Limits of Meaning in Mathematics

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Dedication

For Ludwig Wittgenstein
who has inspired a great deal of my intellectual life and
for Bruce Duffy
whose *The World as I Found It*
introduced me to Ludwig Wittgenstein
all those years ago

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Intentions and Otherwise

A work of critical philosophy clarifies thought and then disappears. Kant's *Critique of Pure Reason* clarified thought. But, looking back on it, we mostly see its faults. The thinking of those who first read it was clarified, sometimes unnoticed by the thinker himself. And this clarity simply became our heritage of thought. Where thought is corrected, we can't imagine ever having thought otherwise. And it would take an astute and historically-minded philosopher to point out precisely what we gained from Kant's criticism of thought. In this work, I try to clarify where, in mathematics as we do it, meaning lies and what boundaries meaning encounters in our work. This is my intention.

There are several things I do not intend. I am not suggesting that mathematics itself be changed in any way. Nor am I trying to reorganize mathematics. I may sort things in order to talk about them. But, in common usage, they should stay where they are. I am certainly not trying to put any new and (again) unnecessary foundation under mathematics. It does not need any more of that. Even more emphatically, I am not re-opening the Formalist vs. Intuitionist exchange. We can just let that go, as well.

This view of meaning does not supply any new analysis of mathematics. It only clarifies what we say about mathematics and its use. It establishes conceptual boundaries for clarity's sake.

People might ask you (or you might ask yourself), "What does this mathematic you are practicing mean?" If I succeed in my intention, you will have a clear and certain basis for answering that question. This basis will seem as completely natural as the rest of your own thoughts. Nothing will have changed except your perception of what that question means. And this text and I will have disappeared.

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Universal Footnote

Actually, I did not intend to write this book. I intended to re-read all the works of Ludwig Wittgenstein and take notes on everything which bore on mathematics. But then he and I got into a rather exciting discussion about the nature of meaning and mathematics. In the heat of this rapid and prolonged interchange of thought, I lost track of who said what and of where I had modified his writings to fit the discussion. Coming to my senses, it was too late to sort these things out. But as I would never wish to be credited with anything which was not clearly my own work, I offer this footnoteless solution for any attribution you may wish to make:

1. Everything in this text which is in italics, unless attributed explicitly to someone else, is Wittgenstein's. He gets the credit for all such italicized text.

2. If you find something in this text which is correct or which causes you, sympathetically, to agree with it, give Wittgenstein credit for that, too. If he didn't write it, he inspired it and deserves your acknowledgment.

3. The remainder -- all that is incorrect, unsympathetic, or simply appalling -- is to be attributed to me.

On the whole, in the absence of footnotes, this heuristic will lead to essentially correct attribution. And if it leads to my not being recognized for some bit of original work -- mea culpa.

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My sources are these two PDFs, both freely available on-line:

The Collected Works of Ludwig Wittgenstein
1998 Blackwell Publishers

Tractatus Logico-Philosophicus - Ludwig Wittgenstein
1922 Kegan Paul, Trench, Trubner

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Schema

The sections of this book have their basis in this schema:

truth grounds => expression => equation => representation => picture => world
[true] [true or false] [true]

Everything that fails, fails in the middle.

Mathematically, it would seem natural to build up this critique from the truth grounds. Philosophically, it would seem more natural to begin with the world and work downwards. Personally, I would have preferred to begin with the viewpoint that arises in the middle and then work out to both ends from there.

None of these was actually very satisfying in practice.

With my original note-taking, I began with the Tractatus Logico-Philosophicus. And it begins with the world. Unable to find an attractive alternative, I have let Wittgenstein's schema for his book determine the direction of mine. This may simply be laziness on my part. Or cowardice.

I should add that some of Wittgenstein's entries below are retained because, in spite of a sense that I should exclude them and of my inability to explain them in a satisfying way, I suspect he knew something I don't yet grasp. Like Ramanujan's equations sent to Hardy, they are too suggestive of truth for me to delete.

In general, his entries are there because they evoked a response in me. Whether or not I agree or disagree with Wittgenstein in each instance is for you to determine. But I wouldn't bother with that if I were you. In many cases, this is something I have myself not yet fully determined. In a philosophical investigation, one ends up with more loose ends than one begins with. It is only one's standpoint which has hopefully improved.

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I should also add that Wittgenstein's punctuation and syntax can be bizarre. Some of this can be blamed on translators who tried to be "true to the original." Some is simply his strange (possibly over-hasty) punctuation.

It may appear to some readers as if many of the quotations below have been wrenched out of context to support some argument I am making. I make no argument, have nothing to defend. I am suggesting a point of view. A philosophical investigation is an attempt to move consciousness from one point of view to another, hopefully, better one within easy reach. All quotations in this text are here to **show** how other thinkers have expressed the ideas under investigation.

In Michael Polanyi's *Personal Knowledge*, I came upon this, to me, honest description of what it means for us to investigate anything:

The process of examining any topic is both an exploration of the topic, and an exegesis of our fundamental beliefs in the light of which we approach it; a dialectical combination of explanation and exegesis. Our fundamental beliefs are continuously reconsidered in the course of this process, but only within the scope of their own basic premises.

I concur with this description. It very accurately describes my participation in this investigation.

At this point, I encourage you to check your philosophical baggage and proceed unencumbered. There are no schools of thought beyond this point (in the spirit of Louis Cha's character Feng Qingyang.) Certainly, Wittgenstein neither adhered to nor created any school. He arrived at no definite conclusion. I, too, have none to propose. The final chapter was also unintended, outside the schema. But conversations and philosophical investigations lead us where they will. And I was happy to follow.

If I were asked to describe this text, in retrospect, I believe it was an instance of Hermann Hesse's Glass Bead Game.

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The World

The world is everything which is the case. The totality of facts in logical space is the world.

This is Wittgenstein trying to close a gap.

Totality of facts -- thoughts -- not things.

Possibility lies in the object, not in the facts, external or internal.

*Matter makes itself known to us by the testimony of the senses. We see it, hear it, smell it, taste it, touch it. But observe, that, after all, this is indirect testimony. These impressions are all of them simply brain impressions. We see, hear, smell, taste, touch, in our consciousness only. We cannot assert therefore that matter exists apart from this consciousness. Science has nothing to say about the ultimate nature of matter. Science studies matter simply as a **fact** of human experience.* -- *Henderson and Woodhull* [emphasis added]

That objects are matter is a point of view. That matter is a mathematical formula is another. That matter is the limited and inverted perception of a spiritual reality is another. Each can be proven in a practical manner. But not to everyone's satisfaction.

To assert ['There are physical objects'] or its opposite is a misfiring attempt to express what can't be expressed like that. And that it does misfire can be shewn; but that isn't the end of the matter. We need to realize that what presents itself to us as the first expression of a difficulty, or of its solution, may as yet not be expressed at all.

We do not prove an equation or an existence. We prove a law.

All proof takes place within an ideology. The proofs of mathematics take place within, at least, the deepest ideology of the current long-term culture. Even in mathematics, other ideologies make inroads.

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The deepest ideology is very hard to detect.

Most people, being formless themselves and being unable to attain to any Gestalt, strive to deprive objects of their Gestalt and reduce everything to chaotic matter, in which category they themselves belong. They reduce everything to its so-called effect. Everything is relative in their sight; so they relativize everything except nonsense and triteness, which hold absolute sway, as is to be expected. -- Goethe

I really want to say that scruples in thinking begin with (have their roots in) instinct. Or again: a language-game does not have its origin in ***consideration****. Consideration is part of a language-game.*

There is a gap between the impressions of the world and the reality of the world. This is the gap of realization. The truth of the only world is realizable through understanding and demonstration.

If intuition is an inner voice -- how do I know ***how*** *I am to obey it? And how do I know it doesn't mislead me? For if it can guide me right, it can also guide me wrong.*

There is no intuition. It falls by Occam's Razor. Intuition is realization viewed egoistically.

The old idea of ***intuition*** *in mathematics. Is this intuition the seeing of the complexes in different aspects?*

Realization is right apprehension of reality. What we do with it is another thing altogether. It cannot come to us prematurely. But it can be obscured and distorted by ideology.

The ***truth*** *of certain empirical propositions belongs to our frame of reference.*

Realization, to some extent, lifts thought out of ideology into truth.

Realization is a faculty of consciousness.

A man can pretend to be unconscious; but ***conscious****?*

There is no subconsciousness or unconsciousness with any consciousness in them. Consciousness is always conscious. We say, "The answer just came to me." Indeed it did. But it didn't come

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from us because, as we can attest, we didn't have it.

Similar mythologies: subconscious, soul, id, ego, superego, electro-chemical consciousness ... . A metaphor is not a proof of existence.

Striving prepares realization.

Let me borrow a metaphor from the Hopi Native Americans without implying any literal truth thereby: we strive to open the door at the top of our head.

For some, like Ramanujan, the door is simply open. One perceives the truth without any context for interpreting it.

Then there are those, like Einstein, whose realization cannot be expressed until someone else supplies the mode of expression.

And there are those, like Newton, who strove at his desk, against the door, for nineteen hours a day.

Most people deny the door.

"I have consciousness," is not a proposition.

Descartes, quoting St. Augustine, said, "I think, therefore I am."

With less ideology: "I am conscious, therefore consciousness is."

*In consideration of our problems, one of the most dangerous ideas is the idea that we think **with**, or **in**, our heads.*

It is possible to doubt or even deny the self but not the activity of consciousness and not the world.

Consciousness is not separate from the world. It exists in the world.

To doubt the world is to doubt even a place for doubt. Mapping everything to the empty set is a caricature of philosophy.

All experience is world and does not need the subject.

To hold that there is an unrealizable reality violates Wittgenstein's version of Occam's Razor:

Occam's Razor: An unnecessary sign is meaningless.

Everything could be otherwise. But not in this only world.

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Realization is never complete. Its receptivity expands, like the world, from a boundless basis. Because realization is boundless expansion, grasping reality is an infinite advance with "quantum leaps" bridging great numbers of the entries of the immediately prior monotonic series of perceptions and concepts.

Trisecting any angle was **unsolved** in Euclidean geometry. Only from a higher point of view was it shown to be impossible within the Euclidean context.

Even after impossibility was shown, men like Gauss and Todhunter probed the problem, providing "least non-Euclidean solutions."

There is a natural and right resistance to "impossibility." This is apart from the tenacity of ignorance. It is the correct sense that realization makes the solution possible.

It is not inconceivable that someone may someday realize a way to map or project any angle onto an intermediate trisectable structure and then map the intermediate back into the original angle as a true trisection.

Proofs of impossibility are always proofs that **all known roads** will not arrive at a solution. A proof of impossibility in therefore a mere historical comment.

Realization is outside understanding and demonstration.

Realization makes these possible.

A human being is part of a whole, called by us -- universe, -- a part limited in time and space. He experiences himself, his thoughts and feelings as something separated from the rest ... a kind of optical delusion of his consciousness. This delusion is a kind of prison for us, restricting us to our personal desires and to affection for a few persons nearest to us. Our task must be to free ourselves from this prison by widening our circle of compassion to embrace all living creatures and the whole of nature in its beauty. -- Einstein

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Compassion is the starting point on the path he is indicating. Such gestures are the modus of this text -- indicating beginnings by pointing in a direction.

Objects are the substance of the world. They fix the form of the world which exists independently of the facts. Substance determines form, not material properties.

Objects lie in their infinite space. *Their form is their possibility of occurrence in facts.*

Only facts have significance. Facts can be described, not named. Facts do not imply or deny each other.

Hold nothing as certain save what can be demonstrated. -- Newton

That for which you can find no demonstrable verification is mere belief.

Believing is not thinking.

Belief is one's ideology at low idle.

What something "means" is shown by our demonstrable realization of the facts. The totality of our demonstration is all that is meaningful to us.

If we understand a sentence, it has a certain depth for us.

A fact is a complex of objects, their relations, and the history of these. All this is the data of the fact as brought into understanding.

This data is not acquired by infinite regress or accretion. The requirements of understanding are not limited, but are at each point finite and within our reach.

More data on the fact does not increase understanding. Understanding must be prepared by realization in order to recognize more significance in the data.

Understanding a fact -- idea -- is the way its usage *meshes with our life.*

Understanding is the demonstrable measure of our sense of dominion in life, the extent to which our life meshes with the world.

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We can describe what is understood. We must show what is realized.

Understanding is outside of time. We understand immediately.

Understanding is not a process of digestion.

Bringing something into understanding is realization. Striving aids realization. Processes may assist in attaining realization but are not necessary. Certainly, no prescribed process leads to realization.

If process produced understanding, we would simply process ourselves into it. Compare: if diet led to spiritual attainment, we would simply all be vegans and attain.

Much of thinking is like staring at a brick wall. You stare and wait. You repeat yourself while you wait. Sometimes this wears you out.

Immediate experience cannot contain any contradiction. If it is beyond all speaking and contradicting, then the demand for an explanation cannot arise either: the feeling that there must be an explanation of what is happening, since otherwise something would be amiss. [after the colon, clarifying "demand for an explanation"]

If something is not demonstrably in the world, it has no meaning.

