Little Mathematics Library

P. P. KOROVKIN

INEQUALITIES

Mir Publishers · Moscow

ИЗДАТЕЛЬСТВО · МИР ·

ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

П. П. Коровкин

НЕРАВЕНСТВА

ИЗДАТЕЛЬСТВО «НАУКА»

LITTLE MATHEMATICS LIBRARY

P. P. Korovkin

INEQUALITIES

Translated from the Russian
by
Sergei Vrubel

MIR PUBLISHERS
MOSCOW

First Published 1975

На английском языке

CONTENTS

Preface 6

Chapter 1. Inequalities 7

1.1. The Whole Part of a Number 7
1.2. The Arithmetic Mean and the Geometric Mean 12
1.3. The Number 19
1.4. The Bernoulli Inequality 23
1.5. The Mean Power of Numbers 27

Chapter 2. Uses of Inequalities 32

2.1. The Greatest and the Least Function Values 32
2.2. The Hölder Inequality 40
2.3. The Use of Inequalities for Calculation of Limits 43
2.4. The Use of Inequalities for Approximate Calculation of Quantities 49

Solutions to Exercises 58

PREFACE

In the mathematics course of secondary schools students get acquainted with the properties of inequalities and methods of their solution in elementary cases (inequalities of the first and the second degree).

In this booklet the author did not pursue the aim of presenting the basic properties of inequalities and made an attempt only to familiarize students of senior classes with some particularly remarkable inequalities playing an important role in various sections of higher mathematics and with their use for finding the greatest and the least values of quantities and for calculating some limits.

The book contains 63 problems, 35 of which are provided with detailed solutions, composing thus its main subject, and 28 others are given in Sections 1.1 and 2.1, 2.3, 2.4 as exercises for individual training. At the end of the book the reader will find the solutions to the given exercises.

The solution of some difficult problems carried out individually will undoubtedly do the reader more good than the solution of a large number of simple ones.

For this reason we strongly recommend the readers to perform their own solutions before referring to the solutions given by the author at the end of the book. However, one should not be disappointed if the obtained results differ from those of the patterns. The author considers it as a positive factor.

When proving the inequalities and solving the given problems, the author has used only the properties of inequalities and limits actually covered by the curriculum on mathematics in the secondary school.

P. Korovkin.

CHAPTER 1

Inequalities

The important role of inequalities is determined by their application in different fields of natural science and engineering. The point is that the values of quantities defined from various practical problems (e.g. the distance to the Moon, its speed of rotation, etc.) may be found not exactly, but only approximately. If x is the found value of a quantity, and Δx is an error of its measurement, then the real value y satisfies the inequalities

$$x - |\Delta x| \leqslant y \leqslant x + |\Delta x|.$$

When solving practical problems, it is necessary to take into account all the errors of the measurements. Moreover, in accordance with the technical progress and the degree of complexity of the problem, it becomes necessary to improve the technique of measurement of quantities. Considerable errors of measurement become inadmissible in solving complicated engineering problems (i.e., landing the moon-car in a specified region of the Moon, landing spaceships on the Venus and so on).

1.1. The Whole Part of a Number

The whole (or integral) part of the number x (denoted by $[x]$) is understood to be the greatest integer not exceeding x. It follows from this definition that $[x] \leqslant x$, since the integral part does not exceed x. On the other hand, since $[x]$ is the greatest integer, satisfying the latter inequality, then $[x] + 1 > x$.

Thus, $[x]$ is the integer (whole number) defined by the inequalities

$$[x] \leqslant x < [x] + 1.$$

For example, from the inequalities

$$3<\pi<4,\ 5<\frac{17}{3}<6,\ -2<-\sqrt{2}<-1,\ 5=5<6$$

it follows that

$$[\pi]=3,\ \left[\frac{17}{3}\right]=5,\ [-\sqrt{2}]=-2,\ [5]=5.$$

The ability to find the integral part of a quantity is an important factor in approximate calculations. If we have the skill to find an integral part of a quantity x, then taking $[x]$ or $[x]+1$ for an approximate value of the quantity x, we shall make an error whose quantity is not greater than 1, since

$$0 \leqslant x-[x]<[x]+1-[x]=1,$$
$$0<[x]+1-x \leqslant [x]+1-[x]=1.$$

Furthermore, the knowledge of the integral part of a quantity permits to find its value with an accuracy up to $\frac{1}{2}$. The quantity $[x]+\frac{1}{2}$ may be taken for this value.

Yet, it is important to note, that the ability to find the whole part of a number will permit to define this number and, with any degree of accuracy. Indeed, since

$$[Nx] \leqslant Nx \leqslant [Nx]+1,$$

then

$$\frac{[Nx]}{N} \leqslant x \leqslant \frac{[Nx]}{N}+\frac{1}{N}.$$

Thus, the number

$$\frac{[Nx]}{N}+\frac{1}{2N}$$

differs from the number x not more than by $\frac{1}{2N}$. With large N the error will be small. The integral part of a number is found in the following problems.

Problem 1. Find the integral part of the number

$$x=1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}.$$

Solution. Let us use the following inequalities

$$1 \leqslant 1 \leqslant 1,$$
$$0.7 < \sqrt{\frac{1}{2}} < 0.8,$$
$$0.5 < \sqrt{\frac{1}{3}} < 0.6,$$
$$0.5 \leqslant \sqrt{\frac{1}{4}} \leqslant 0.5,$$
$$0.4 < \sqrt{\frac{1}{5}} < 0.5$$

(which are obtained by extracting roots (evolution) with an accuracy to 0.1 in excess or deficiency). Combining them we get

$$1 + 0.7 + 0.5 + 0.5 + 0.4 < x < 1 + 0.8 + 0.6 + 0.5 + 0.5,$$

that is, $3.1 < x < 3.4$, hence, $[x] = 3$.

In this relation, it is necessary to note that the number 3.25 differs from x not more than by 0.15.

Problem 2. Find the integral part of the number

$$y = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{1\,000\,000}}.$$

Solution. This problem differs from the previous one only by the number of addends (in the first, there were only 5 addends, while in the second, 1 000, 000 addends). This circumstance makes it practically impossible to get the solution by the former method.

To solve this problem, let us investigate the sum

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{n}}$$

and prove that

$$2\sqrt{n+1} - 2\sqrt{n} < \frac{1}{\sqrt{n}} < 2\sqrt{n} - 2\sqrt{n-1}. \qquad (1)$$

Indeed, since

$$2\sqrt{n+1} - 2\sqrt{n} = \frac{2(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{2}{\sqrt{n+1} + \sqrt{n}}$$

and

$$\sqrt{n+1} > \sqrt{n},$$

it follows that

$$2\sqrt{n+1} - 2\sqrt{n} < \frac{2}{2\sqrt{n}} = \frac{1}{\sqrt{n}}.$$

Thereby proof has been made for the first part of the inequality (1); its second part is proved in a similar way.

Assuming in the inequalities (1) $n = 2, 3, 4, \ldots, n$, we get

$$2\sqrt{3} - 2\sqrt{2} < \frac{1}{\sqrt{2}} < 2\sqrt{2} - 2,$$

$$2\sqrt{4} - 2\sqrt{3} < \frac{1}{\sqrt{3}} < 2\sqrt{3} - 2\sqrt{2},$$

$$2\sqrt{5} - 2\sqrt{4} < \frac{1}{\sqrt{4}} < 2\sqrt{4} - 2\sqrt{3},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$2\sqrt{n+1} - 2\sqrt{n} < \frac{1}{\sqrt{n}} < 2\sqrt{n} - 2\sqrt{n-1}.$$

Adding these inequalities, we get

$$2\sqrt{n+1} - 2\sqrt{2} < \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{n}} < 2\sqrt{n} - 2.$$

Adding 1 to all parts of the obtained inequalities, we find

$$2\sqrt{n+1} - 2\sqrt{2} + 1 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{n}} < 2\sqrt{n} - 1. \quad (2)$$

Since $2\sqrt{2} < 3$, and $\sqrt{n+1} > \sqrt{n}$, it follows from the inequalities (2) that

$$2\sqrt{n} - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{n}} < 2\sqrt{n} - 1. \quad (3)$$

Using the inequalities (3) we can easily find the integral part of the number

$$y = 1 + \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{1\,000\,000}}.$$

Thus, taking in the inequalities (3) $n = 1\,000\,000$, we get $2\sqrt{1\,000\,000} - 2 <$

$$< 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{1\,000\,000}} < 2\sqrt{1\,000\,000} - 1,$$

or

$$1998 < y < 1999.$$

Hence, $[y] = 1998$.

