 40$ and $x^2 + (y - \frac{336}{97})^2 + (z - \frac{465}{97})^2 = 49$.
9. $x^2 + y^2 + z^2 = 4$.
10. $x^2 - 2x + y^2 + 6y + z^2 - 8z = 55$.
12. $x^2 - 2x + y^2 - 4y + z^2 + 2z = 18$; $(1, 2, -1)$; $2\sqrt{6}$.
13. $x^2 - 2x - 4z + 5 = 0$.

Art. 103. Page 211.

1. (*a*) $y^2 + z^2 = x^2$. (*b*) $4x^2 + y^2 + 4z^2 = 16$.
   (*c*) $x^2 + y^2 = 4z$. (*d*) $x^2 + y^2 - z^2 = 4$.
   (*e*) $x^2 + z^2 - y^2 + 2y = 1$.
2. (*a*) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$. (*b*) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1$.
3. $y^2 + z^2 = 16$.
6. $x^2 + y^2 + z^2 - 14z + 33 = 0$.

Art. 112. Page 220.

2. $x^2 + 2y^2 + z^2 = 4$.
5. $x^2 + y^2 - z^2 + 4z + 14 = 0$.

Art. 118. Pages 224 and 225.

2. (*a*) $2x + 9y + 3z = 11$.
   (*b*) $7x + 2y + 10z = 3$.
   (*c*) $x + 2y + 7z = 8$.
   (*d*) $21x + 35y - 15z = 105$.
   (*e*) $2x - y \pm 2z = 15$.
   (*f*) $3x + 2y - z = 7$.
   (*g*) $z = 3$ and $3x - 6y + 2z = 21$.
   (*h*) $x + y + z = \pm 2\sqrt{3}$.
   (*i*) $8z - 6x = 9$.
   (*j*) $xy_1 - x_1y = 0$.
   (*k*) $2x - y + z = 3$.
5. $\frac{7}{5}$.
6. (*a*) $\cos^{-1}\frac{16}{45}$.
   (*b*) $\cos^{-1}\frac{20}{91}$.
7. $x + 2y - 5z = 15$ and $5x - 5y - z = 0$.
8. (*a*) $K = \pm 4$.
   (*b*) $K = 1$.

Art. 123. Pages 227, 228 and 229.

1. (*a*) $\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$. (*b*) $\frac{x-1}{2} = \frac{y-4}{-6} = \frac{z}{3}$.
   (*c*) $\frac{x-1}{7} = \frac{y-5}{4} = \frac{z-1}{-4}$.
2. (*b*) $(1, 4, 0)$, $(\frac{7}{3}, 0, 2)$, $(0, 7, -\frac{3}{2})$.
3. (*a*) $\frac{x-3}{2} = \frac{y}{5} = \frac{z-1}{\pm 4}$. (*b*) $\frac{x-5}{1} = \frac{y+3}{-2} = \frac{z-1}{1}$.
   (*c*) $\frac{x-3}{3} = \frac{y}{4} = \frac{z+1}{12}$. (*d*) $\frac{x}{2} = \frac{y}{5} = \frac{z}{-1}$.

5. (*a*) $45°$. (*b*) $\cos^{-1}\dfrac{10}{7\sqrt{241}}$.

10. 2.

11. 7.

13. $7x = 5,\ 14y + 7z + 8 = 0$.

14. (*a*) $2x + y = 1$. (*b*) $x + y - 5z + 1 = 0$. (*c*) $11x - 5y - z = 14$.

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