The concept of 'meaning' will serve to distinguish those linguistic formations that might be called capricious from those that are essential, inherent in the very purpose of language. ... Human nature determines what is capricious.

All contradictions take place within a language. If something simply isn't conceivable, *then neither is its opposite.*

The world is independent of my will.

I cannot will a connection between mathematics -- my expression of it -- and the only world.

All mathematics is a personal expression.

'Mathematics' is not a sharply delimited concept. ... For mathematics is after all an anthropological phenomenon.

Its generality comes from our individual efforts to reflect the same

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world, at some remove, in our representations, equations, and expressions. This common effort produces congruence and, to some extent, a common picture.

There is the world and there is my world and there is the attempt to express the first through the second.

To perform this kind of critique in logic, as Wittgenstein did, would be much more difficult due to the ambiguities of language. In mathematics, the truth is tautologically true and says nothing.

This simplifies things somewhat.

The range of general equations is their agreement with the world in their application, their approach to truth through specificity.

Any internal, unrecognized falsehood maps to the empty set.

Propositions have a sense which is independent of their truth or falsehood.

No. Their truth or falsehood is their sense. No analysis can take you beyond the result. The form of a thought simply brings you back to its consequence.

Truth is the basis of logic. It is the demand of mathematics.

***Before** the proposition, the concept is still pliable.*

As false logic is not logic, false consciousness is not consciousness.

In so far as we realize the truth, we can determine falsehood.

Fear is a kind of false science. And conversely.

Nothing is more difficult than to look at concepts without prejudice. For prejudice is a kind of understanding. And to forgo it, when it is so full of consequences for us . . .

Logic is a restriction on mathematical language that what we say must be true. In mathematics, falsehood and contradiction are excluded.

Where logical propositions can be contradicted, mathematical equations can only be corrected or discarded.

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All that is thinkable is possible. We cannot think what is unlogical. To present in language what contradicts logic is impossible ... the co-ordinates of a non-existent point.

What we cannot realize cannot be expressed in language. This void is felt everywhere as a hunger for understanding.

*The propositions of logic are **laws of thought** because **they bring out the essence of human thinking** -- to put it more correctly: because they bring out, or show, the essence, the technique, of thinking. They show what thinking is and also show kinds of thinking. ... [Thought's] essence, logic, presents an order ... the order of **possibilities**, which must be common to both the world and thought.*

Logic is the form of consciousness, of its expression of the truth, in as much as we grasp it. Logic is answerable to the world.

Mathematics is our imitation, our expression, of the order within consciousness.

Mathematics is answerable to us.

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The Picture

We make to ourselves pictures of facts, models of reality, scales applied to reality, with a representing relation joining the two.

The limits of what can be said is the limit of our model of the world. The world and life are one. I delimit the model of my world and, thereby, my experience of the world itself.

Empirical reality is limited by the totality of objects. ... Our empirical propositions do not form an homogeneous mass.

This boundary appears again as the limits of our ability to picture the world.

No part of our experience, no order of things, is a priori. ... Because it seems so to me -- or to everybody -- it does not follow that it ***is*** *so. ... That it seems so to men is their criterion for its* ***being*** *so.*

The limits of my ignorance, my lack of demonstration, are not the limits of the only world. They are the temporary limits of my world.

My picture of the world excludes from my experience everything true which my picture does not include.

Only as we look beyond the picture can we fill the picture in.

This is realization.

At bottom, the whole Weltanschauung of the moderns involves the illusion that the so-called laws of nature are explanations of natural phenomena. In this way, they [the moderns] stop short of the laws of nature as at something ***impregnable*** *as men of former times did of God and fate. And both are right and wrong. The older ones are indeed clearer in the sense that they acknowledge a clear terminus, while with the new system, it is supposed to look as if* ***everything*** *had a foundation.*

The picture is a relation between the world and our understanding.

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In the picture, the equation consists of its component expressions. Their specificities attach it to our picture of the world. Experience and understanding judge the relation of the picture and reality.

Understanding precedes the creation of the picture, follows from the reading of the picture's interpreted result. Limited understanding expresses itself in the picture's creation.

Objects acquire significance here in relation to our unfolding understanding.

This relation is expressed by the picture's form of representation.

In application, the picture and the method of representation are inseparable. Neither works alone. Alone, the picture is a primitive proposition unanalysable by truth-values. Alone, the representation is tautological.

We treat this pictorial language as a description of reality.

*The form of a picture [is] that in which the picture **must** agree with reality (in order to be capable of portraying it at all.)*

The form of the picture is the form of a concept.

The concept of a world of consciousness. We people a space with impressions. ... A concept forces itself on one. (This is what you must not forget.)

Should I say: Our concepts are determined by our interest, and therefore by our way of living?

But it is forced through the sieve of ideology. It cannot otherwise enter. If we are not free of ideology, only that which is stripped of meaning is free of ideology. And such a thing is neither concept nor representation.

Concepts are not functions but frameworks. ... It is as if one had brought a concept to what one sees, and one one sees the concept along with the thing. It is itself hardly visible, and yet it spreads an ordering veil over the objects.

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The values in the general equation become specific by falling under the concept.

What is essential to an hypothesis is that it arouses an expectation, i.e. its confirmation is never completed. It has a different formal relation to reality than that of verification -- belief in the uniformity of events.

You cannot expect nonsense. The picture shows our expectation.

The equation merely resolves into its solution. The picture says, "The world functions this way too." Reality, for us, is never more than the demonstrable limit of the picture.

Each proposition connected with a fact *makes at least one element of an hypothesis unhypothetical*.

And yet, if all elements of an hypothesis were so verified, one might from that still determine that the hypothesis was false. The affirmation would be realized as nonsense.

Things look different in the light.

An hypothesis is a law for forming propositions ... for forming expectations.

Under Newton, the picture appeared to reach its limit: "This is how things really are." Einstein pushed the limit back: "In these cases, however ..."

We demonstrate as much as we **can**.

Expectation is immediately connected with reality. ... The expectation is completely determined in the grammar.

We test the picture. Judgment then adjusts the picture, the representation, the specificities, the equation as necessary.

I want to say: we use judgments as principles of judgment.

Mathematics remains in the equation. Outside it, in judgment, is the relation to the world.

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In mathematics, there can only be mathematical troubles, there can't be philosophical ones.

So there is no distinction between pure and applied mathematics beyond this: your metaphysic comes under more scrutiny in the latter than in the former.

A metaphysical question is always in appearance a factual one, although the problem is a conceptual one.

A practical metaphysic is a demonstrable metaphysic by which the concept is shown to be a true and demonstrable realization of aspects of the fact.

*When we first begin to **believe** anything, what we believe is not a single proposition, it is a whole system of propositions. (Light dawns gradually over the whole.) ... The difficulty is to realize the groundlessness of our believing.*

A non-demonstrable metaphysic is mere belief.

No duality in the world. Duality is always in us.

There are no mathematical metaphors.

The purity of letters and numbers keeps mathematics apart from the world. All it can do is assert that our pictures are "formally" correct.

But the picture can still be totally false.

The picture's meaning is the magnitude of truth captured by its form. By truth, we can only mean its constant and increasing approximation as shown by our demonstration.

The meaning of a question is the method of answering it.

The question shows what you expect. And what you expect must be within your point of view. We might interpret a valid answer as nonsense the first time it came into view -- if it were **too** unexpected.

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[We] must begin with the distinction between sense and nonsense. [We] can't give it a foundation.

The picture must make sense now.

We can only foresee what we ourselves construct.

The gap between the picture and the world is the gap of demonstration.

The purpose of the picture is to realize and demonstrate new truths. These truths are waiting in the wings, in our hypotheses. But upon arrival, they may not appear to be what we expected.

No picture is true a priori. An a priori true thought is one whose possibility guarantees its truth, without needing a [fact] to compare it to.

Before I use "a priori" again, I should define it. This is Arthur Cayley: *The idea of order with its subordinate ideas of number and figure, we must not call innate ideas, if that phrase be defined to imply that all men must possess them with equal clearness and fullness; they are, however, ideas which seem to be so far born with us that the possession of them in any conceivable degree is only the development of our original powers, the unfolding of our proper humanity.*

That which follows the semicolon is what I mean by "a priori." So not **strictly** Kantian.

The totality of our true thoughts is our picture of the world.

That which is known to be false, we exclude from the picture. That which is falsely thought to be true will eventually be corrected.

Thought contains nothing more than was put into it.

Our concepts are our mental frameworks concerning related facts.

Their complexity is not in mathematics but in our mode of understanding the world, as shown by our pictures.

Complexity is explained by its intended use.

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False thoughts are either undemonstrable concepts or premature frameworks. We think we know enough to proceed. But our false concepts assert our misunderstanding of the world.

The incomplete picture is, if we compare it with reality, right or wrong, whether or not reality agrees with what can be read off from the picture.

All pictures are then incomplete.

In logic, this agreement with reality is often apparent. In mathematics, quite the opposite. Everything in the picture can be true and nothing actually mapped to reality.

Mortal mind sees what it believes as certainly as it believes what it sees. -- Mary Baker Eddy

We can draw conclusions from the false use of an equation. And we do -- to our detriment. But our culture of thought may preserve this falsehood if its cause is inescapable due to our deepest cultural ideology.

We're used to a particular classification of things. ... These are the fixed rails along which all our thinking runs, and so our judgment and action goes according to them too.

It is very hard to imagine concepts other than our own because we never become aware of certain very general facts of nature. It doesn't occur to us to imagine them differently from what they are. But if we do, then even concepts which are different from the ones we're used to no longer seem unnatural to us.

A concept cannot be true without the true existence of all its facts and their objects.

Because a concept implies or asserts all that follows from it, false concepts project their falsehood upon our world.

We can assert anything that can be checked in practice. It's a question of the possibility of checking.

The only sense of "a thing in itself" is that this constant projection

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stands between us and that which we perceive. As the projection is corrected, the perception adjusts towards an understanding of reality.

Eventually, nothing stands between. The world in not an infinite regress.

(My) doubts form a system. ... Doubt comes ***after*** *belief.*

Idealist philosophy creates a series of objects between consciousness and the world, when these objects, if actual, are simply aspects of relatively simple relations.

(Idealism ≡ Barber Occam's next customer)

Materialist philosophy, as we can see from its extensive effects, leads either to some kind of solipsism or to gradations of nihilism or both. One is the exclusion of all that is outside the limited false sense of self. The other is the exclusion of a realized future. These two exclusions are an excellent description of death.

(Materialism ≡ embracing of death)

Doubt can only exist where a question exists; a question can only exist where an answer exists; and this only exists where something ***can*** *be* ***said****.*

Doubt can only exist where truth exists but is yet unrealized. Where truth is demonstrated and realized, no doubt remains. Or if doubt remains, undiscovered aspects of truth remain.

Doubt gradually loses its sense.

Meaning occupies the ground of doubt.

Truth's (false) opposite only appears in the presence of truth. The falsehood is a herald of its own destruction.

You cannot doubt or falsely apprehend what is beyond your grasp of the truth. The student of Euclid cannot doubt or falsely apprehend Riemannian geometry which is to him, if anything, nonsense.

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A false picture has no hegemony in the world.

But if accepted by consciousness, it can make an awful mess of our experience in our world.

And here, the only world is not to blame.

What cannot be imagined cannot even be talked about.

That which is excluded by our point of view cannot appear in our concepts. So relevant facts are always obscured. Our point of view, to some extent, absolutely filters out the truth.

It is from our concepts that we construct our pictures of the world.

In order to discover whether the picture is true or false we must compare it with reality. True or false cannot be discovered from the picture alone.

In the picture, psychological investigations have no meaning.

Meaning: *We seemed to ask about the state of mind of a man who says a sentence, whereas the idea of meaning we arrived at was not that of a state of mind.*

Even in application, mathematics is outside the modes of personality.

The picture can represent every reality whose form it has.

In the picture, the equation as representation is itself a picture of a possible situation.

But the picture, where falsehood enters, is not mathematical.

Tell me how you are searching and I will tell you what you are looking for.

The picture has the defining context.

Only in the context of a picture has a name meaning.

If the facts, to which the representation points, guided by the context of the picture, do not exist, the picture is nonsense.

The underlying contexts, those of equations and the expressions

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composing them, are subjugated by the context of the picture. These underlying contexts are only judged in the light of the picture.

Searching presupposes the elements of the [fact], but not the combination [one] was looking for.

Given an equation, its pair of expressions may be found in various contexts. The intersection of these contextual expressions is the scope of possible interpretation. This is the equation's possibility of "sense."

As we increase specificity, refining sense, we send out feelers towards the world. But only in interpretation do the feelers touch, and then only idealistically, conceptually.