From the inequalities (2) it follows that the number 1998.6 differs from y not more than by 0.4. Thus, we have calculated the number y with an accuracy up to $\frac{40}{1998.4}\% = 0.02\%$. The numbers 1998 and 1999 differ from the number y not more than by unity, and the number 1998.5 differs not more than by 0.5.

Now let us examine the next problem of somewhat different pattern.

Problem 3. Prove the inequality

$$x = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{99}{100} < \frac{1}{10}.$$

Solution. Suppose

$$y = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{100}{101}.$$

Since

$$\frac{1}{2} < \frac{2}{3}, \quad \frac{3}{4} < \frac{4}{5}, \quad \frac{5}{6} < \frac{6}{7}, \ldots, \frac{99}{100} < \frac{100}{101},$$

it follows that $x < y$ and, consequently,

$$x^2 < xy = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdot \frac{6}{7} \cdots \frac{99}{100} \cdot \frac{100}{101} = \frac{1}{101}.$$

Finding the square root of both members of the inequalities yields

$$x < \frac{1}{\sqrt{101}} < 0.1.$$

Exercises

1. Prove the inequalities

$$2\sqrt{n+1}-2\sqrt{m}<\frac{1}{\sqrt{m}}+\frac{1}{\sqrt{m+1}}+\dots$$
$$\dots+\frac{1}{\sqrt{n}}<2\sqrt{n}-2\sqrt{m-1}.$$

2. Prove the inequalities

$$1{,}800<\frac{1}{\sqrt{10{,}000}}+\frac{1}{\sqrt{10{,}001}}+\dots+\frac{1}{\sqrt{1{,}000{,}000}}<$$
$$<1{,}800.02.$$

3. Find $[50z]$, where

$$z=\frac{1}{\sqrt{10{,}000}}+\frac{1}{\sqrt{10{,}001}}+\dots+\frac{1}{\sqrt{1{,}000{,}000}}.$$

Answer. $[50z] = 90{,}000$.

4. Prove the following inequality using the method of mathematical induction

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\dots\frac{2n-1}{2n}\leqslant\frac{1}{\sqrt{3n+1}}.$$

5. Prove the inequality

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\dots\frac{99}{100}<\frac{1}{12}.$$

1.2. The Arithmetic Mean and the Geometric Mean

If $x_1, x_2, \dots, x_n$ are positive numbers, then the numbers formed with them

$$a=\frac{x_1+x_2+\dots+x_n}{n},$$
$$g=\sqrt[n]{x_1x_2\dots x_n}$$

are called, respectively, *the arithmetic mean* and *the geometric mean* of the numbers $x_1, x_2, \dots, x_n$. At the beginning of the last century, the French mathematician O. Cauchy has established for these numbers the inequality

$$g\leqslant a,$$

often used in solving problems. Before proving the inequality we have to establish the validity of an auxiliary assertion

Theorem 1. *If the product n of the positive numbers* x_1, x_2, . . ., x_n *is equal to* 1, *then the sum of these numbers is not less than n*:

$$x_1x_2, \ldots, x_n = 1 \Rightarrow x_1 + x_2 + \ldots + x_n \geqslant n.$$

Proof. Use the method of mathematical induction[1]. First of all check up the validity of the theorem for $n = 2$, i.e. show that

$$x_1x_2 = 1 \Rightarrow x_1 + x_2 \geqslant 2.$$

Solving the question, examine the two given cases separately:

(1) $$x_1 = x_2 = 1.$$

In this case $x_1 + x_2 = 2$, and the theorem is proved.

(2) $$0 < x_1 < x_2.$$

Here $x_1 < 1$, and $x_2 > 1$, since their product is equal to 1. From the equation

$$(1 - x_1)(x_2 - 1) = x_2 + x_1 - x_1x_2 - 1$$

it follows that

$$x_1 + x_2 = x_1x_2 + 1 + (1 - x_1)(x_2 - 1). \qquad (4)$$

The equation (4) has been established without limitations to the numbers x_1 and x_2. Yet, taking into account, that $x_1x_2 = 1$, we get

$$x_1 + x_2 = 2 + (1 - x_1)(x_2 - 1).$$

At length, since $x_1 < 1 < x_2$, then the last number is positive and $x_1 + x_2 > 2$. Thus, for $n = 2$ the theorem is already proved. Notice, that the equation

$$x_1 + x_2 = 2$$

is realized only when $x_1 = x_2$. But if $x_1 \neq x_2$, then

$$x_1 + x_2 > 2.$$

Now, making use of the method of mathematical induction, assume that the theorem is true for $n = k$, that is, sup-

---

[1] More detailed information concerning mathematical induction is published in the book by I. S. Sominsky "The Method of Mathematical Induction", Nauka, Moscow, 1974.

pose the inequality

$$x_1 + x_2 + x_3 + \ldots + x_k \geqslant k$$

occurs, if $x_1x_2x_3 \ldots x_k = 1$, and prove the theorem for $n = k + 1$, i.e. prove that

$$x_1 + x_2 + x_3 + \ldots + x_k + x_{k+1} \geqslant k + 1,$$

if $x_1x_2x_3 \ldots x_kx_{k+1} = 1$, for $x_1 > 0,\ x_2 > 0,\ x_3 > 0, \ldots,$

$$x_k > 0,\ x_{k+1} > 0.$$

First of all, it is necessary to notice that if

$$x_1x_2x_3 \ldots x_kx_{k+1} = 1,$$

then there may be two cases:

(1) when all the multipliers $x_1, x_2, x_3, \ldots, x_k, x_{k+1}$ are equal, that is

$$x_1 = x_2 = x_3 = \ldots = x_k = x_{k+1},$$

(2) when not all multipliers are equal.

In the first case every multiplier is equal to unity, and their sum equals $k + 1$, that is

$$x_1 + x_2 + x_3 + \ldots + x_k + x_{k+1} = k + 1.$$

In the second case, among the multipliers of the product $x_1x_2 \ldots x_kx_{k+1}$, there may be both numbers greater than unity and numbers less than unity (if all the multipliers were less than unity, then their product as well would be less than unity).

For example, suppose $x_1 < 1$, and $x_{k+1} > 1$. We have

$$(x_1x_{k+1})\, x_2x_3 \ldots x_k = 1.$$

Assuming $y_1 = x_1x_{k+1}$, we get

$$y_1x_2x_3 \ldots x_k = 1.$$

Since here the product k of positive numbers is equal to unity, then (according to the assumption) their sum is not less than k, that is

$$y_1 + x_2 + x_3 + \ldots + x_k \geqslant k.$$

But

$$x_1 + x_2 + x_3 + \ldots + x_k + x_{k+1} =$$
$$= (y_1 + x_2 + x_3 + \ldots + x_k) + x_{k+1} - y_1 + x_1 \geqslant$$
$$\geqslant k + x_{k+1} - y_1 + x_1 = (k+1) + x_{k+1} - y_1 + x_1 - 1.$$

Remembering that $y_1 = x_1 x_{k+1}$ we get

$$x_1 + x_2 + x_3 + \ldots + x_k + x_{k+1} \geqslant$$
$$\geqslant (k+1) + x_{k+1} - x_1 x_{k+1} + x_1 - 1 =$$
$$= (k+1) + (x_{k+1} - 1)(1 - x_1).$$

Since $x_1 < 1$, and $x_{k+1} > 1$, then $(x_{k+1} - 1)(1 - x_1) > 0$ and, hence,

$$x_1 + x_2 + x_3 + \ldots + x_k + x_{k+1} \geqslant$$
$$\geqslant (k+1) + (x_{k+1} - 1)(1 - x_1) > k + 1.$$

Thus the theorem is proved.

Problem 1. Prove, that if $x_1, x_2, x_3, \ldots, x_n$ are positive numbers then

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \ldots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \geqslant n,$$

the equality being valid only when

$$x_1 = x_2 = x_3 = \ldots = x_n.$$

Solution. Since

$$\frac{x_1}{x_2} \cdot \frac{x_2}{x_3} \ldots \frac{x_{n-1}}{x_n} \cdot \frac{x_n}{x_1} = 1,$$

then the inequality follows from Theorem 1, the sign of equality holds only when

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \ldots = \frac{x_{n-1}}{x_n} = \frac{x_n}{x_1} = 1,$$

namely, when $x_1 = x_2 = x_3 = \ldots = x_n$.