You can only search within a system. And so there is necessarily something you ***can't*** *search for. ... A system is, so to speak, a world. ... The system is not so much a point of departure as the element in which arguments have their life.*

We have a colour system and a number system. Do the systems reside in ***our*** *nature or in the nature of things? How are we to put it? --* ***Not*** *in the nature of numbers or colours.*

A system, producing a picture's sense, in no way assures agreement with reality. The feelers of our representation might never approximate the world. (This might be the fault of the physicist using the equation, for example. His poor choice which followed from deeper unconsidered assumptions.)

It is this mathematical sense we hold up to the picture. To the extent this sense satisfies the picture, to this extent we judge our result mathematically "meaningful."

These feelers can touch my world. But the judgment of meaning and of the correlation of model to reality lie outside mathematics.

The sense of a picture lies in the equations which express its form. there is no "meta-sense" beyond this in the picture.

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If we have determined anything arbitrarily, then something else must be the case.

The thought is the significance of the picture.

Thoughts are in the same space as the things that admit of doubt.

The former measure the latter.

Language disguises the thought, concealing its form.

The picture is a model of reality as we believe reality to be. The equation in the picture shows its sense, shows how thought stands, **if** it is true. The logic of the representation projects this sense into the understanding as the affirmation or denial of the picture.

Our understanding tries to connect the sense of the equations in the context of the picture with the thoughts they represent. If this adds to our understanding, the significance of the objects (for us) is increased. This is the meaning of the picture.

Only in the stream of thought and life do words have meaning.

Significance relies upon the deepest cultural ideology.

Under the influence of Greek scientific thought the Egyptians discovered the steamship. And didn't care.

The realization of steam power had, for them, no significance.

The trouble starts when we notice that the old model is inadequate, but then instead of altering it, [we] sublimate it.

All ideology -- belief -- excludes the truth from our experience.

Judgment determines the truth of the equation in the picture, as best it can, filtered by ideology.

An equation defines the intersection of two tautological expressions. If the expressions are well-formed they simply say what they say.

Their intersection, the result, is mathematically correct, true, and tautological.

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The equation expresses the intersection of two mathematical truths. And there it stops.

In a + b = c, the intersection of these tautologies is very large and very trivial. Understanding this sense of the intersection is all there is to understanding this equation.

If the intersection of the tautologies points to demonstrable significance, we judge that our picture is true, or that the fact is existent.

If judgment cannot establish the connection, we say our picture is false or the fact non-existent.

Our judgment has no hegemony in the only world.

The picture only looks like reality from the inside, which shows what part mathematics plays in the picture. It is like a mechanism of relations for relations it cannot contain. We bring the relations with us to the table.

Obviousness is no measure of truth.

The equations in the picture express the truth-possibilities of their expressions. But only in so far as their specificity points into the objects of understanding can there be significance.

Of themselves, equations only express a tautological sense. This sense aids in bridging the gap of demonstration.

But the meaning of the demonstration can never be explained by this mathematical sense.

Every question that can be decided at all can be decided without further trouble.

Else we are either premature or looking in the wrong direction.

In our pictures, equations cannot assert the existence of their interpreted results. The interpretation is outside of the equations.

Inference of significance takes place a posteriori. The a priori truths of mathematics are not in the space of significance.

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A phenomenon isn't a symptom of something else; it is the reality. [It] isn't a symptom of something else which makes the [picture] true or false; it itself is what verifies the [picture]. (Here I assert that the [picture] is Wittgenstein's "proposition.")

From one fact to another disjoint fact, no inference can be made, even if they seem to sit well with each other. Nothing about water, alone, says anything about dirt.

One can describe [what is demonstrable in] *the world completely with general propositions*, *i.e. without from the outset coordinating any name with a definite object.*

But this is only a mechanical description, a treatise of general representations awaiting judgment.

As we gain understanding, we alter the underlying expressions and recombine them into equations in order to gain more understanding.

Equations cannot be negated. They uniformly assert a tautological intersection of expressions.

Pictures cannot be negated. They can only be corrected or replaced.

Induction (logical induction) *itself is a significant proposition, not an a priori law. We know a priori the possibility of a logical form.*

Induction is the process of assuming the simplest law that can be made to harmonize with our experience. *Its foundation is psychological.*

There is no intersection between psychology and realization or demonstration. These are outside the modes of personality.

The image and reality are in one space. ... You can only search in a space. For only in space do you stand in relation to where you are not.

All pictures are not of equal value.

Our worlds, our pictures of the only world, lie in the world. If our

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worlds lack meaning, it is because we fail to understand the only world.

Significance is the connection of our worlds to the unitary one.

The reality that is perceived takes the place of the picture.

A successful picture must agree with something. But that something need not be made up of physical realities.

Any realities will do.

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Representation

*In life, it is never a mathematical proposition we need, but we use mathematical propositions [equations] **only** in order to infer from propositions which do not belong to mathematics to others which equally do not belong to mathematics.*

The use of mathematics takes place in the representation. This is the space of experiment. The picture as a progressive log of our demonstrable understanding.

A mathematical method can be considered "numerical" or "spatial." By this I mean: establishing results by equations or by logical argument; establishing a solution or establishing a territory and its properties in a space.

Examples: pure geometry is spatial; basic algebra, numerical; linear algebra, both by turns.

These are not absolute categories. More broadly considered, the numerical springs from the spatial. The latter is the proof which gives new powers to the former, which is a calculus.

A proof need not provide a new calculus or new methods for an existing calculus. But a hierarchy of proofs without a calculus is a space without a method of moving about in that space.

In mathematics, there is only one space. Each spatial mathematic shows its power by the extent of movement it allows among the forms of number.

*The geometry of visual space is the syntax of the propositions about objects in visual space. Euclidean geometry is the syntax of assertions in Euclidean space. And these objects are not lines, planes and points, but **bodies**.*

Geometry only enters the representation as equations, as numerical mathematics.

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You cannot set geometric propositions equal to each other. You can either move from one to the other or you can't. And the possibility of doing so says nothing about its necessity.

In a picture, an equation is the picture's projective relation to the world, its representation. This relation is the expression of our understanding.

*Could one say that arithmetical or geometrical problems can always look, or can be falsely conceived , as if they referred to objects in space itself? By "space" I mean what one can be **certain** of while searching.*

But the way of representing is already formed in the equation. And the picture is an interpretation of the equation's relation to the world.

We cannot share what we understand. We can only offer a point of view that leads someone else to realization. Or not.

We can, of course, share the effects of understanding: *One picture throws light on another.*

I have to judge the world, to measure things. I read this as "in order to measure things."

In rigorous application, mathematics and extreme specificity attempt to create a reasonably true representation according to a plan, i.e. Newtonian mechanics, Einstein's relativity.

How a magnitude is measured is what it is. -- Einstein

Equations, in their general form, are tautologies. They say what they say. Only through specificity can they enter the representation.

Context, assignment of constants, assignment of variables, these make an equation more and more specific.

It is through specificity that we try to link the equation to the picture as means of representation. This is the specific sense of the equation, pointing it at the world.

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Clearly, specificity both complicates and refines the mathematical representation.

The specificity attempts to join the mathematical equation to the measurements of the world.

In such measurement, do we perform an experiment or ... only establish internal relations [of measurement] and the physical result of our operations proves nothing?

What the physicist measures in visual space is measured by instruments whose readings do not distinguish conceptual types of space: Euclidean, Riemannian, etc.

The mathematic tries to reach the measurement. And the measurement tries to reach the world.

Physics differs from phenomenology in that it is concerned to establish laws. Phenomenology only establishes the possibilities. Thus, phenomenology would be the grammar of the description of these facts on which physics builds its theories.

Agreement of the picture and the world is a judgment. So the solution is the mathematical form of a possible agreement. Its form could be false to begin with, if its truth cannot be mapped onto the world.

What Mach calls a thought experiment ... at bottom ... is a grammatical investigation.

The gap between representation and picture is the gap of understanding.

The logic of the world cannot be represented by the logic of mathematics. The equation does not represent the facts.

You cannot use language to go beyond the possibility of evidence.

The furthest mathematics can go is the limit of the surface of a formal conceptual framework of our world, which is not even the only world but simply our own.

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The moment we try to apply exact concepts of measurement to immediate experience, we come up against a peculiar vagueness in this experience. But that only means a vagueness relative to these concepts of measurement.

Ideology interposes itself between measurement and the world. Mathematics, at best, reaches the measurement. At worst, only the ideology.

Admittedly, the words 'rough,' 'approximate,' etc. have only a relative sense, but they are still needed and they characterize the nature of our experience; not as rough and vague in itself, but still rough and vague in relation to our techniques of representation.

Only intention brings representation to the picture. We intend that this equation with these expressions relates to that object and its relations in the picture.

What can be shown cannot be said. Form is shown. Sense is said.

The equation has the sense of its general form, expressed tautologically, overlaid with the specificity of this representation.

Meaning enters at the point of representation. We intend for a representation, composed of equations, to help us demonstrate the significance (to us) of the objects in the world.

By means of representations, we explain ourselves.

Why do you demand explanations? If they are given you, you will once more be facing a terminus. They cannot get you farther than you are at present. [There is a kind of echo of the New Testament in this.]

The foundation of our explanations are ideological. If we are unaware of this, we mistake the nature of ideology. We ignore our blinders.

Ideology is unjustifiable certainty of thought.

Ideology is outside of understanding. It is often a denial of the

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possibility of understanding, a falsehood as prophylactic preventing understanding.

The meaning of an idea is its totality in my understanding.

I convey an idea to you with the intention of transferring, ideally, that totality. You will probably get some of my meaning. But even if I convey a tautological equation to you, you will not get all of my meaning because my understanding is greater than the tautology.

What you do get may combine with your understanding so that your understanding of the idea is greater than mine.

I could have a better understanding of something or better express my understanding of it or you could grasp what I am conveying more completely. But *it makes no sense to talk of a more complete expression* as all these factors are always completely all that they are right now.

I can only show a picture or create a representation of what I understand. I show the best picture, build the best representation, I can.

A meaningless representation is a lack of meaningful intention.

*The intention is already expressed in the way I **now** compare the picture with reality.*

Representations are "numerical."

The equation brings to the picture the form of its sense.

In the picture, equations act as arrows pointing at the facts.

Mathematical representation *can represent the object and contains no suggestion of a subject*. And yet our presence as subject always alters the object experimentally and the picture and representation ideologically.

The picture tries to obliterate all personal sense. But the subject continues to *make the space asymmetrical*.

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We compare incomplete systems, not single propositions, with reality.

*To have meaning **means** to be true or false.*

This might be true of logic. In the world, the false has no meaning. The false is merely a void waiting to be filled with the truth.

A false representation is a representation of the void. It therefore maps to the empty set.

If the world had no substance, then whether a representation had meaning or not would depend on whether an equation was interpreted as **meaningfully** true in this way within the picture.

But an equation is only a tautological intersection of sense. It would then be impossible to form a picture of the world (true or false) if the substance of the world were not true.

Meaning always points into the world as our relation of understanding, of our demonstrable grasp of reality. Sense remains in the conceptual construct and cannot escape it.

Meaning and sense, in this text, are names of two aspects of relations within consciousness. I attempt through description to show what I am naming.

After I chose these labels, I discovered Wittgenstein also chose "Bedeutung" and "Sinn" in this way. Frege apparently made a similar choice.

So this dichotomy is arguably a natural choice, even if, in some sense, it is hard to see what other people mean by it.

It should be clarified that in this investigation "sense" is **not** "sense data."

Names are points, designating objects. Objects can be named but not asserted. Names are primitives, unanalysable.

A property is internal if it is unthinkable that the object does not possess it.

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We sometimes see an internal relation between facts by an internal relation between the equations presenting the facts.

But this, only in so far as the representation holds true. The internal relations of equations are formal relations of tautological expressions. We are mapping tautological mathematical sense to conscious meaning.

It is entirely paradoxical to say that, as a general process, we can start from equations having a meaning, and arrive at equations having a meaning by passing through equations which have no meaning. -- Alexander Macfarlane on Boole's methods

The mistake here is that none of the equations have meaning. All of the equations have sense. Meaning informs our construction of the equation and its mathematic and determines their sense.

Consciousness unceasingly maps sense to meaning.

So the meaning begins with the initial equations but does not enter into them. The equations are put through their tautological dance. When the music stops, we have our intersection of solution. And then, from outside mathematics, we ask, "What does this mean?"

There are no hypothetical internal relations.

The perceived meaning is our attempt to close a gap of understanding through demonstration.

In the equations, *"x" is the proper sign of the concept "objects" ... "x, y" are two objects. To go further into objects is nonsense.*

a = b -- same object, two names
a = a -- no significance

The form of representation must allow for both the falsehood and the truth of implied results. No cheating. No filling the void with willful lies.

Falsehood in mathematics comes from inappropriate use of elements in equations and from false metaphysics brought in from outside mathematics.

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The method of mathematics is not a metaphysic. Our ability to realize the truths of the only world is not mathematical.

Calculation with letters is not a theory.

Two tautological expressions can produce an inappropriate equation. Think of Euler's early work with series. This is inappropriate use of operators.

Mathematics does not overlap with its application.

An application is not the extension of a calculation.

A significant representation asserts something. The demonstration it leads to shows the assertion true or false.

The picture is a transcendental reflection of the world.

By "transcendental" here, I mean that we know exactly what we mean by the picture. And yet we never expect to encounter exactly this pure picture in the world.

I believe this to be precisely Kant's meaning of "transcendental" in spite of the uses that have been made of it.

The question, how simple a representation is yielded by assuming a particular hypothesis, is directly connected, I believe, with the question of probability.

Equations, as representation, represent a fact, a situation. Expressions represent what can be in a fact. If there is no solution, then there was no such situation, no such state of things, and no meaning attributable to the sense of the expressions. So we try other pieces.

The probability of an hypothesis has its measure in how much evidence is needed to make it profitable to throw it out.

The choice of expressions is correct or incorrect. The equations' results are then true or false as mapped through representation into our world.

Our world is never the singular unity of reality, the only world.

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But mathematics deals only with the correct and the true. It offers only tautological results.

The links in the chain of reasons come to an end, at the boundary of the game.

Mathematics is only betrayed by lingering ideas from false metaphysics.

A true, or practical, metaphysic could not undermine mathematics.

Once, when I was a student struggling to understand modern algebra, I was told to view this subject as an intellectual chess game, with conventional moves and prescribed rules of play. I was ill-served by this bit of extemporaneous advice, and vowed never to perpetuate the falsehood that mathematics is purely -- or primarily -- a formalism. ... I have devoted a great deal of attention to bringing out the ***meaningfulness*** *.... -- Charles Pinter*

The meaning of a rule is its application.

Mathematics, apart from a senseless instance of formalism, is impossible without our imbuing the activity of our doing mathematics with meaning. Yet, there is nothing but tautology in the symbols.

We bring meaning in with us, establishing the sense of the signs within their context and the meaning of the representation into our choice of equations.

Sense and meaning are ideas, expressed by means of signs but never entering that which has no consciousness.

It is meaningful to participate in the intelligent activity of the only world. But the means of each activity have no more meaning than a chisel, a brick, or a two-by-four. It is the participants who are meaningful, more full of meaning with each realization.

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Equations

It seems to me that you may compare equations only with significant propositions, not with tautologies.

I emphatically disagree with this.

And yet, with so many of the implications of this assertion, both on his side and from my opposite view, Wittgenstein and I are in complete agreement.

Or, at least, we completely agree within the bounds of the ambiguity of language.

Probably much more than that. Language isn't all that ambiguous.

The propositions of mathematics are equations, and therefore pseudo-propositions.

With "pseudo-propositions" never defined by Wittgenstein, this may account for our agreement.

[An equation's] truth or falsity must be contained in it as its sense. (I will often exchange "equation" for "proposition" in his statements.)

Wittgenstein gives as examples 3×3=9 and 3×3=11. I consider the latter as not an equation. Or, that which has the form of an equation and is shown to be false is not an equation. I might call such a thing a "pseudo-equation" but the fewer labels the better.

An equation is a priori true, a tautology. Its two expressions are themselves tautologies. The equation correctly defines their tautological intersection. At the level of "complete" generality, nothing has been said. But we do learn how the expressions interact. They make **this** intersection. This kind of **this** is what mathematics is built upon.

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Equations are a kind of number. (That is, they can be treated similarly to the numbers.)

A solution is the result of a calculation or stipulation which then is shown to be a tautology and we judge its sense, i.e. we ask, "Was the calculation that led here sufficient, was the stipulation meaningful, are our ideas here aligned with the only world? Are we satisfied with our work?"

Consider some well-known mathematical equation from statics that expresses the load on a column generated by weight from above. The mathematical expression of this "acknowledged truth" does not contain that truth. One can study the signs themselves and never discover their relation to statics. In such a case, meaning is only attained through education. The equation itself, completely analyzed, is a generic tautology suggestive only of algebra or basic analysis.

Neither is such an equation a statement of some absolute meaning among the "educated." One has only to compare the columns of ancient Egypt and of Persia before the Alexandrine invasion with our own to see that the truth of statics and of its architectural application fundamentally differ among cultural contexts.

If it were possible to offer our "truth" of statics to those cultures, it might be accepted as technical innovation or simply rejected as meaningless on grounds of deep ideological prejudice. And any argument that there has been some monolithic evolutional progress in statics is absurd. The heights of one culture are unachievable by any other culture.

The equation, in itself, is a tautology. In application, as representation, its meaning is a cultural phenomenon.

Grammatical rules determine a meaning and are not asnwerable to any meaning they could contradict. ... The rules of grammar are arbitrary in the same sense as a unit of measurement.

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Grammar is composed of morphology and syntax. Morphology is the structure of words, or a language's basic elements. In spoken languages, this structure is invariant. Syntax is the rules of combining words into larger structures. In some languages there is total freedom of syntax. Others have very strict syntax. (Not that Wittgenstein was ever bound by these definitions.)

Grammar is for us a pure calculus (not the application of a calculus to reality.)

"The rules of [mathematics] are arbitrary" means: the concept '[mathematics]' is not defined by the effect [it] is supposed to have on us.

He speaks here of language, not mathematics. But this shows his usage of "arbitrary." And the statement is apropos of mathematics.

The rules of grammar may be called "arbitrary," if that is to mean that the ***aim*** *of the grammar is nothing but that of the language.*

The rules are arbitrary but are selected from limited choices.

Reality limits meaningful choice.

Every rule is general.

Grammar, for Wittgenstein, consists of the rules of a language, in this case mathematics. So when we look at the results of mathematics, we are not looking at intended effects but at a calculus.

Grammar describes the use of words in the language.

Is there a distinction between use and purpose in mathematics?
I think not.

Grammar prescribes the formation of expression and equation.

Grammar proscribes contradiction, the only falsehood available to tautologies.

What is hidden must be capable of being found. (Hidden contradictions.) Also, what is hidden must be completely

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describable before it is found, no less than if it had already been found.

In most cases, we recognize contradiction in mathematics as nonsense. Only deep ideologies cause us to cling to contradiction.

Equations are significant "propositions" in that they assert a relation between expressions and their (re)solution and, perhaps, their remote reflection of form in the world. Their significance is the extent to which they increase our understanding in the world, at a tautological remove.

Wittgenstein also says: *An equation is a rule of syntax.* I can't make sense of this. And he doesn't develop the idea. Perhaps he was trying it on for size.

An equation is an allusion to a proof. ... The proof is part of the grammar of the [equation] ... a ***new*** *paradigm. ... It isn't something behind the proof, but the proof, that proves.*

Without a form of representation, an equation's significance reduces to purely mathematical tautology.

The picture containing a representation is not always intended as a picture of the world. There are also pictures of our (formal) understanding. Progress here allows us to move about in the related ideas.

An equation can be a description of a fact only in so far as its sense can be mapped onto a representation. But this mapping is outside of mathematics.

With an equation so mapped, *true or false, one can draw conclusions from it.*

And this, even if you cannot clearly distinguish true from false.

"To understand p" means to know its system. If p appears to cross over from one system to another, it has in fact changed its sense.

To understand an equation means to know what its sense signifies with regard to the one or more mathematical contexts the equation

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arises within. Looking in this direction is looking away from the world. Connecting sense to representation looks towards the world.

The sense of the equation is internal. It is not asserted externally.

Sense is the understanding we have of our own mathematics, viewed as a construct of our own culture, without judging its relation to the world.

Sense has significance as its growth in understanding leads to excellence in our work. But this significance has meaning only when it reaches the world through being shown or shared.

All of mathematics, as we use it, is a construct of our culture. To think we can even imagine how the Greeks or the medieval schoolmen thought is delusional.

The past can only be viewed and judged **anachronistically**.

We make or make over all mathematics in our own image.

The ideologies of other cultures remain in those cultures. We bring our own blinders to the party.

An equation shows nothing; rather it shows that its sides show something.

The sense of an equation is the consequence of its expressions.

One can practice mathematics with absolute correctness and misunderstand everything else. (Just go listen to a non-mathematical talk by the mathematically educated.)

An equation is a function of the expressions contained in it.

The analysis of an equation is unique. Process and result are equivalent. No surprises.

An equation *is completely analyzed if its grammar is made clear -- in whatever idiom.*

Equations express the intersection of their left-hand side and right-hand side expressions. The intersection is its own proof. (This assumes one is working with proven processes.)

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This solution is asserted by all pairs of expressions which produce just this intersection. Interpretation and significance lie outside the result.

There is a mathematical equivalence of such same-resultant pairs. But their contexts suggest different interpretations of the picture, through representation. Some might be absurd. Or strangely revealing.

What makes it difficult for us ... is our craving for generality. This craving for generality is the resultant of a number of tendencies connected with particular philosophical confusions.

We cannot talk of contexts in general but only of this or that context.

There is no general context.

Generality is an ideal. For some, it is a false ideology.

In mathematics there isn't any such thing as a generalization whose application to particular cases is still unforeseeable. ... The distinction between the general truth that we can know and the particular that one doesn't know, or between the known description of the object and the object itself one hasn't seen, is another example of something that has been taken over into logic from the physical description of the world. And that is where we get the idea that our reason can recognize questions but not their answers. ("or between ..." as clarification, not as second example.)

The only assertion of an equation is the result.

Equations tautologically show their result.

We then assert that the result means something.

What is the connection of sign and thing signified? That we regard only their logical content.

There is no reason to overload "solution" with the burden of "meaning." The solution can be interpreted mathematically with no picture involved. This is its sense.

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Sense is not formalistic abstraction. The sense of a solution expands our understanding of this equation with these expressions in this context (which could be as general as possible.)

There is always something of a context wherever an equation arises.

The progressive sense within a mathematical context simply restates what we already knew about mathematical progress. Realization works inwardly as well as outwardly.

The sign of a rule, like any other sign, is a sign belonging to a calculus; its job isn't to hypnotize people into accepting an application, but to be used in the calculus in accordance with a system.

Mathematics can be expressed without reference to what is outside mathematics. What is outside comes only into our doing of mathematics and, even then, not into the equations themselves.

Nothing rises above its source. Composed of tautologies, equations can only be tautologies, expressing only truth in their limited way.

The mathematician creates essences. ... The mathematician is an inventor, not a discoverer.

No: the mathematician discovers the forms in which the essence of number can be expressed. The essence of number forces its forms upon us through one discovery after another.

A form is discovered, not invented, because just **this** form forces itself upon us and brooks no alteration.

Forms are inspired by realization, by the apprehension of reality. This inspiration of meaning informs the sense of the equation.

The equation shows the boundaries of its sense. Sense is the possibility of meaning. Bounded by its expressions, it suggests, "Just this intersection, properly interpreted, could be true."

It is one of the most deeply rooted mistakes of philosophy to see possibility as a shadow of reality.

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Equations are not equations of probability. Probability is a context. (One with a modern, and false, metaphysic.)

Thinkers like Charles Peirce and Michael Polanyi grasped at chance as a defense against Laplacean mechanism. But mechanism is only a partial understanding, a child's drawing of the universe, and exists only in the finite mind. Probability, as mathematic, is a declaration of ignorance.

Heisenberg's Uncertainty Principle assumes an infinite regress which is itself the manifestation of a false metaphysic. It assumes an infinity of smaller and faster particles and an eternal inadequacy of measurement.

Infinity is expansion, not contraction. You cannot penetrate that which is always infinitely more in order to reach its non-existent finite beginning.

On the other hand, finity has an end. When the smallest particle is reached and the instruments of measurement catch up, Heisenberg's principle is no longer even a useful heuristic.

Well, if everything speaks for an hypothesis and nothing against it -- is it then certainly true? One may designate it as such. -- But does it certainly agree with reality, with the facts? -- With this question you are already going round in a circle.

The Big Bang Theory is a paradox, a contradiction arising from running a finite model of astronomy and thermodynamics backwards into an infinite antinomy of reason.

In their late phases, all cultures reframe their deepest ideologies in terms of what they understand to be "reason". It wasn't "God", it was the "Singularity"; it wasn't "Divine Wisdom", it was "Chance"; it wasn't 6000 years ago, it was 10 to the 6000, or similar.

***Very** intelligent and well-educated people believe in the story of creation in the Bible, while others hold it as proven false, and the grounds of the latter are known to the former.*

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Both of these creation theories ask for a leap of faith that a reasonable person could decline to make.

*What must a man be called, who cannot understand the concept 'God,' cannot see how a reasonable man may use this word seriously? Are we to say he suffers from some **blindness**?*

Refusal to place a bet is neither affirmation nor denial. And yet one may emphatically refuse.

Introducing a metric into a mythical space does not make it less mythical.

What this language describes is a picture. What is to be done with this picture, how it is to be used, is still obscure. Quite clearly, however, this must be explored if we want to understand the sense of what we are saying. But the picture seems to spare us this work; it already points to a (very) particular use. This is how it takes us in.

Religion and materialism, theism and atheism, both peer down on the same narrow ground of a culture's deepest ideology. They are, in a sense, equivalent.

Ideology is opposed to realization.

Reality lies outside ideology.