Problem 2. Prove the inequality

$$\frac{x^2+2}{\sqrt{x^2+1}} \geqslant 2.$$

Solution. We have

$$\frac{x^2+2}{\sqrt{x^2+1}} = \frac{x^2+1}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}} = \sqrt{x^2+1} + \frac{1}{\sqrt{x^2+1}}.$$

Since the product of addends in the right-hand member of the equality equals unity, then their sum is not less than 2. The sign of equality holds only for $x = 0$.

Problem 3. Prove that for $a > 1$

$$\log a + \log_a 10 \geqslant 2.$$

Solution. Since $\log_a 10 \cdot \log a = 1$, then

$$\log a + \log_a 10 = \log a + \frac{1}{\log a} \geqslant 2.$$

Problem 4. Prove the inequality

$$\frac{x^2}{1+x^4} \leqslant \frac{1}{2}.$$

Solution. Divide by x^2 the numerator and denominator of the left-hand member of the inequality:

$$\frac{x^2}{1+x^4} = \frac{1}{\frac{1}{x^2}+x^2}.$$

Since $\frac{1}{x^2} \cdot x^2 = 1$, then $\frac{1}{x^2} + x^2 \geqslant 2$ and, hence,

$$\frac{1}{\frac{1}{x^2}+x^2} \leqslant \frac{1}{2}.$$

Now let us prove the statement made at the beginning of the section.

Theorem 2. *The geometric mean of positive numbers is not greater than the arithmetic mean of the same numbers.*

If the numbers $x_1, x_3, \ldots, x_n$ are not all equal, then the geometric mean of these numbers is less than their arithmetic mean.

Proof. From the equality $g = \sqrt[n]{x_1 x_2 \ldots x_n}$ it follows that

$$1 = \sqrt[n]{\frac{x_1}{g} \frac{x_2}{g} \ldots \frac{x_n}{g}}, \text{ or } \frac{x_1}{g} \frac{x_2}{g} \ldots \frac{x_n}{g} = 1.$$

Since the product n of the positive numbers equals 1, then (Theorem 1) their sum is not less than n, that is

$$\frac{x_1}{g} + \frac{x_2}{g} + \ldots + \frac{x_n}{g} \geqslant n.$$

Multiplying both members of the last inequality by g and dividing by n, we get

$$a = \frac{x_1 + x_2 + \ldots + x_n}{n} \geqslant g.$$

Notice, that the equality holds only when $\frac{x_1}{g}=\frac{x_2}{g}=\ldots$ $\ldots=\frac{x_n}{g}=1$, that is $x_1=x_2=\ldots=x_n=g$. But if the numbers $x_1, x_2, \ldots, x_n$ are not equal, then

$$a > g.$$

Problem 5. From all parallelepipeds with the given sum of the three mutually perpendicular edges, find the parallelepiped having the greatest volume.

Solution. Suppose $m=a+b+c$ is the sum of the edges and $V=abc$ is the volume of the parallelepiped. Since

$$\sqrt[3]{V}=\sqrt[3]{abc}\leqslant\frac{a+b+c}{3}=\frac{m}{3},$$

then $V\leqslant\frac{m^3}{27}$. The sign of equality holds only when $a=$ $=b=c=\frac{m}{3}$, that is, when the parallelepiped is a cube.

Problem 6. Prove the inequality

$$n!<\left(\frac{n+1}{2}\right)^n, \quad n\geqslant 2. \tag{5}$$

Solution. Using Theorem 2, we get

$$\sqrt[n]{n!}=\sqrt[n]{1\cdot 2\cdot 3\ldots n}<\frac{1+2+3+\ldots+n}{n}=$$
$$=\frac{(n+1)\,n}{2n}=\frac{n+1}{2}.$$

Raising to the nth power both parts of the last inequality, we get the inequality (5).

Definition. The number

$$c_\alpha=\left(\frac{a_1^\alpha+a_2^\alpha+\ldots+a_n^\alpha}{n}\right)^{\frac{1}{\alpha}}$$

is termed *the mean power* of numbers $a_1, a_2, \ldots, a_n$ of the order α. Particularly, the number

$$c_1=\frac{a_1+a_2+\ldots+a_n}{n}$$

is the arithmetic mean of the numbers $a_1, a_2, \ldots, a_n$, the number

$$c_2=\left(\frac{a_1^2+a_2^2+\ldots+a_n^2}{n}\right)^{\frac{1}{2}}$$

is named *the root-mean-square*, and the number

$$c_{-1}=\left(\frac{a_1^{-1}+a_2^{-1}+\ldots+a_n^{-1}}{n}\right)^{-1}=\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}}$$

is called *the harmonic mean* of the numbers $a_1, a_2, \ldots, a_n$.

Problem 7. Prove that if $a_1, a_2, \ldots, a_n$ are positive numbers and $\alpha < 0 < \beta$, then

$$c_\alpha \leqslant g \leqslant c_\beta, \tag{6}$$

that is, the mean power with a negative exponent does not exceed the geometric mean, and the mean power with a positive exponent is not less than the geometric mean.

Solution. From the fact, that the geometric mean of positive numbers does not exceed the arithmetic mean, we have

$$\sqrt[n]{a_1^\alpha a_2^\alpha \ldots a_n^\alpha} \leqslant \frac{a_1^\alpha+a_2^\alpha+\ldots+a_n^\alpha}{n}.$$

Raising both parts of the last inequality to a power $\frac{1}{\alpha}$ and taking into consideration, that $\frac{1}{\alpha} < 0$, we get

$$g=\sqrt[n]{a_1a_2\ldots a_n} \geqslant \left(\frac{a_1^\alpha+a_2^\alpha+\ldots+a_n^\alpha}{n}\right)^{\frac{1}{\alpha}}=c_\alpha.$$

So the first part of the inequality (6) is proved; the second is proved in a similar way.

From the inequality (6) it follows, in particular, that the harmonic mean c_{-1} does not exceed the arithmetic mean c_1.

Problem 8. Prove that if $a_1, a_2, \ldots, a_n$ are positive numbers, then

$$(a_1+a_2+\ldots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right) \geqslant n^2.$$

Solution. Since $c_{-1} \leqslant g \leqslant c_1$, then

$$c_{-1}=\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}} \leqslant \frac{a_1+a_2+\ldots+a_n}{n}=c_1.$$

It follows from this inequality that

$$n^2 \leqslant (a_1+a_2+\ldots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right).$$

Problem 9. Prove the inequality

$$na_1a_2 \ldots a_n \leqslant a_1^n + a_2^n + \ldots + a_n^n, \qquad (7)$$

where $a_1 > 0$, $a_2 > 0$, ..., $a_n > 0$.

Solution. Since the geometric mean does not exceed the arithmetic mean, then

$$a_1a_2 \ldots a_n = \sqrt[n]{a_1^n a_2^n \ldots a_n^n} \leqslant \frac{a_1^n + a_2^n + \ldots + a_n^n}{n}.$$

Multiplying both members of this inequality by n, we shall get the inequality (7).

From the inequality (7) it follows, that

$$2a_1a_2 \leqslant a_1^2 + a_2^2, \quad 3a_1a_2a_3 \leqslant a_1^3 + a_2^3 + a_3^3,$$
$$4a_1a_2a_3a_4 \leqslant a_1^4 + a_2^4 + a_3^4 + a_4^4,$$

that is, *the doubled product of two positive numbers does not exceed the sum of their squares, the trebled product of three numbers does not exceed the sum of their cubes and so on.*

1.3. The Number e

The number e plays an important role in mathematics. We shall come to its determination after carrying out the solution of a number of problems in which only Theorem 2 is used.

Problem 1. Prove that for any positive numbers a, b, $(a \neq b)$ the inequality

$$\sqrt[n+1]{ab^n} < \frac{a + nb}{n+1}$$

is true.

Solution. We have

$$\sqrt[n+1]{ab^n} = \sqrt[n+1]{a\underbrace{bb \ldots b}_{n}} < \frac{a + \overbrace{b + b + \ldots + b}^{n}}{n+1} = \frac{a + nb}{n+1},$$

and that suits the requirement.