If one accepts the doctrine of probability and believes one's self to be in an infinite structure, it is infinitely more likely that one is in the middle of the structure than that one is a countable distance from one of its putative "ends."

You cannot count your way out of infinity. You can't count your way into it either. The infinite has no beginning.

The form of the natural numbers is the infinite expressed from a finite standpoint. Counting makes the picture finite.

Infinity as a centerless expansion. (The way squirrels propagate over an area -- but on a much larger scale.)

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A temporal First Cause, at whatever remove, is an antinomy of reason, a religious belief.

Infinite something from nothing is, as it were, the god of materialism. The scripture for such a religion would be H. G. Well's *An Outline of World History.* (Postgate's addendum as *New Testament.*) Further, infinite something from nothing has uniquely Judeo-Christian origins. No other culture could conceive it.

Probability and statistics make the ignorance of the finite mind a principle of the universe.

That is not the universe. And a metaphysic that enshrines ignorance is not a practical metaphysic.

Equations say nothing about the world.

Equations show our understanding of the world within the limits of our representations.

A law of the calculus that I do not know is not a law. Only what I ***see*** *is a law; not what I* ***describe****. That is the only thing standing in the way of my expressing more in my signs than I can understand.*

Equations express truly the intersection of their equated expressions. There are no laws of operations beyond this.

Additions of tautologies to a tautology serve to narrow the sense.

Inference of mathematical results takes place within mathematics. Interpretation of mathematics comes from outside mathematics.

Any equation shows by its internal relations the contexts its stands within. No further "foundation" adds anything to this.

An equation bounds the space of its negation, the negation of its sense. And *negation has only the multiplicity of the negated*.

The propositions of logic are tautologies. One can perceive by symbol alone their truth or falsehood. Real propositions' truth comes from outside their form. Logical propositions present the scaffolding of the world.

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Logic, being formed of language, often lies closer to the world than mathematics.

Tautology is shared by all mathematical expressions which, in themselves, disregarding context, have nothing in common with each other.

Tautology is the substanceless center of all [equations].

Equations are independent of the truth about the only world. They speak only to the truth of conceptual relations.

Mathematics more powerfully relates to the world the more clearly it reflects the realities, or truth, of consciousness.

Truth is not relative. Only our relation to it is relative.

In the picture, mathematical context, form of representation, and judgment come together to bridge this gap of truth.

We bring out an equation's relevance to our thought by choosing the expressions it arises from.

The sense of a proposition is interpreted from its result.

The sense of an equation is interpreted from its intersection of expressions, its result.

An equation sometimes begins as an expression to solve:

$$\int x^2 \, dx = \text{what?}$$

and we supply the what:

$$\int x^2 \, dx = \tfrac{1}{3}x^3 + C$$

Here the sense of the solution or intersection is very narrow.

Or an equation can set two expression as equivalents:

$$x^2 + y^2 - x - y = xy + x + y + 39$$

In either case, once there is a LHS and a RHS, an intersection -- the solution -- is established.

Two equations are the same if their solutions are forms of the same thing. The interpretation of their meaning within a context or

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within a representation is another thing altogether.

Process and result are equivalent; there are no surprises.

The intersection can be empty.

The intersection can be simple or compound.

In any case, the intersection is a tautology determined by the expressions on either side.

A general method is in itself a clarification of the essence (nature) of the equation. Again, it isn't an incidental device for discovering an extension, it's a goal in itself.

General equations show their generality, which bounds the scope of specificity.

Generality in mathematics does not stand to particularity in mathematics in the same way as general to particular elsewhere.

The general [equation] does not enumerate the particular [equations]. It defines their space. *It's generality lies in the grammatical property of its variables.*

Expressions are possibilities of structure. General equations show what structures can be valid.

The equation delimits the range of its expressions.

If it were impossible for a general solution to be applied to anything less than a complete generality, its validity would vanish and the equation itself be without possible significance.

Equations can only be thought of as completely general, so long as we ignore the mathematical context they arise from. If we do so, they have no sense, no significance to us at all.

This is false formalism. True formalism is much more than this.

There are no completely general equations.

If there were general equations we would have to define where specificity begins. This would require more than putting two expressions around an equals sign and saying, "There, that's

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completely general," as this says absolutely nothing.

Every equation arises from some mathematical context. The equation inherits something of the specificity of the context.

As we assign values, specificity increases. But the general equation is the initial specificity.

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Expressions

Algebraic expressions are stipulations. The system of algebraic expressions corresponds to a system of [mathematical] inductions. ... Algebra deals with the substitutability of other parts of speech.

The expression has both form and sense.

Expressions are tautologies, significant to the extent that they express their sense within some mathematic, but saying nothing. They must justify themselves. They assert only their own reflection of form.

In logic, a tautology is true for all truth values. (We are talking of truth tables here.) A contradiction is false for all truth-values.

Tautology (and contradiction) show that they say nothing. They are without meaning, are not pictures of the world, present no possible facts or objects.

In mathematics, everything must be justified. But *the primitive language-game we originally learned needs no justification, and the false attempts at justification, which force themselves on us, need to be rejected.*

A game, a language, a rule, is an institution.

Some expressions can be tentative as they await confirmation by a "consensus of the elect." This is not the same as submitting a proof and awaiting confirmation. It has to do with communal choices of extension, i.e, √-1.

By "consensus of the elect" is meant the agreement of those who determine the direction of the institutions, in a larger sense, of mathematics.

All institutions age into the errors of ecclesiasticism.

This sense [of an expression]*, its content, is a kind of free variable.*

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All expressions are number -- either a single number or a set of numbers.

The sense of an expression is that which it takes from the mathematic in which it arises and, beyond that, its sense in those mathematics in which it is applicable.

In Euclid, one can begin with a proportion of lengths:

lenA : lenB :: lenC : lenD

and arrive at ratios of length and area:

lenA : areaA :: lenC : areaC

One operation on proportion is a : b - a :: c : d - c:

lenA : areaA - lenA :: lenC : areaC - lenC

Here we would have to ask what sense has "area minus length." And our answer would be influenced by the meaning we could realize in the result. If we were unable to supply a sense of this subtraction, backed by our understanding and our grasp of meaning and yet had a meaningful result, we might have to expand our understanding of subtraction.

Lack of sense comes from lack of meaningful intention.

[Algebra's] generality doesn't lie in itself, but in the possibility of correct application.

With expressions, there can be no classification, no side-by-side comparison, no "more general and more special."

Expressions are numerically indifferent to the values assigned them.

An elementary expression asserts the truth of a simple tautology. It is merely a concatenation of names and operators. And it cannot be contradicted by another elementary expression.

An expression cannot possibly assert of itself that anything beyond its sense and form are true.

Expressions can be analyzed for clarity but not for content. Full analysis presents the simplest form.

So there is no question of "atomic" or "molecular" expressions.

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A simple expression is an expression in its simplest form and we recognize it as such. For, if we could make it simpler, we would -- for our own convenience.

We simplify an expression so long as the result continues to be equivalent to the initial state. In its simplest state, if it is not unitary, we can decompose it into the simpler expressions it contains.

But this lower hierarchy is meaningless. The original simplest form is the totality of the expression.

A complex expression reduced to its simplest form might appear to change contexts. This is simply a matter of usage: "But we don't say it that way here."

There are hierarchies of expression. Distinctions of type arise in contexts. But once in use within equations, they rise to their natural level of generality.

The sense of an expression is "these variables and constants are subject to these operations." This sense suggests a picture. But more understanding forms a truer picture.

Such pictures are only possibilities of equations using these expressions in a given representation.

If we arrange an expression unmathematically, it ceases to be an expression.

Only malformed expressions are senseless. And these are hard to produce because mathematics continually brings to the meaningless sign an appropriate contextual meaning, i.e. zero, negative numbers, complex numbers, *a genealogical tree of different number concepts.*

We are led to find extensions to our basic concepts that will eliminate the exceptions. -- Richard Courant.

We do this by stating, mathematically, the properties of the extensions. Such statements should exclude metaphysics.

Every expression of mathematics must explain itself.

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Our disease is one of wanting to explain.

The laws of composing expressions are the expressions' only explanation. We cannot go wrong in mathematics within mathematics. In this sense, falsehood (eventually) excludes itself.

The internal sense of the expression points to possible external connections. But this internal sense cannot explain the connection or its object, both of which lie in meaning and understanding.

Mathematics is at best strongly suggestive of meaning.

False premises come from outside mathematics.

In mathematics, a tautology simply reduces to its simplest form. Its general form expresses its mathematical sense and nothing else.

In logic, what is not a tautology or contradiction is subject to truth values. *Truth conditions determine the range which is left to the facts in the proposition.*

Mathematical expressions have a range limited only by reason, understanding.

In "numerical" mathematics, there are no contradictions or even propositions.

The level of specificity of an expression in an equation determines the range of its containing equation's solution.

An expression is a name. Its negation denies that name. Such a negation is non-mathematical. The expression is simply seen as incorrect in a particular usage.

In an equation, expressions are used to assert that, in the context of the representation, these names do or do not exist.

The name can be general (x) and later used to point to an object.

An expression cannot be negated in a logical sense. The expression "a+b" is all of "a+b", "-a+b", "a-b", and "-a-b" until we begin to be specific.

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Negation only takes place in the picture. It says, "No, this expression has no significance here."

Both negation and correction are then possible -- outside mathematics.

Specificity narrows the intersection of expressions in an equation -- either in size or through context. Both have an impact on meaning, according to their weight in the judgments made:

Solution in a general context: we have just these values.
Same solution in analytical geometry: we have just these points.

General form of a proposition: such and such is the case.
General form of an expression: upon these variables, these operations will perform as expected.

An expression can be the result of an operation which produces it from other expressions.

All operations produce a tautological result.

We don't know what an expression will say, in representation, until it is made specific. But we know how it will say it and that it will say the truth regarding its sense. And the equation containing it will define the range of that truth.

Operation gives expression to the differences between forms. It does not characterize forms or sense. Nor does it assert.

The context does not enter into the expression. It remains outside, as a point of view.

We are able to project the sense -- the ideas we built the expression from -- out of their given context into their forms in another context. And within every context they can be changed into equivalent general or specific expressions, with different senses and implications of meaning.

Because expressions arise in mathematical contexts and those contexts are often addressing relations and objects in the world, the expressions, therefore, naturally suggest a picture.

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We do not assert this or that with expressions. *The nature of the essentially necessary sign does the asserting.* Or rather, it does all the asserting it can. It asserts its own form.

The form and the rules of syntax are equivalent. So if I change the rules -- seemingly supplement them, say -- then I change the form, the meaning.

Any sense the expression has is purely mathematical and is further specified by the specific equation. None of this sense crosses the gap of understanding. It merely points in the direction of possibilities.

An expression is not a picture. *It is a neutral thing, waiting in the wings.*

General expression: the idea of all lines:

$$ax + by + c$$

Restated as general equation:

$$ax + by = -c$$

$$y = (-a/b)x + (-c/b)$$

No form is pre-eminent. Specificity is the narrowing or focusing of the sense of the form.

Equation of one point on all lines:

$$ax + b = 0$$

Assigning values increases specificity.

The equation of one line:

$$y = 3x + 1$$

Of one point on that line:

$$3x + 1 = 0$$

Assigning values to all the free variables maximizes specificity.

All this by way of defining terms: general, specific. Nothing surprising or hidden.

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Expressions also have forms related to context. But anywhere the above expressions arise, the lines are lines and the points are points. And this is so even if there are no points or lines in the intended form of representation or picture.

In short, $ax + by + c$ is always subject to the laws of lines in any context. Lines have all the senses of lines.

All senses are latent in the "general" expression.

Similarly, with

$$x^n + y^n = z^n$$

The expression $x + y$ stands in some relation (>=<) to z. And if

$$x + y > z$$

then x, y, and z fall under all of the laws of triangles, not only in algebraic geometry or in pure geometry but anywhere else this relation arises. (You should now be able to solve Fermat's Last Theorem in six to eight lines.)

All forms are latent in the general expression.

Each context from which an expression arises gives that expression a sense peculiar to the context. These different senses lend themselves in the representation to different interpretations, different forms.

All this by way of defining the form of number, which is a basis of praxis.

Every sign is capable of [subject to] *interpretation; but the* ***meaning*** *mustn't be capable of interpretation. It is the* [result of the] *last* [final] *interpretation.*

And with number, its form is its meaning.

One can more fully and easily express mathematics the more fully and easily one can recognize equivalent forms and the principles these forms fall under in disparate contexts.

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Just as there are no pre-eminent mathematics, there are no pre-eminent expressions, contexts, or forms. Each provides a different perspective to a given idea. Each gives rise to its own possibilities.

The most basic forms are the most powerful.

Sense is therefore in no way inferior to **meaning**. Understanding the form of number and the consequences of mathematic's truth grounds are just as significant as the realization of reality in the only world. Mathematics, as we do it, occurs in the only world.

Form of expression: *such that we cannot say, "it could not have been foreseen that there was such a thing as this." For that would mean we had a new experience and that it took that to make this form possible.*

Form follows from realization.