Problem 2. Prove that with the increase of the number n the quantities

$$x_n = \left(1 + \frac{1}{n}\right)^n \text{ and } z_n = \left(1 - \frac{1}{n}\right)^n$$

increase, i.e.

$$x_n < x_{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+1},$$

$$z_n < z_{n+1} = \left(1 - \frac{1}{n+1}\right)^{n+1}.$$

Solution. Setting in the inequality of the previous problem $a = 1$, $b = 1 + \frac{1}{n}$, we get

$$\sqrt[n+1]{1 \cdot \left(1 + \frac{1}{n}\right)^n} < \frac{1 + n\left(1 + \frac{1}{n}\right)}{n+1} = \frac{n+2}{n+1} = 1 + \frac{1}{n+1}.$$

Raising both parts of the inequality to the $(n+1)$th power, we shall obtain

$$\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}, \text{ that is } x_n < x_{n+1}.$$

The second inequality is proved in a similar way.

Problem 3. Prove that

$$y_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

decreases with the increase of the number n, that is

$$y_n > y_{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+2}.$$

Solution. We have

$$y_n = \left(1 + \frac{1}{n}\right)^{n+1} = \left(\frac{n+1}{n}\right)^{n+1} = \frac{1}{\left(\frac{n}{n+1}\right)^{n+1}} = \frac{1}{\left(1 - \frac{1}{n+1}\right)^{n+1}} = \frac{1}{z_{n+1}}$$

(see designations of Problem 2). Since z_n increases with the increase of the number n, then y_n decreases.

In Problems 2 and 3 we have proved that

$$x_1 = \left(1 + \frac{1}{1}\right)^1 = 2 < x_2 = \left(1 + \frac{1}{2}\right)^2 = 2.25 < x_3 < \ldots < x_n < \ldots,$$

$$y_1 = \left(1 + \frac{1}{2}\right)^2 = 4 > y_2 = \left(1 + \frac{1}{2}\right)^3 = 3.375 > y_3 > \ldots > y_n > \ldots.$$

On the other hand,

$$2 = x_1 < x_n = \left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n}\right)^{n+1} = y_n < y_1 = 4.$$

Thus, the variable x_n satisfies two conditions:

(1) x_n monotonically increases together with the increase of the number n;

(2) x_n is a limited quantity, $2 < x_n < 4$.

It is known, that monotonically increasing and restricted variable has a limit. Hence, there exists a limit of the variable quantity x_n. This limit is marked by the letter e, that is,

$$e = \lim_{n \to \infty} x_n = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$

As the quantity x_n increases reaching its limit, then x_n is smaller than its limit, that is

$$x_n = \left(1 + \frac{1}{n}\right)^n < e. \tag{8}$$

It is not difficult to check that $e < 3$. Indeed, if the number n is high, then

$$x_n < y_n < y_5 = \left(1 + \frac{1}{5}\right)^6 = 2.985984.$$

Hence,

$$e = \lim_{n \to \infty} x_n \leqslant 2.985984 < 3.$$

In mathematics, the number e together with the number π is of great significance. It is used, for instance, as the base of logarithms, known as *natural logarithms*. The logarithm of the number N at the base e is symbolically denoted by $\ln N$ (reads: logarithm natural N).

It is common knowledge that the numbers e and π are irrational. Each of them is calculated with an accuracy of up to 808 signs after the decimal point, and

$$e = 2.7182818285490 \ldots .$$

Now, let us show that the limit of the variable y_n also equals e. Indeed,

$$\lim y_n = \lim \left(1 + \frac{1}{n}\right)^{n+1} = \lim \left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right) =$$
$$= e \cdot 1 = e.$$

Since y_n diminishes coming close to the number e (Problem 2), then

$$\left(1+\frac{1}{n}\right)^{n+1} > e. \qquad (9)$$

Problem 4. Prove the inequality

$$n! > \left(\frac{n}{e}\right)^{n}. \qquad (10)$$

Solution. We shall prove the inequality (10) using the method of mathematical induction. The inequality is easily checked for $n = 1$. Actually,

$$1! = 1 > \left(\frac{1}{e}\right)^{1}.$$

Assume, that the inequality (10) is true for $n = k$, that is

$$k! > \left(\frac{k}{e}\right)^{k}.$$

Multiplying both members of the last inequality by $k+1$, we get

$$(k+1)\,k! = (k+1)! > \left(\frac{k}{e}\right)^{k}(k+1) = \left(\frac{k+1}{e}\right)^{k+1} \frac{e}{\left(1+\frac{1}{k}\right)^{k}}.$$

Since, according to the inequality (8) $\left(1+\frac{1}{k}\right)^{k} < e$, then

$$(k+1)! > \left(\frac{k+1}{e}\right)^{k+1} \frac{e}{e} = \left(\frac{k+1}{e}\right)^{k+1},$$

that is the inequality (9) is proved for $n = k + 1$. Thus the inequality (9) is proved to be true for all values of n.

Since $e < 3$, it follows from the inequality (9) that

$$n! > \left(\frac{n}{3}\right)^{n}.$$

By means of the last inequality, it is easy to prove that

$$300! > 100^{300}.$$

Indeed, setting in it $n = 300$, we get

$$300! > \left(\frac{300}{3}\right)^{300} = 100^{300}.$$

The inequality

$$n! < e\left(\frac{n+1}{e}\right)^{n+1}$$

is proved completely the same way as it is done with the inequality of Problem 4.

1.4. The Bernoulli Inequality

In this section, making use of Theorem 2 we shall prove the Bernoulli inequality which is of individual interest and is often used in solving problems.

Theorem 3. *If* $x \geqslant -1$ *and* $0 < \alpha < 1$, *then*

$$(1+x)^{\alpha} \leqslant 1 + \alpha x. \tag{11}$$

However if $\alpha < 0$ *or* $\alpha > 1$, *then*

$$(1+x)^{\alpha} \geqslant 1 + \alpha x. \tag{12}$$

The sign of equality in (11) *and* (12) *holds only when* $x = 0$.

Proof. Suppose that α is a rational number, bearing in mind that $0 < \alpha < 1$. Let $\alpha = \frac{m}{n}$, where m and n are positive integers, $1 \leqslant m < n$. Since according to the condition, $1 + x \geqslant 0$, then

$$(1+x)^{\alpha} = (1+x)^{\frac{m}{n}} = \sqrt[n]{(1+x)^m \cdot 1^{n-m}} =$$

$$= \sqrt[n]{\underbrace{(1+x)(1+x)\ldots(1+x)}_{m}\cdot\underbrace{1\cdot 1\ldots 1}_{n-m}} \leqslant$$

$$\leqslant \frac{(1+x)+(1+x)+\ldots+(1+x)+1+1+\ldots+1}{n} =$$

$$= \frac{m(1+x)+n-m}{n} = \frac{n+mx}{n} = 1 + \frac{m}{n}x = 1 + \alpha x.$$

The sign of equality occurs only when all multipliers standing under the root sign are identical, i.e., when $1 + x = 1$, $x = 0$. But if $x \neq 0$, then

$$(1+x)^{\alpha} < 1 + \alpha x.$$

Thus, we have proved the first part of the theorem considering the case, when α is a rational number.

Assume now, that α is an irrational number, $0 < \alpha < 1$. Let $r_1, r_2, \ldots, r_n \ldots$ be the sequence of rational numbers, having for a limit the number α. Bear in mind that $0 < r_n < 1$. From the inequalities

$$(1 + x)^{r_n} \leqslant 1 + r_n x, \quad x \geqslant -1, \quad n = 1, 2, 3, \ldots,$$

already proved by us for the case when the exponent is a rational number, it follows that

$$(1+x)^{\alpha} = \lim_{r_n \to \alpha} (1+x)^{r_n} \leqslant \lim_{r_n \to \alpha} (1 + r_n x) = 1 + \alpha x.$$

Thus the inequality (11) is proved for irrational values of α as well. What we still have to prove is that for irrational values of α when $x \neq 0$ and $0 < \alpha < 1$

$$(1 + x)^{\alpha} < 1 + \alpha x,$$

i.e., that when $x \neq 0$ in (11), the sign of equality does not hold. For this reason, take a rational number r such that $\alpha < r < 1$. Obviously, we have

$$(1 + x)^{\alpha} = [(1 + x)^{\frac{\alpha}{r}}]^{r}.$$

Since $0 < \frac{\alpha}{r} < 1$, then as it has already been proved

$$(1 + x)^{\frac{\alpha}{r}} \leqslant 1 + \frac{\alpha}{r} x.$$

Hence,

$$(1 + x)^{\alpha} \leqslant \left(1 + \frac{\alpha}{r} x\right)^{r}.$$

If $x \neq 0$, then $\left(1 + \frac{\alpha}{r} x\right)^{r} < 1 + r \frac{\alpha}{r} x = 1 + \alpha x$, that is

$$(1 + x)^{\alpha} < 1 + \alpha x.$$

Thus the first part of the theorem is proved completely.