Form is latent in the idea. We bring it out according to our understanding.

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Truth Grounds

Mathematics is an assertion that number's consequences can best be described by using our truth grounds and operators. I wanted to add "axioms" but axioms are either a truth-ground or a cheat.

*It is not single axioms that strike me as obvious, it is a system in which consequences and premises give one another **mutual** support.*

The truth grounds are an agreement on how we are to use number. The various mathematics are the consequences of this agreement. The consequent expansion of mathematics **shows** the boundless basis of consciousness.

*Logic and mathematics are not **based** on axioms, any more than a group is based on the elements and operations that define it.*

The technique of our word use is always a tacit presupposition.

Mathematics is interested only in reality but only the reality of the form of consciousness. By this I mean the form which excludes the un-logical, the false.

In logic, the truth grounds are:

1) names: p, q, ...
2) basic operators: not, and, or, implication
3) the simplest truth relations, sixteen in number:

(TTTT)(p,q)	(p=>p & q=>q)
(FTTT)(p,q)	!(p & q)
...	...
(FFFF)(p,q)	(p & !p) & (q & !q)

*The old [19th C.] logic contains more of convention and physics than has been realized. If a noun is the name of a **body**, a verb is to denote movement, an adjective to denote a property of the body.*

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Formal languages, as they arise, arise from the structures of the prevailing metaphysic. They are ideological pictures of consciousness.

Logic must take care of itself.

Mathematics, as well.

If any logic "underlies" mathematics, it is not the modern formal logic. It is the logic found in Euclid which is still used as the justification of "spatial" mathematics. But this logic co-exists with, and does not underlie, mathematics. In this co-existence, the two are inseparable.

'Mathematical logic' has completely deformed the thinking of mathematicians and philosophers by setting up a superficial interpretation of the forms of our everyday language as an analysis of the structures of facts.

When modern logic was trying to put a foundation "under mathematics" (Whitehead, Russell, Frege, et. al.), modern logic was still trying to figure out what its own symbolism was and what the scope was of expressions using those symbols. It was still struggling with the question, "What is a proposition?"

Geometry and grammar always correspond with one another.

Logic was born with geometry. It is a discipline of thought expressed -- a method, not a basis. Set theory is a method, not a basis. You can drive from here to there in either car. Or just walk.

(It might be supposed from this text that Wittgenstein and I are somehow opposed to set theory and other modern mathematics. I do not believe that is true in his case. It is categorically not true in mine.)

Wittgenstein's concept of geometry is a broad and fundamental idea. I do not pretend to entirely grasp his meaning. It is far from trivial and extends beyond mathematics.

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Euclidean geometry: *What is demonstrated can't be expressed by a proposition.*

It is expressed by laws and by a chain of logical consequences leading to a proposition.

To say that logic or set theory underlies mathematics is to say we didn't know what we were doing, in some fundamental sense, until formal logic or set theory came along. But it was only formal logic and set theory we didn't understand.

Set theory builds on a fictitious system, therefore on nonsense. ...
*In mathematics **everything** is algorithm and **nothing** is meaning. ...*
*In set theory, what is calculus must be separated off from what attempts to be (and of course cannot be) **theory**.*

Such bases are chosen, not to support mathematics, but to support and develop a point of view regarding mathematics. These are transient ideological pictures.

Most mathematicians are unaware of the transience of mathematics. The same false ideology behind the thought of **progress** infects our idea of the landscape of mathematics.

A point of view should be judged by its fruits.

Arithmetic is the grammar of numbers.
Arithmetic is a more general kind of geometry.
Arithmetic doesn't talk about numbers, it works with numbers.
Arithmetical expressions are autonomous.

To understand mathematics is *to take in a symbolism as a whole.*

Truth grounds are not propositions. They are stipulations.

Truth grounds are everywhere applicable.

Truth grounds are the intersection of all grammar.

*Axioms are only postulates of the form of expression. ... Something is an axiom, **not** because we accept it as extremely probable, nay certain, but because we assign to it a peculiar function, and one*

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that conflicts with that of an empirical proposition. ... By accepting a proposition as self-evident, we also release it from all responsibility in face of experience.

Axioms, according to Frege, have two significances:

1. The rules by which you play; and
2. The opening positions of the game.

*A game does not just have rules, it has a point. ... Mathematical propositions are not positions in a game. And in **this** way they are not prophecies either.*

When axioms are made the truth grounds, we believe (axiomatically) they are required to be consistent, complete, and independent.

On the former two: *results indicate that such efforts cannot be completely successful, in the sense that proofs for consistency and completeness are not possible within strictly closed systems of concepts. Remarkably enough, all those [Formalist] arguments on foundations proceed by [Intuitionist] methods that in themselves are thoroughly constructive and directed by intuitive patterns. ... [It] would be completely unjustified to infer that the living body of mathematics is in the least threatened by such differences of opinions or by the paradoxes inherent in an uncontrolled drift towards boundless generality. -- Richard Courant*

A consistency proof can't be essential for the application of axioms. For these are the propositions of syntax.

*The fundamental fact here is that we lay down rules, a technique, for a game, and that then when we follow the rules, things do not turn out as we had assumed. That we are as it were entangled in our own rules. This entanglement in our own rules is what we want to understand. It throws light on our concept of **meaning** something.*

"This law was not given with such cases in view." Does that mean that it is senseless?

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If inconsistencies were to arise between the rules of the game of mathematics, it would be the easiest thing in the world to remedy, All we have to do is to make a new stipulation to cover the case in which the rules conflict, and the matter is resolved.

It seemed worthwhile to point out [an aspect of algebraic geometry] for the reassurance of young mathematicians who have just heard of Gödel's Theorem and expect the eminent collapse of mathematics! Whatever may be said of such an attitude on their part, it is certainly indicative of a serious concern for our subject and so should be regarded with sympathetic understanding. -- W. E. Jenner

The truth grounds have consistently led to contradictions -- negative number, negative square roots, even exponential notation was a struggle. At no point have we thrown out the truth grounds. We simply extend them.

There is a difference between a mistake for which, as it were, a place is prepared in the game, and a complete irregularity that happens as an exception.

It still remains true that there are no negative apples and that no negative number has a integer for a square root.

We are not justified in having any more scruples about our [mathematics] than the chess player has about chess, namely none.

If the contradictions in mathematics arise through an unclarity, I can ***never dispel this unclarity with a proof****. The proof only proves what it proves. But it can't lift the fog.*

A lack of clarity indicates the influence of a false metaphysic, usually coming from the deepest cultural ideology.

Our inability to think otherwise, to escape our fundamental cultural assumptions, dims the clarity of the truth.

Mathematics cannot be incomplete, any more than a ***sense*** *can be incomplete. Whatever I understand, I must completely understand.*

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We understand all that we have realized, all which we can demonstrate. But from its boundless basis, understanding can only increase.

Our understanding is a reality in the only world.

No conceivable experience can refute a postulate, even though it may be extremely inconvenient to hang onto it.

Our postulates are those ideas about which we cannot think otherwise. These are ideals.

It is important to distinguish between the ideals of idealism as philosophy and ideals as transcendental ideals. The former are a false reification claiming hegemony for the limited mind. The latter are simply those concepts we understand completely but which cannot be experienced: an infinite plane, infinite parallel lines.

The failure to distinguish between the two led to the Great Panic of Non-Euclidean Geometry. This reaction merely showed the extreme philosophical naiveté of those who panicked. There is always room for more transcendental ideals, so long as they are genuinely meaningful.

The axioms of geometry are not to include any truths. ... The axioms -- e.g. of Euclidean geometry are the disguised rules of syntax. This becomes very clear if you look at what corresponds to them in analytic geometry.

Expansions of mathematics which retain the coherence of the truth grounds increase the possibility of what can be expressed in mathematics. They give to mathematics a larger sense of what can be said. And this increases our powers of representation.

Increasing the senses of the equations expands our possibilities of expressing the world meaningfully in our pictures.

*In abandoning the **meaning** of symbols, we abandon the words which describe them. -- Augustus De Morgan*

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For De Morgan, abandoning the meaning was the basis of his construction of Double Algebra, now vector algebra. But meaning was only abandoned in order to create a new sense upon old symbols.

As for remaining in meaninglessness: *We are always in danger of giving a mythology of the symbolism.*

In mathematics, the truth grounds are the natural numbers, the four operators, and the sign of equivalence. Everything else is forced into existence by these.

Numbers are pictures of the extension of concepts.

Numbers are the results of extended concepts but show no trace of the concept.

Mathematical operations are not logical operations. They guide the alteration of internal relations.

The truth grounds of logic enable us to express our understanding of the world. The truth grounds of mathematics enable us to express our understanding of tautological sense.

We can more completely grasp the tautologies of mathematics because they themselves are transcendental ideals.

Reality is not a transcendental ideal. Neither is the only world.

Thought contains nothing more than was put into it.

Nothing underlies the truth grounds.

You can't get behind the rules, because there isn't any behind.

It does nothing to look behind primitives. Looking for what underlies "thought," you find only psychological and metaphysical preconceptions, themselves "thought."

And what you look **with** is "thought" too.

Analysis has its dead level. Beyond this, self-delusion.

Everything which does not belong to the number calculus is mere decoration.

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The truth grounds of ab or a + b are a, b, and the operations. Nothing lies below this. There are no foundations of the truth grounds.

Truth grounds are a choice. But everyone must agree to this choice. This agreement is always based upon the deepest ideology of a culture.

Our truth grounds are not those of the Greeks, Chinese, Egyptians, or Indians. In Greek geometry, nothing moved. All was static, solid.

Euclid was even squeamish about superposition, using it reluctantly.

Primitive ideas are independent of one another. A new primitive is introduced at once, everywhere it can occur.

Truth grounds are completely general and free of all context.

A rule of syntax corresponds to the position in the game. (Can the rules of syntax contradict each other?) Syntax cannot be justified.

Symbolism obeys the laws of mathematical grammar, expressing mathematical syntax.

Geometry, as measurement of the world, can be inconsistent due to choice of syntax. But if the inconsistency is outside our point of view, for us it is out of sight. The state of play is consistent.

In mathematics, the signs themselves do mathematics, they don't describe it. The mathematical signs ***are*** *like the beads of an abacus. And the beads are in space. An investigation of an abacus is an investigation of space.*

As we grow, our descriptions of what we understand are shown by our realization to be inconsistent with reality. And so we alter our descriptions to approach more closely to reality.

We cannot make any discoveries in the syntax.

We do not choose the grammar and syntax of reality.

Syntax draws together the expressions that make **one** determination.

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Every correct symbolism is translatable into every other correct symbolism.

Misused signs produce accidental consequences. But *we cannot give a sign the wrong sense.*

The number of necessary fundamental operations depends only upon our notation. Operations are a set of rules.

Operators must at least encompass the rules. But there is an optimum number of operators, below which obscurantism is encouraged, and beyond which clarity is dissipated.

The number system, i.e. the decimal, is not the ***subject matter*** *of number.*

There is no pre-eminent number.

The concept "number" is nothing else but that which is common to all numbers, the general form of number.

The concept of number is the acknowledgement of the constant inclusion of that which can be harmoniously brought under the operations of number.

Kinds of number can only be distinguished by the arithmetical rules relating to them.

An irrational number is a law.

Only a law approaches a value. In the case of approximation by repeated bisection we approach ***every*** *point via* ***rational*** *numbers.*

The expression of the law specifies the number.

There is no adequate law of the square root or we would use it on $\sqrt{2}$. What we have is a method or heuristic for approximating such a square root. And that heuristic conforms to the law we do have of finding the square roots of perfect squares.

We grasp law only from our finite standpoint. And the infinite cannot be expressed from a finite standpoint.

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The objection that 'the finite cannot grasp the infinite' is ***really*** *directed against the psychological act of grasping or understanding.*

This, only if you consider understanding to be a psychological act, which it is not. Understanding and realization are necessarily outside the modes of personality. And psychology is the analysis of personality -- the false sense of individuality.

Individuality has universal intent.

[D]efinition is only possible if it is itself not a proposition. A definition cannot be denied. It is a rule by which we must proceed. ... The definition is a kind of ornament coping that supports nothing.

A definition is an assertion with consequences. It is the entrance of metaphysics: true or false, practical or not.

The system is the logically important thing and not the single symbols.

Symbols are either names, operators, or syntactic candy.

$$f(x) = x^2 + x + 1$$

is not an equation. It is an assertion that the RHS is a function. This is syntactic candy. It might be better to express this as:

$$f(x) := x^2 + x + 1$$

or

$$f(x) \equiv x^2 + x + 1$$

to make the assignment clearer.

$\sqrt{}$ a is syntactic candy for the concept b: (b × b) = a

$$\sum x_i = x_1 + x_2 + \ldots$$

is syntactic convenience candy like f(x). Syntactic candy introduces nothing. It is a form of abbreviation.

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There was initial resistance by some algebraists to using exponent indices because it was not "associative", i.e. $m^n \neq n^m$. But a^n is merely syntactic candy for aaaa...[n times]...aaa.

That operators appear in the use of such candy does not make the candy an operator.

The symbol ∫ is an operator bringing a new context to old operators. We then use old operators to turn $\frac{1}{3}x^3$ into 2x, or vice versa, in this new context.

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Metaphysic

In any serious question, uncertainty extends to the very roots of the problem. One must always be prepared to learn something ***totally*** *new.*

I have found no better expression than "religious" for confidence in the rational nature of reality. ... Whenever this feeling is absent, science degenerates into uninspired empiricism. -- Einstein

Or "religion" ≡ "adherence to rational nature of reality"

If free of **all** ideology, this then **is** a practical metaphysic.

There is no religious denomination in which so much sin has been committed through the misuse of metaphorical expressions as in mathematics.

If you are viewing these propositions in a religious sense, you need to turn around. They point in the opposite direction. Which is not the direction of materialism. (Ideas have more than two dimensions.)

A serious threat to the life of science is implied in the assertion that mathematics is nothing but a system of conclusions drawn from definitions and postulates that must be consistent but otherwise may be created by the free will of the mathematician. If this description were accurate, mathematics could not attract any intelligent person. It would be a game of definitions, rules, and syllogisms without motive or goal. The notion that the intellect can create meaningful postulational systems at its whim is a deceptive half-truth. Only under the discipline of responsibility to the organic whole, only as guided by intrinsic necessity, can the free mind achieve results of scientific value. -- R. Courant

Science is the realization of reality.

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It would be easy to misinterpret this quote. But consider how harmoniously Richard Courant worked with David Hilbert on *Methods of Mathematical Physics.* Then consider Hilbert's works on axiomatic bases. Then consider Courant's acceptance of Godel's Diagonal Proof and Hilbert's eschewal of same.

Here we have a hitherto unknown kind of insanity. -- Frege

It is a mistake to view the truth of mathematics as lying fundamentally on a plane, as if projected there from reality. Reality shows that truth has a profound and practical metaphysical depth. Mathematics models the relations of this depth as best it can.

We simplify the picture of our world by removing the depth of meaning. We project the volume of experience onto a plane of ideology. Then we say two things are alike because they are congruently projected onto **the plane of our choosing**. We do this within and without mathematics.

Mathematics must rely on the same metaphysic by which reality is successively realized and demonstrated.

A main source of our failure to understand is that we do not ***command a clear view*** *of the use of our words.*

The hegemony of the human mind is a fundamental ideology in our culture. Not all cultures have shared this view. Some to their advantage, some to their detriment.

Why is it important to depict anomalies accurately? If someone can't do this, it shows that he isn't quite at home yet among the concepts.

I would like to say that infinity hosts the last lingering metaphysical falsehood in mathematics. But on consideration, existence is also a false metaphysical view in mathematics. And both of these are bound up in the concept of the continuum.

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There are many "figures seven" but there is but one "number seven," because "number seven" is idea, one idea. On the same principle there is but one everything. -- Edward Kimball

There is but one infinity. As infinity, it must have infinite aspects. Mathematically, it is only the abstraction: "and so on." But even this has its necessary consequences.

The infinite is that whose essence is to exclude nothing finite.

- ≡ *that which cannot be divided into parts*
- ≡ *that which is not affected by division*
- ≡ *that which offers infinite opportunity for division*

These very real aspects of mathematical infinity begin to show the paradoxical effect of the infinite on the finite standpoint.

Space has no extension, only spatial objects are extended but infinity is a property of space.

Or rather, a property of thought. We can imagine having the standpoint that space must always end. The passage from indefinitely to endlessly is a choice of standpoint.

Consciousness is infinite. There is no close-able set of all thoughts.

Let us momentarily distinguish between "thoughts" and "ideas." Let thoughts be the quotidian elements of consciousness ("Where's my cellphone?" "It's eight o'clock.") and ideas, the creative, expansive elements of understanding which demonstrate our realization.

Then thoughts may well be a not-terribly-large finite set. ("Where's my cellphone?" replacing the earlier "Where's my pager?" replacing the earlier "Where's my change for the pay-phone?")

Then only the expression of individuality through realization and demonstration makes consciousness infinite. And this is what we model in mathematics as infinity.

Here again, we impose the ideological necessity of a "first thought."

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Imagine a tribe of people who can accept the idea of any natural number no matter how great but consider infinity to be absurd. They could accept theoretically any rational number but none of the irrationals. They could say, "Pi must end somewhere."

This shows a limited idea of number.

The space of human movement is infinite in the same way as time. Infinitely long isn't a measure of distance.

That an object is infinitely extendable says only that the object and the space are homogeneous. Nothing hinders.

"And so on" symbolized by "..." is one of the most important [concepts] of all and ... infinitely fundamental. Without this concept we should be stuck at the primitive signs and could not go "on."

"And so on" and "∞" say no more than the signs themselves show. They do *not harbour a secret power.*

The definition of a word is not an analysis of what goes on inside me (or what should go on) when I utter it.

"And so on" or "..." is not "∞" and trouble arises from conflating the two.

The concept "and so on" and the concept of the operation are equivalent. ... The "and so on" is not a sign of incompleteness.

Consider Cantor's Diagonal Proof of the nondenumerability of the reals.

The infinite number series is only the infinite possibility of a finite series of numbers. The signs themselves only contain the possibility and not the reality of their repetition. Mathematics can't even try to speak about their possibility. If it tries to ***express*** *their possibility, i.e. when it confuses this with their reality, we ought to cut it down to size.*

Let us alter Cantor's rectangle of numbers so as to order, so far as we are able, the numbers in ascending order: those at the top less than those below. Those which are beyond our capacity for

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ordering can simply be sent to the bottom of the rectangle. After ordering the numbers as best we can, we simply have to send to the bottom any which are either less than the one we are considering or are indeterminable. This we can do as we come to them. And if we get so far as to realize a need for something discarded, we can move it back up.

If we **can** create a diagonal, we **can** do all this.

By definition, with infinity, we cannot survey its entirety. It doesn't have an end "infinitely far away." It has no end.

So as we move down the diagonal, altering digits, we are dealing only with numbers beginning with zeroes. The density of the rationals gives us infinitely many of these, each beginning with as many zeroes as we need. If we mistakenly sent a needed one to the bottom, we can bring it back up.

*We may also say: there is no path to infinity, **not even an endless one**.*

Let our first choice of digits for Cantor's Diagonal be 1 and the second 2. We are now building a number which begins as 0.12. This number itself lies below us in our somewhat-ordered list. We are therefore also choosing a number between 0.12 and 0.13. Each choice we make specifies another number below our current position on a monotonically decreasing interval.

*I grasp an infinite stretch in a different way from an endless stretch. A proposition about it cannot be verified by a putative endless striding, but only in **one** stride.*

Making an infinite number of choices is clearly impossible. There is always another choice, **actually** another infinity of choices, to make. But Cantor has created the rules of this game. So let us complete our choices just as he did. In the end, we have selected precisely the limit of those monotonically decreasing intervals. And such choice is some irrational number: 0.12... .

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*It isn't just impossible "for us men" to run through the natural numbers one by one; it's **impossible**. It means nothing. The totality is only given as a concept.*

This limit of decreasing intervals is simply the flip side of Cantor's Diagonal Proof coin. If one side is legitimate, the other side is legitimate. Cantor's "diagonal" does not -- cannot -- reach the putative corner of his infinite rectangle of number. It reaches, if it reaches anything at all, the limit of his sequence. According to him, the limit isn't there on the continuum. Imagine the chaos ....

The rules of the foreground make it impossible to recognize the rules in the background.

Are the irrationals simply the unsorted (unsortable?) rationals in this process? Of course not. But this cannot be shown empirically. Or by any finite picture, no matter how suggestive of our desires.

*I can surely imagine a wheel spinning and never coming to rest. Which is a peculiar argument: "**I can imagine** ..."*

Cantor's Diagonal Proof is not a proof at all. It is simply the world's shortest antinomy of reason. Like all antinomies, it can be played both ways, with infinity open then closed or closed then open, and either way, in the end, it means nothing. And this, because we are using contradictory concepts of infinity. One true and one false.

A mathematical proof incorporates the mathematical proposition into a new calculus, and alters its [the proposition's] *position in mathematics.*

In mathematics, the only **propositions** are proofs asserting that **these** laws we accept have **these** consequences.

Logical proof stands and falls with its geometrical cogency.

A proof says nothing about other possibilities, those outside the proof. It operates upon a well-defined and logically closed space.

*Understanding the use of a word in **one** context does not relieve us from investigating its grammar in another.*

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Induction proof of infinite series: we say that even beyond our finite standpoint, the elements maintain the form of the n^{th} term with these finite or infinite consequences.

The only reason why you can't say there are infinitely many things is that there aren't.

A picture, combined with sleight-of-hand, asserts no law.

A mathematical proof could not have been described before it was discovered.

Cantor's proof was a pre-described solution to a persistent difficulty.

If you derive a theory from the proof, then the sense of the theory must be independent of the proof; for otherwise the theory could never have been separated from the proof.

The interesting question to me is: Why did the mathematical world affirm Cantor? Why not Russell and Whitehead's classes of infinities which tried to push the identical antinomy out of reach? My working hypothesis is that Cantor's Diagonal picture was like a sudden vaccination. One shot and it was over. Just turn away for a moment and close your eyes. Everyone else's pseudo-solution required one to think about the problem.

And this problem is not thinkable.

This is the nature of error. The mark of ignorance is on its forehead, for it neither understands nor can be understood. -- Mary Baker Eddy

This points to there never being an explanation of the false. (No explanation of 25×25=637.) There is only the destruction of the falsehood by the truth.

False metaphysics, lingering medieval ecclesiastical thought, naive philosophies, unquestioned imaginings, all these block our view of the truth.

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The explanation of the Dedekind cut pretends to be clear when it says there are 3 cases: either the class R has a first member and L has no last member, etc. In fact, two of these 3 cases cannot be imagined, unless the words "class", "first member", "last member", altogether change the everyday meanings they are supposed to have retained.

Dedekind's popularity seems to have ridden this same picture-wave begun by Cantor. The only knife that can make a Dedekind cut is a bladeless one. And a bladeless knife without a handle was all the rage in those paradox years. (Although, that knife was invented by Lichtenberg.)

Something surprising, a paradox, is a paradox only in a particular, as it were, defective surrounding. One needs to complete this surrounding in such a way that what looked like a paradox no longer seems one.

There is **no meaning** in any paradox.

Nothing is more likely than that the verbal expression of the result of a mathematical proof is calculated to delude us with a myth. I am trying to say something like this: even if the proved mathematical proposition seems to point to a reality outside itself, still it is only the acceptance of a new measure [as in measuring device] *(of reality). ... For the mathematical proposition is to show us what it makes SENSE to say.*

A question as yet unanswered: "Is there a limit in using ten digits such that beyond this limit, a pattern must emerge and repeat? In trying to avoid a repeated pattern, would you run through the possibilities of avoidance and begin again?"

You do not have infinite choices. You have ten.

Eternity, the necessary period for completing an irrational number, is a long time to avoid repeating yourself. We are only playing with metaphors here. But such are the metaphors still plaguing mathematical thought.

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*There is no such thing as **all** numbers.*

It would be easy (facile) to interpret Wittgenstein as being opposed to the idea of infinity in mathematics. It is not that simple.

Exhibit A: *Ought the word 'infinite' be avoided in mathematics? Yes; where it appears to confer a meaning upon the calculus; instead of getting one from it.*

Exhibit B: *Finitism and behaviourism are quite similar trends. ... Both deny the existence of something, both with a view to escaping from a confusion.*

Exhibit C: *The 'actual infinite' is 'mere word.' It would be better to say: for the time being this expression merely produces a picture -- which so far hangs in the air; you still owe us its application.*

There is no clarity without sufficient criticism being adequately answered.

Some modern mathematics seek to avoid criticism by claiming to be "about nothing." As if "anything" were outside their scope.

*There can't be possibility and actuality in mathematics. It's all on **one** level. And is in a certain sense, **actual**.*

The concept of infinity is the expression of a possibility.

The concept of an infinite set is the expression of an inclusive ideal.

The temptation is to close it: "I have a set, therefore I have all of it."

There is not enough room on our planet for you to **have** even one of what are quite modestly-sized rational numbers, compared to how large they might be.

If you **had** an infinite set, there would be no room in the universe for anything else. (This is not an argument for constructivism. Simply more metaphors.)

"Infinite class" and "finite class" are different logical categories; what can be significantly asserted of one category cannot be

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significantly asserted of the other.

We are not comfortable with contradiction but we are perfectly *willing to live in the neighborhood of a contradiction.*

All of the contradictions arising from infinity arise from the false belief that infinity can be contained in a closed set. Where this is not used, contradiction does not appear. Where it is used, a cascade of contradictions can be produced at will.

Infinity is that which is always expanding, that which essentially lies entirely outside our finite standpoint.