Now, move on to proving the second part of the theorem.

If $1 + \alpha x < 0$, then the inequality (12) is obvious, since its left part is not negative, and its right part is negative.

If $1 + \alpha x \geqslant 0$, $\alpha x \geqslant -1$, then let us consider both cases separately.

Suppose $\alpha > 1$; then by virtue of the first part of the theorem proved above we have

$$(1 + \alpha x)^{\frac{1}{\alpha}} \leqslant 1 + \frac{1}{\alpha} \alpha x = 1 + x.$$

Here the sign of equality holds only when $x = 0$. Raising both parts of the last inequality to the power α we get

$$1 + \alpha x \leqslant (1 + x)^{\alpha}.$$

Now let us suppose $\alpha < 0$. If $1 + \alpha x < 0$, then the inequality (12) is obvious. But if $1 + \alpha x \geqslant 0$, then select the positive integer n, so that the inequality $-\frac{\alpha}{n} < 1$ would be valid. By virtue of the first part of the theorem we get

$$(1+x)^{-\frac{\alpha}{n}} \leqslant 1 - \frac{\alpha}{n} x,$$

$$(1+x)^{\frac{\alpha}{n}} \geqslant \frac{1}{1 - \frac{\alpha}{n} x} \geqslant 1 + \frac{\alpha}{n} x$$

$\left(\text{the latter inequality is true, since } 1 \geqslant 1 - \frac{\alpha^2}{n^2} x^2\right)$. Raising both parts of the latter inequality to the nth power we get

$$(1+x)^{\alpha} \geqslant \left(1 + \frac{\alpha}{n} x\right)^{n} \geqslant 1 + n \frac{\alpha}{n} x = 1 + \alpha x.$$

Notice, that the equality is possible only when $x = 0$. Thus, the theorem is proved completely.

Problem 1. Prove, that if $0 > \alpha > -1$, then

$$\frac{(n+1)^{\alpha+1} - n^{\alpha+1}}{\alpha+1} < n^{\alpha} < \frac{n^{\alpha+1} - (n-1)^{\alpha+1}}{\alpha+1}. \qquad (13)$$

Solution. Since $0 < \alpha + 1 < 1$, then according to the inequality (11) we have

$$\left(1 + \frac{1}{n}\right)^{\alpha+1} < 1 + \frac{\alpha+1}{n},$$

$$\left(1 - \frac{1}{n}\right)^{\alpha+1} < 1 - \frac{\alpha+1}{n}.$$

Multiplying these inequalities by $n^{\alpha+1}$, we obtain

$$(n+1)^{\alpha+1} < n^{\alpha+1} + (\alpha+1) n^{\alpha},$$

$$(n-1)^{\alpha+1} < n^{\alpha+1} - (\alpha+1) n^{\alpha}.$$

The inequalities (13) easily follow from these inequalities.

Problem 2. Prove, that if $0 > \alpha > -1$, then

$$\frac{(n+1)^{\alpha+1}-m^{\alpha+1}}{\alpha+1} <$$

$$< m^{\alpha}+(m+1)^{\alpha}+\ldots+n^{\alpha} < \frac{n^{\alpha+1}-(m-1)^{\alpha+1}}{\alpha+1}. \qquad (14)$$

Solution. Setting in the inequalities (13) $n=m$, $m+1, \ldots, n$, we get

$$\frac{(m+1)^{1+\alpha}-m^{1+\alpha}}{1+\alpha} < m^{\alpha} < \frac{m^{1+\alpha}-(m-1)^{1+\alpha}}{1+\alpha},$$

$$\frac{(m+2)^{1+\alpha}-(m+1)^{1+\alpha}}{1+\alpha} < (m-1)^{\alpha} < \frac{(m+1)^{1+\alpha}-m^{1+\alpha}}{1+\alpha},$$

$$\frac{(m+3)^{1+\alpha}-(m+2)^{1+\alpha}}{1+\alpha} < (m+2)^{\alpha} < \frac{(m+2)^{1+\alpha}-(m+1)^{1+\alpha}}{1+\alpha},$$

. . . .

$$\frac{(n+1)^{1+\alpha}-n^{1+\alpha}}{1+\alpha} < n^{\alpha} < \frac{n^{1+\alpha}-(n-1)^{1+\alpha}}{1+\alpha}.$$

Adding these inequalities we shall get the inequality (14).

Problem 3. Find the integral part of the number

$$x=\frac{1}{\sqrt[3]{4}}+\frac{1}{\sqrt[3]{5}}+\frac{1}{\sqrt[3]{6}}+\ldots+\frac{1}{\sqrt[3]{1{,}000{,}000}}.$$

Solution. Setting in the inequality (14) $m=4$, $n=1{,}000{,}000$, $\alpha=-\frac{1}{3}$, we get

$$\frac{1{,}000{,}001^{\frac{2}{3}}-4^{\frac{2}{3}}}{\frac{2}{3}} < x < \frac{1{,}000{,}000^{\frac{2}{3}}-3^{\frac{2}{3}}}{\frac{2}{3}},$$

that is

$$\frac{3}{2}\cdot 1{,}000{,}001^{\frac{2}{3}}-\frac{3}{2}\cdot 4^{\frac{2}{3}} < x < \frac{3}{2}\cdot 1{,}000{,}000^{\frac{2}{3}}-\frac{3}{2}\cdot 3^{\frac{2}{3}}.$$

Since

$$\frac{3}{2}\cdot 1{,}000{,}001^{\frac{2}{3}} > \frac{3}{2}\cdot 1{,}000{,}000^{\frac{2}{3}}=\frac{3}{2}\cdot 10{,}000=15{,}000,$$

$$\frac{3}{2}\sqrt[3]{16}=\sqrt[3]{54}<4, \quad \frac{3}{2}\sqrt[3]{9} > \frac{3}{2}\sqrt[3]{8}=3,$$

then

$15{,}000-4 < x < 15{,}000-3$, that is $14{,}996 < x < 14{,}997$. From these inequalities it follows that $[x] = 14{,}996$.

1.5. The Mean Power of Numbers

In Sec. 1.2 before Problem 7 we have already named the number

$$c_\alpha = \left(\frac{a_1^\alpha + a_2^\alpha + \ldots + a_n^\alpha}{n}\right)^{\frac{1}{\alpha}}$$

the mean power of order α of the positive numbers a_1, a_2, . . ., a_n. In the same problem, it has been proved, that $c_\alpha \leqslant c_\beta$, if $\alpha < 0 < \beta$.

Here, should be proved the validity of the inequality $c_\alpha \leqslant c_\beta$ any time when $\alpha < \beta$. In other words, the mean power of order α is monotonically increasing together with α.

Theorem 4. *If $a_1, a_2, \ldots, a_n$ are positive numbers and $\alpha < \beta$, then $c_\alpha \leqslant c_\beta$, and $c_\alpha = c_\beta$, only when $a_1 = a_2 = \ldots = a_n$.*

Proof. For the case, when the numbers α and β have different signs the theorem has been proved above (refer to Problem 7, Sec. 1.2 and the definition prior to it). Thus, we have to prove the theorem only for the case when α and β have the same signs.