A paradox, a paradox, a most amazing paradox. -- Gilbert and Sullivan, Pirates of Penzance

Innumerable contradictions can be produced by playing with the contradiction of the openness and closure of infinity. One could assign to a class of graduate students the production of an antinomy within Cantor's proof and receive as many antinomies as students.

[A] ***mysteriousness*** *about some mathematical concept is not* ***straight away*** *interpreted as an erroneous conception, as a mistake of ideas; but rather as something that is at any rate not to be despised, is perhaps even rather to be respected.*

This is in fact the correct approach. All new concepts should be respected so long as clarity prevails in their results.

But all intrusions of false metaphysics should be rooted out.

And now each bad analogy gets explained by another bad one, so that in the end only weariness releases us from these ineptitudes.

When one accepts one side of an antinomy of reason and then accepts the other side, one is led into paradox.

I would like to say: By the law of the excluded middle, an infinite set is either open or closed and closure brings mysteriousness in its train. (But I'm not entirely convinced of the universal applicability of that law.)

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"Infinite" is an adverb.

Infinite means "passing beyond a finite standpoint."

Because of the density of the rationals, our somewhat-ordered list of numbers can produce not only one, but infinite irrationals through choices effecting only numbers beginning with zero.

Infinity is not one-dimensional. The natural numbers, **N**, are one-dimensional. Infinity is **N**-dimensional. (At least.)

Our picture of infinity is still that of the medieval school-men. This picture supported the infinitesimals. Weierstrass' expression of the limit destroyed that expression of a false metaphysic. But its basis lives on in our general picture of infinity.

We interpret the enigma of our misunderstanding as the enigma of an incomprehensible process.

We do not need the false picture of Cantor for our expression of the nature of infinity. But we do need a clear picture to replace the pseudo-picture he created.

Such a picture can only be a "consensus of the elect." And such an effort may be beyond us.

In the superstition that m = 2n correlates [an infinite] class with its [infinite] subclass, we merely have yet another case of ambiguous grammar. ... m = 2n contains the ***possibility of correlating any number*** *with another,* ***but doesn't correlate all numbers with others****. ... The word "possibility" is of course misleading, as someone will say, let what is possible become actual. And in thinking this, we always think of a temporal process and infer from the fact that mathematics has nothing to do with time, that in its case possibility is (already) actuality.*

An infinite set does not contain infinite subsets equal to itself in size. There is only one infinite set of each dimension and its elements can be renamed at will from the finite standpoint.

$$m = 2\times n \equiv m = q\times n$$

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If we take as q the first integer of which mankind can have no experience, due to its size and the impending death of our sun, we can correlate n to qn and, as we correlate, have a set of 1/q of the numbers mankind will always be ignorant of and at any point have a set of numbers without any of the numbers man can have a knowledge of. This set correlates, paradoxically, to the empty set. Therefore, the empty set has the same cardinality as the natural numbers, from the limited standpoint of mankind.

Although here I have done nothing Cantor did not do, there will be mathematicians who will say: "We don't do it that way." But we can't do it in any case from here.

I can only count what is actually there, not possibilities.

We cannot correlate anything infinitely without playing Cantor's false trump card: "I am done."

Always, with infinite anything, there is more to do. But all the doing is finite. Completion is out of our reach. Even conceptually.

There is no system of irrational numbers -- but also no super-system, no 'set of irrational numbers' of higher-order infinity.

In infinity, that which cannot be described is that which makes the concept complete.

The theory of aggregates attempts to grasp the infinite at a more general level than a theory of rules. It says that you can't grasp the actual infinite by means of arithmetical symbolism at all and that therefore it can only be described and not represented. The description would encompass it in something like the way in which you carry a number of things that you can't hold in your hands by packing them in a box. Then they are invisible but we still know that we are carrying them (so to speak, indirectly) . The theory of aggregates is a pig in a poke. Let the infinite accommodate itself in this box as best it can. ... The point of this method is to make everything amorphous and treat it accordingly.

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And this is precisely what topology, abstract algebra, and numerous other mathematics of the 20th Century are based upon. They show both the value and the limits of the amorphous point of view regarding the form of number.

If an amorphous theory of infinite aggregates is possible, it can describe and represent only what is amorphous about those aggregates.

The danger of using these is to forget that you are viewing mathematics amorphously.

Most people, being formless themselves and being unable to attain to any Gestalt, strive to deprive objects of their Gestalt and reduce everything to chaotic matter, in which category they themselves belong. They reduce everything to its so-called effect. Everything is relative in their sight; so they relativize everything except nonsense and triteness, which hold absolute sway, as is to be expected. --Goethe

The actual truth is in non-amorphous reality and its infinite details.

There is a proof, using Bolzano's theorem twice, which proves that any two figures in the plane can be bisected by one line. The first use of Bolzano's theorem, to find the bisector of the first region, is fine. But the second use, at best, gives only a partial truth. At worst, it is in itself false and this falsity is concealed by the innate truth of the proposition.

If a figure in the plane is symmetrical, it has a center of symmetry. To bisect two symmetrical regions, produce a line on their centers of symmetry.

If a figure, and in this proof we are considering only closed curves, is not symmetrical, its bisectors form a closed curved of the bisectors as tangents. So if one region is symmetrical and the other not, we have two bisectors of both. If both are not symmetrical, we have four bisectors.

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Are we sure that a proposition that had been proved by transfinite methods can never be refuted by concrete numerical calculation? That's the mathematical problem of consistency. -- Friedrich Waismann

Even the above direct, non-amorphous analysis, is a bit amorphous in my mind. "All closed curves in the plane" is a set which could contain some surprises. I can correct this lack of knowledge by replacing the "two" and "four" above with "at least two" and "at least four." But even this not entirely satisfying vagary is an improvement on the method of amorphous infinites.

The question my above comparison of methods raises in my mind is: "How many of the 20th century's results, based on amorphous analysis, are either half-truths or truths-in-spite-of-themselves?"

The danger would be that a hierarchy of amorphous results would necessarily exaggerate the amorphous quality of the resultant picture. As in, each picture being more blurred than the last.

Set theory is wrong because it apparently presupposes a symbolism that doesn't exist instead of one that does exist (is alone possible). It builds on a fictitious symbolism, therefore on nonsense.

The question is not: "Are topology, set theory, group theory, et al. valid mathematics?" The question is: "When are these mathematics mis-applied?"

Certainly, the intuitive idea of a continuum has a psychological reality in the human mind. -- Richard Courant

Here again we get the same thing as in set theory: the form of expression we use seems to have been designed for a god, who knows what we cannot know; he sees the whole of each of those infinite series and he sees into human consciousness. For us, of course, these forms of expression are like pontificals which we put on, but cannot do much with, since we lack the effective power that would give these vestments meaning and purpose.

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More metaphors. Imagine an actual continuum. Let each number be marked with a 0.5mm pencil and leave at least a 0.2mm buffer around each mark. Then one million numbers would require at least a seven kilometer continuum.

Has mankind explicitly expressed one million distinct numbers yet? Will we ever need a 7000km continuum?

We concern ourselves with the existence of a number on the continuum. Can this mean anything beyond "this is a number"?

The present idea of the continuum is a more or less Platonic ideal.

The truth of the continuum is a transcendental ideal.

The falsity of the former could be replaced by the apodictical truths of the latter.

A line isn't composed of anything at all. It is a law.

Nothing is composed of points.

So the question would really be: Can the continuum be described? As Cantor and others tried to do. A form cannot be described; it can only be presented.

Some of our ideas about number, regarding the continuum, run counter to the form of number which is the law of the continuum. We say that

1.000...

cannot be distinguished from

0.999...

The latter is absolutely in the form of a 9 pushed away from the decimal point by an infinitude of nines. Changing our notation to place infinity in the middle, the numbers descending from 1.000... are:

1.000...000
0.999...999
0.999...998
0.999...997

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To claim that the first two cannot be distinguished is to assert that the second cannot be distinguished from the first and third, the third from the second and fourth, and so on.

If this is the case, we are left with little but the rationals.

But these numbers are all distinguishable elements of the continuum. Clearly, one can order as many numbers in such a notation as we can in the normal notation. The problem for us, with our inherent finite standpoint, is the infinite "middle" here which usually poses as an "end."

Let us note that the idea of infinity or "..." functions identically in either notation.

No matter how the rule [of creating number] is formulated, when I translate it into geometrical notation, everything is of the same type [rational number].

This is simply the limitation of our finite standpoint. If we consider all the numbers mankind can express, we have only an infinitesimal selection of the rationals.

We do not limit the form of number with our finite standpoint. The form of number imposes, apodictically, its full implication which includes the irrational number.

In no sense do we need to run away from these implications as the Greeks fled from the square root of two.

A real number lies in the substratum of operations out of which it is born.

The series of approximations of $\sqrt{2}$ **is** $\sqrt{2}$. But not the series. The law of the process.

An irrational number isn't the extension of an infinite decimal fraction [as series of approximation], it's a law.

Existence **is** a number produced by a law. We can produce $\sqrt{2}$ as easily and as "lawfully" as 2.

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Existence ≡ producible according to law.
Uniqueness ≡ demonstrating law as unambiguous.

That a law was shown to be ambiguous might indicate, not contradiction, but a need for more clarity or for the expansion of ideas.

The construction of a real number must be conceivable. The construction corresponds to the unity of the law.

Asking if the number exists, having already created a space for it to pre-exist in a falsely idealistic continuum, is questionable mathematics.

The question is not: "Does this number exist?" The question is: "Has this number been produced by the laws of number as we find them and presently understand them?"

If we later have a better understanding, we can throw out what is discovered to be false. But it is more likely that our increased understanding will include more of the form of number rather than less.

Given a number, you ask: Can the known laws produce this number? If not, you ask: Can the laws be extended to produce it? Is this mathematics justified?

A real number is what can be compared with the rationals.

Or -- Def. real number: *general method of comparison with the rationals.*

In no case do we produce a number according to law and then carry it over to the continuum where we verify its existence by discovering it already in the continuum.

In every case, we are able to conceptually position our new number in the continuum through comparison with the rationals. And we can do this because, in every case, the numbers we discover are within the focal distance of our finite standpoint.

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In our only possible comparison, *the rational number given is either equal to, less than, or greater than, the interval that has so far been worked out.*

This is all that can be true of irrational number. We can always distinguish an irrational from such an interval if we extend the interval sufficiently for our purposes. Which throws us back on comparing rationals to intervals in praxis.

Pre-Weierstrass, the syntax of mathematics contained a great many phrases in the nature of "as small as you please" or "as close as you please." I appreciate the accomplishment of Weierstrass. But neither he nor any other thinker can free us from the continuing applicability of these phrases.

Only a law approaches a value. ... The absence of a limit is not a limit.

If two real numbers are identical up to the number of digits expressed, they still can't be compared. So are they the same numbers? It's as if they share the same point on the continuum and we are waiting (experimentally?) for them to separate.

It must make sense to ask: Can this number be π? And the question must be answered in the affirmative for π to be located in the continuum.

And yet we can only locate a number accurately to the extent that our realization of the form of number allows.

Because the form of number is both true and infinite, there will always be more truth to realize. And a great deal must remain beyond our reach.

In mathematics, nothing can be inferred unless it can be seen.

The constant danger comes from our belief in the hegemony of the limited mind. We imagine we understand more than we have realized and can demonstrate.

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*Tolstoy: '**the meaning (importance) of something lies in its being something everyone can understand.**' That is both true and false. What makes the object hard to understand -- if it is ignificant, important -- is not that you have to be instructed in abstruse matters in order to understand it, but the antithesis between understanding the object and what most people **want** to see.*

The imagination lurks as the most powerful foe. It has an irresistible affinity for the absurd. Even cultured individuals are subject to this impulse to a high degree. -- Goethe

But this speaks only to the misuse of imagination. Imagination is one creative power behind our unfolding understanding of mathematics, our use of mathematics as representation, and of our ability to create the pictures of **our** world which attempt to capture the reality of the **only** world.

Paragraph three [of the Critique of Pure Reason] seems to harbor a major deficiency which makes itself felt in the whole development of that philosophy. Here the major faculties of the mind are listed as sensation, understanding, and reason. But the imagination is overlooked, causing an irreparable gap. Imagination is the fourth major faculty of our mental constitution. It supplements sensation in the form of memory. It submits a pattern of the world to the understanding, in the form of experience. It creates or finds sensuous shapes corresponding to the ideas of reason. Thus it gives life to the totality of the self, which would otherwise stagnate. --Goethe

To think of infinity as other than infinite expansion, or of mathematical "existence" as other than law, or the continuum as other than a transcendental ideal, is the false use of imagination.

*What will distinguish the mathematicians of the future from those of today will really be greater sensitivity, and **that** will -- as it were -- prune mathematics; since people will then be more intent on absolute clarity than on the discovery of new games.*

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Or: the future holds the possibility of clarity.

But complexity, for its own sake, seems to be a persistent ideal in our culture.

I believe that what is essential is for the activity of clarification to be carried out with COURAGE; without this it becomes merely a clever game.

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