Assume, that $0 < \alpha < \beta$, and let

$$k = c_\alpha = \left(\frac{a_1^\alpha + a_2^\alpha + \ldots + a_n^\alpha}{n}\right)^{\frac{1}{\alpha}}.$$

Dividing c_β by k, we get

$$\frac{c_\beta}{k} = \frac{c_\beta}{c_\alpha} = \left(\frac{\left(\frac{a_1}{k}\right)^\beta + \left(\frac{a_2}{k}\right)^\beta + \ldots + \left(\frac{a_n}{k}\right)^\beta}{n}\right)^{\frac{1}{\beta}}.$$

Now, supposing

$$d_1 = \left(\frac{a_1}{k}\right)^\alpha, \quad d_2 = \left(\frac{a_2}{k}\right)^\alpha, \ldots, d_n = \left(\frac{a_n}{k}\right)^\alpha,$$

we obtain

$$\frac{c_\beta}{k} = \left(\frac{d_1^{\frac{\beta}{\alpha}} + d_2^{\frac{\beta}{\alpha}} + \ldots + d_n^{\frac{\beta}{\alpha}}}{n}\right)^{\frac{1}{\beta}}. \qquad (15)$$

Since

$$\left(\frac{d_1+d_2+\ldots+d_n}{n}\right)^{\frac{1}{\alpha}}=$$

$$=\left(\frac{\left(\frac{a_1}{k}\right)^{\alpha}+\left(\frac{a_2}{k}\right)^{\alpha}+\ldots+\left(\frac{a_n}{k}\right)^{\alpha}}{n}\right)^{\frac{1}{\alpha}}=$$

$$=\frac{1}{k}\left(\frac{a_1^{\alpha}+a_2^{\alpha}+\ldots+a_n^{\alpha}}{n}\right)^{\frac{1}{\alpha}}=\frac{1}{k}c_{\alpha}=\frac{1}{c_{\alpha}}c_{\alpha}=1,$$

then

$$\frac{d_1+d_2+\ldots+d_n}{n}=1,\quad d_1+d_2+\ldots+d_n=n.$$

Suppose

$$d_1=1+x_1,\quad d_2=1+x_2,\ \ldots,\quad d_n=1+x_n.$$

From the equality $d_1+d_2+\ldots+d_n=n$ it follows that

$$x_1+x_2+\ldots+x_n=0.$$

On the basis of Theorem 3 $\left(\text{notice, that } \frac{\beta}{\alpha}>1\right)$ we have

$$\left.\begin{aligned}
d_1^{\frac{\beta}{\alpha}}&=(1+x_1)^{\frac{\beta}{\alpha}}\geqslant 1+\frac{\beta}{\alpha}x_1,\\
d_2^{\frac{\beta}{\alpha}}&=(1+x_2)^{\frac{\beta}{\alpha}}\geqslant 1+\frac{\beta}{\alpha}x_2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
d_n^{\frac{\beta}{\alpha}}&=(1+x_n)^{\frac{\beta}{\alpha}}\geqslant 1+\frac{\beta}{\alpha}x_n.
\end{aligned}\right\}\qquad (*)$$

Adding these inequalities, we get

$$d_1^{\frac{\beta}{\alpha}}+d_2^{\frac{\beta}{\alpha}}+\ldots+d_n^{\frac{\beta}{\alpha}}\geqslant n+\frac{\beta}{\alpha}(x_1+x_2+\ldots+x_n)=n. \quad (16)$$

From the inequalities (15) and (16) it follows that

$$\frac{c_{\beta}}{k}\geqslant\left(\frac{n}{n}\right)^{\frac{1}{\beta}}=1,\quad c_{\beta}\geqslant k=c_{\alpha}.$$

It is necessary to note that $c_\beta = k = c_\alpha$ only when the signs of equality occur everywhere in (*), that is when $x_1 = x_2 = \ldots = x_n = 0$ (Theorem 3). In this case $d_1 = d_2 = \ldots = d_n = 1$ and, hence, $a_1 = a_2 = \ldots = a_n = k$. But if the numbers $a_1, a_2, \ldots, a_n$ are not identical, then

$$c_\beta > c_\alpha.$$

Thus Theorem 4 is proved regarding the case when $0 < \alpha < \beta$.

If $\alpha < \beta < 0$, then $0 < \frac{\beta}{\alpha} < 1$. Reasoning the same way as before, we get in (*) and (16) the opposite signs of inequalities. But since $\beta < 0$, then from the inequality

$$\frac{d_1^{\frac{\beta}{\alpha}} + d_2^{\frac{\beta}{\alpha}} + \ldots + d_n^{\frac{\beta}{\alpha}}}{n} \leqslant 1$$

it follows that

$$\frac{c_\beta}{k} = \left(\frac{d_1^{\frac{\beta}{\alpha}} + d_2^{\frac{\beta}{\alpha}} + \ldots + d_n^{\frac{\beta}{\alpha}}}{n}\right)^{\frac{1}{\beta}} \geqslant 1^{\frac{1}{\beta}} = 1,$$

that is

$$c_\beta \geqslant k = c_\alpha.$$

Thus, Theorem 4 is proved completely.

Further on we shall name the geometric mean by *mean power of the order zero*, that is, we shall assume $g = c_0$.

Notice, that Theorem 4 is applicable in this case as well, since (see Problem 7, Sec. 1.2) $c_\alpha \leqslant g = c_0$, if $\alpha < 0$, and $c_\beta \geqslant g = c_0$, if $\beta > 0$.

From the proved theorem it follows, in particular, that

$$c_{-1} \leqslant c_0 \leqslant c_1 \leqslant c_2,$$

i.e. the harmonic mean does not exceed the geometric mean, the geometric mean in its turn does not exceed the arithmetic mean, while the arithmetic mean does not exceed the root-mean-square of positive numbers. For example, if

$a_1 = 1$, $a_2 = 2$, $a_3 = 4$, then

$$c_{-1} = \left(\frac{a_1^{-1}+a_2^{-1}+a_3^{-1}}{3}\right)^{-1} = \frac{3}{\frac{1}{1}+\frac{1}{2}+\frac{1}{4}} = \frac{12}{7} = 1.7\ \ldots,$$

$$c_0 = \sqrt[3]{a_1a_2a_3} = \sqrt[3]{1\cdot 2\cdot 4} = 2,$$

$$c_1 = \frac{1+2+4}{3} = \frac{7}{3} = 2.3\ \ldots,$$

$$c_2 = \left(\frac{a_1^2+a_2^2+a_3^2}{3}\right)^{\frac{1}{2}} = \sqrt{\frac{1+4+16}{3}} = \sqrt{7} = 2.6\ \ldots$$

and therefore

$$c_{-1} = 1.7\ldots < 2 = c_0 < 2,3\ldots = c_1 < 2.6\ldots = c_2.$$

Problem 1. Prove, that $x^2 + y^2 + z^2 \geqslant 12$, if

$$x + y + z = 6.$$

Solution. Since the arithmetic mean does not exceed the root-mean-square, then

$$\frac{x+y+z}{3} \leqslant \left(\frac{x^2+y^2+z^2}{3}\right)^{\frac{1}{2}},$$

that is

$$x^2+y^2+z^2 \geqslant \frac{(x+y+z)^2}{3}.$$

In our problem $x^2 + y^2 + z^2 \geqslant \frac{6^2}{3} = 12$. The sign of equality holds only when $x = y = z = 2$.

Problem 2. Prove, that if x, y, z are positive numbers and $x^2 + y^2 + z^2 = 8$, then

$$x^3+y^3+z^3 \geqslant 16\sqrt{\frac{2}{3}}.$$

Solution. Since $c_2 \leqslant c_3$, then

$$\left(\frac{x^2+y^2+z^2}{3}\right)^{\frac{1}{2}} \leqslant \left(\frac{x^3+y^3+z^3}{3}\right)^{\frac{1}{3}}.$$

In our problem

$$\left(\frac{x^3+y^3+z^3}{3}\right)^{\frac{1}{3}} \geqslant \sqrt{\frac{8}{3}},$$

that is

$$x^3+y^3+z^3 \geqslant 3\cdot\frac{8}{3}\sqrt{\frac{8}{3}} = 16\sqrt{\frac{2}{3}}.$$

Problem 3. Prove, that for positive numbers a_1, a_2, a_3, ..., a_n, the following inequalities are true

$$(a_1+a_2+\ldots+a_n)^\alpha \leqslant n^{\alpha-1}(a_1^\alpha+a_2^\alpha+\ldots+a_n^\alpha),\quad \alpha \geqslant 1, \quad (17)$$

$$(a_1+a_2+\ldots+a_n)^\alpha \geqslant$$
$$\geqslant n^{\alpha-1}(a_1^\alpha+a_2^\alpha+\ldots+a_n^\alpha),\quad 0<\alpha\leqslant 1. \quad (18)$$

Solution. If $\alpha > 1$, then

$$c_\alpha = \left(\frac{a_1^\alpha+a_2^\alpha+\ldots+a_n^\alpha}{n}\right)^{\frac{1}{\alpha}} \geqslant \frac{a_1+a_2+\ldots+a_n}{n} = c_1.$$

The inequality (17) follows easily from this inequality. The inequality (18) is proved in exactly the same way. In particular, from the inequalities (17) and (18) it follows that

$$(x+y)^\alpha \leqslant 2^{\alpha-1}(x^\alpha+y^\alpha),\quad \alpha\geqslant 1,\quad x>0,\quad y>0,$$
$$(x+y)^\alpha \geqslant 2^{\alpha-1}(x^\alpha+y^\alpha),\quad 0<\alpha<1,\quad x>0,\quad y>0.$$

Problem 4. Prove, that if $x^3+y^3+z^3=81$, $x>0$, $y>0$, $z>0$, then

$$x+y+z\leqslant 9.$$

Solution. Since

$$(x+y+z)^3 \leqslant 3^2(x^3+y^3+z^3) = 9\cdot 81 = 729$$

(the inequality (17)), then

$$x+y+z\leqslant \sqrt[3]{729} = 9.$$

CHAPTER 2

Uses of Inequalities

The use of inequalities in finding the greatest and the least function values and in calculating limits of some sequences will be examined in this chapter. Besides that, some important inequalities will be demonstrated here as well.

2.1. The Greatest and the Least Function Values

A great deal of practical problems come to various functions. For example, if x, y, z are the lengths of the edges of a box with a cover (a parallelepiped), then the area of the box surface is

$$S = 2xy + 2yz + 2zx,$$

and its volume is

$$V = xyz.$$

If the material from which the box is made is expensive, then, certainly, it is desirable, with the given volume of the box, to manufacture it with the least consumption of the material, i.e., so that the area of the box surface should be the least. We gave a simple example of a problem considering the maximum and the minimum functions of a great number of variables. One may encounter similar problems very often and the most celebrated mathematicians always pay considerable attention to working out methods of their solution.

Here, we shall solve a number of such problems, making use of the inequalities, studied in the first chapter[1]. First of all, we shall prove one theorem.

[1] Concerning the application of inequalities of the second degree to solving problems for finding the greatest and the least values see the book by I.P. Natanson "Simplest Problems for Calculating the Maximum and Minimum Values", 2nd edition, Gostekhizdat, Moscow, 1952.

Theorem 5. *If* $a>0$, $\alpha>1$, $x\geqslant 0$, *then the function* $x^{\alpha}-ax$ *takes the least value in the point* $x=\left(\frac{a}{\alpha}\right)^{\frac{1}{1-\alpha}}$, *equal to* $(1-\alpha)\left(\frac{a}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}$.

Proof. The theorem is proved very simply for the case when $\alpha=2$. Indeed, since

$$x^2-ax=\left(x-\frac{a}{2}\right)^2-\frac{a^2}{4},$$

the function has the least value when $x=\frac{a}{2}>0$, this value being equal to $-\frac{a^2}{4}$.

In case of arbitrary value of $\alpha>1$ the theorem is proved by using the inequality (12), demonstrated in Theorem 3. Since $\alpha>1$, then

$$(1+z)^{\alpha}\geqslant 1+\alpha z, \quad z\geqslant -1,$$

the equality holding only when $z=0$. Assuming here, that $1+z=y$, we get

$$y^{\alpha}\geqslant 1+\alpha(y-1), \quad y^{\alpha}-\alpha y\geqslant 1-\alpha, \quad y\geqslant 0,$$

the sign of equality holds only when $y=1$. Multiplying both members of the latter inequality by c^{α}, we get

$$(cy)^{\alpha}-\alpha c^{\alpha-1}(cy)\geqslant(1-\alpha)c^{\alpha}, \quad y\geqslant 0.$$

Assuming

$$x=cy \text{ and } \alpha c^{\alpha-1}=a, \quad c=\left(\frac{a}{\alpha}\right)^{\frac{1}{\alpha-1}},$$

we get

$$x^{\alpha}-ax\geqslant(1-\alpha)c^{\alpha}=(1-\alpha)\left(\frac{a}{\alpha}\right)^{\frac{\alpha}{\alpha-1}},$$

here the equality occurs only when $x=c=\left(\frac{a}{\alpha}\right)^{\frac{1}{\alpha-1}}$.

Thus, the function

$$x^{\alpha}-ax, \quad \alpha>1, \quad a>0, \quad x\geqslant 0,$$

takes the least value in the point $x=\left(\frac{a}{\alpha}\right)^{\frac{1}{1-\alpha}}$, equal to $(1-\alpha)\left(\frac{a}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}$. The theorem is proved.

In particular, the function $x^2 - ax$ ($\alpha = 2$) takes the least value in the point $x = \left(\frac{a}{2}\right)^{\frac{1}{2-1}} = \frac{a}{2}$, equal to $(1-2)\left(\frac{a}{2}\right)^{\frac{2}{2-1}} = -\frac{a^2}{4}$. This result is in accordance with the conclusion, obtained earlier by a different method. The function $x^3 - 27x$ takes the least value in the point

$$x = \left(\frac{27}{3}\right)^{\frac{1}{3-1}} = 3, \text{ equal to } (1-3)\left(\frac{27}{3}\right)^{\frac{3}{3-1}} = -54.$$

Note. Let us mark for the following, that the function

$$ax - x^\alpha = -(x^\alpha - ax),$$

where $\alpha > 1$, $a > 0$, $x \geqslant 0$, takes the greatest value in the point

$$x = \left(\frac{a}{\alpha}\right)^{\frac{1}{\alpha-1}},$$

equal to

$$(\alpha - 1)\left(\frac{a}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}.$$

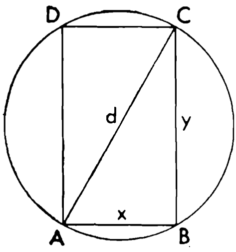

Fig. 1

Problem 1. It is required to saw out a beam of the greatest durability from a round log (the durability of the beam is directly proportional to the product of the width of the beam by the square of its height).

Solution. Suppose $AB = x$ is the width of the beam, $BC = y$ is its height and $AC = d$ is the diameter of the log (Fig. 1). Denoting the durability of the beam by P, we get

$$P = kxy^2 = kx(d^2 - x^2) = k(d^2x - x^3).$$

The function $d^2x - x^3$ takes the greatest value when

$$x = \left(\frac{d^2}{3}\right)^{\frac{1}{3-1}} = \frac{d}{\sqrt{3}}, \quad y^2 = d^2 - x^2 = \frac{2}{3}d^2,$$

$$y = \frac{d}{\sqrt{3}}\sqrt{2} = x\sqrt{2}.$$

Thus, the beam may have the highest (greatest) durability if the ratio of its height to its width will be equal to $\sqrt{2} \approx \approx 1.4 = \frac{7}{5}$.

Problem 2. Find the greatest value of the function

$$y = \sin x \sin 2x.$$

Solution. Since $\sin 2x = 2 \sin x \cos x$, then $\sin x \sin 2x = 2 \cos x \sin^2 x = 2 \cos x (1 - \cos^2 x) = 2 (z - z^3)$, where $z = \cos x$ and, hence, $-1 \leqslant z \leqslant 1$. The function $z - z^3 = z(1 - z^2)$ takes a negative value when $-1 \leqslant z < 0$,

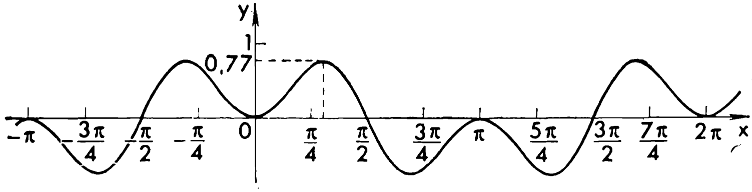

Fig. 2

is equal to 0 when $z = 0$ and takes a positive value when $0 < z \leqslant 1$. Therefore, the greatest value of the function is gained in the interval $0 < z \leqslant 1$.

It is shown in Theorem 5 that the function $z - z^3$, $z \geqslant 0$, takes the greatest value in the point

$$z = \left(\frac{1}{3}\right)^{\frac{1}{3-1}} = \frac{1}{\sqrt{3}}.$$

In this point

$$\sin x \sin 2x = 2z(1 - z^2) = \frac{2}{\sqrt{3}}\left(1 - \frac{1}{3}\right) = \frac{4}{3\sqrt{3}}.$$

So, the function $y = \sin x \sin 2x$ takes the greatest value in those points, where $z = \cos x = \frac{1}{\sqrt{3}}$ and this value is equal to $\frac{4}{3\sqrt{3}} \approx 0.77$. The graph of the function $y = \sin x \sin 2x$ is shown in Fig. 2.

Problem 3. Find the greatest value of the function

$$y = \cos x \cos 2x.$$

Solution. The function $y = \cos x \cos 2x$ does not exceed 1, since each of the cofactors $\cos x$ and $\cos 2x$ does not exceed 1. But in the points $x = 0, \pm 2\pi, \pm 4\pi, \ldots$

$$\cos x \cos 2x = 1.$$

Thus, the function $y = \cos x \cos 2x$ takes the greatest value of 1 in the points $x = 0, \pm 2\pi, \pm 4\pi, \ldots$. The graph of the function $y = \cos x \cos 2x$ is drawn in Fig. 3.

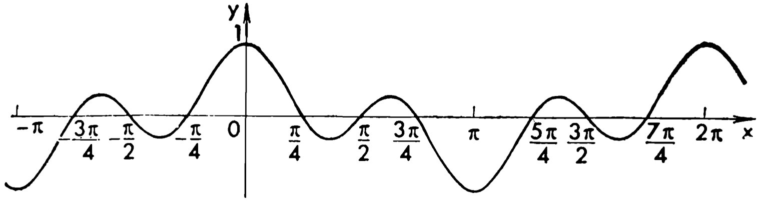

Fig. 3

Problem 4. Find the least value of the function

$$x^{\alpha} + ax,$$

where $a > 0$, $\alpha < 0$, $x \geqslant 0$.

Solution. Since $\alpha < 0$, then according to the inequality (12)

$$(1 + z)^{\alpha} \geqslant 1 + \alpha z,$$

and the sign of equality holds only when $z = 0$. Assuming $1 + z = y$, $z = y - 1$, we get

$$y^{\alpha} \geqslant 1 + \alpha (y - 1), \quad y \geqslant 0,$$

the sign of equality occurring only when $y = 1$. From the last inequality it follows, that

$$y^{\alpha} - \alpha y \geqslant 1 - \alpha, \; (cy)^{\alpha} - \alpha c^{\alpha - 1} (cy) \geqslant (1 - \alpha) c^{\alpha}.$$

Assuming $a = -\alpha c^{\alpha - 1}$, $x = cy$, we get

$$x^{\alpha} + ax \geqslant (1 - \alpha) c^{\alpha} = (1 - \alpha) \left(\frac{a}{-\alpha} \right)^{\frac{\alpha}{\alpha - 1}},$$

the equality holding only when $x = c = \left(\frac{a}{-\alpha} \right)^{\frac{1}{\alpha - 1}}$.

Thus, the function $x^{\alpha}+ax$ takes the least value in the point

$$x=\left(\frac{a}{-\alpha}\right)^{\frac{1}{\alpha-1}},$$

equal to $(1-\alpha)\left(\frac{a}{-\alpha}\right)^{\frac{\alpha}{\alpha-1}}$.

For example, the function

$$\frac{1}{\sqrt[3]{x}}+27x, \quad x>0,$$

takes the least value in the point

$$x=\left(\frac{27}{\frac{1}{3}}\right)^{\frac{1}{-\frac{1}{3}-1}}=\frac{1}{27}.$$

This value equals

$$\left(1+\frac{1}{3}\right)\left(\frac{27}{\frac{1}{3}}\right)^{\frac{-\frac{1}{3}}{-\frac{1}{3}-1}}=4.$$

Problem 5. Find the optimum dimensions of a cylindrical tin having a bottom and a cover (dimensions of a vessel are considered to be the most profitable, if for a given volume the least amount of material is required for its manufacture, that is, the vessel has the least surface area).

Solution. Let $V = \pi r^2 h$ be the volume of the vessel, where r is the radius, h is the height of the cylinder. The total surface area of the cylinder is

$$S = 2\pi r^2 + 2\pi rh.$$

Since $h=\frac{V}{\pi r^2}$, then

$$S=2\pi r^2+2\pi r\frac{V}{\pi r^2}=2\pi r^2+\frac{2V}{r}.$$

Assuming $x=\frac{1}{r}$, we get

$$S=2\pi x^{-2}+2Vx=2\pi\left(x^{-2}+\frac{V}{\pi}x\right).$$

The function $x^{-2}+\frac{V}{\pi}x$, according to the solution of the previous problem, takes the least value when

$$x=\left(\frac{V}{2\pi}\right)^{\frac{1}{-2-1}}=\sqrt[3]{\frac{2\pi}{V}}.$$

Returning back to our previous designations, we find

$$\frac{1}{r}=\sqrt[3]{\frac{2\pi}{V}},\quad r^3=\frac{V}{2\pi}=\frac{\pi r^2 h}{2\pi},\quad r=\frac{h}{2},$$

$$h=2r=d.$$

Thus, the vessel has the most profitable dimensions, if the height and diameter of the vessel are equal.

Exercises

6. Find the greatest value of the function $x(6-x)^2$ when $0<x<6$.

Indication. Suppose $y=6-x$.

7. From a square sheet whose side is equal to $2a$ it is required to make a box without a cover by cutting out a square at each vertex and then bending the obtained edges,

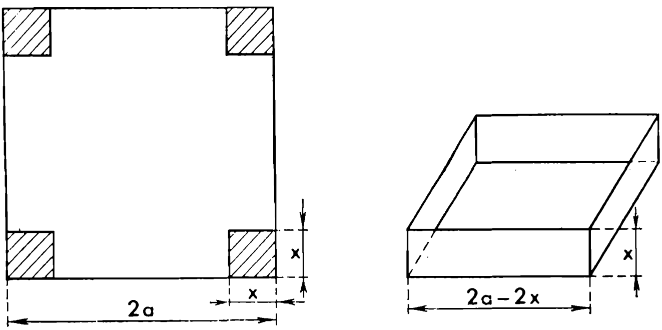

Fig. 4

so that the box would be produced with the greatest volume (Fig. 4). What should the length of the side of the cut-out squares be?

8. Find the least value of the function

$$x^6+8x^2+5,$$

9. Find the least value of the function

$$x^6 - 8x^2 + 5.$$

10. Find the greatest value of the function

$$x^\alpha - ax \quad \text{when} \quad 0 < \alpha < 1, \quad a > 0, \quad x \geqslant 0.$$

11. Prove that, when $x \geqslant 0$, the following inequality is true

$$\sqrt[4]{x} \leqslant \frac{3}{8} + 2x.$$

12. Prove that, when $n \geqslant 3$, the following inequality is true

$$\sqrt[n]{n} > \sqrt[n+1]{n+1}.$$

Indication. Make use of the inequality (8).

13. Find the greatest of the numbers

$$1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \sqrt[5]{5}, \ldots, \sqrt[n]{n}, \ldots .$$

14. Prove the inequality

$$\sqrt[n]{n} < 1 + \frac{2}{\sqrt{n}}.$$

15. Prove the inequality

$$(1 + a_1)(1 + a_2) \ldots (1 + a_n) \geqslant$$
$$\geqslant 1 + a_1 + a_2 + \ldots + a_n,$$

if the numbers a_i are of the same sign and are not less than -1.

16. Prove the inequality

$$(a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 \leqslant$$
$$\leqslant (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2). \quad (19)$$

Indication. First prove, that the polynomial

$$(a_1x - b_1)^2 + (a_2x - b_2)^2 + \ldots + (a_nx - b_n)^2 =$$
$$= x^2(a_1^2 + a_2^2 + \ldots + a_n^2) -$$
$$- 2x(a_1b_1 + a_2b_2 + \ldots + a_nb_n) +$$
$$+ (b_1^2 + b_2^2 + \ldots + b_n^2)$$

cannot have two different real roots.

17. Using the inequality (19), prove, that the arithmetic mean is not greater than the root-mean-square.

18. Prove the inequality

$$\frac{1}{\sqrt{n}} < \sqrt{n+1} - \sqrt{n-1}.$$

19. Using the inequality of Exercise 18, prove the inequality

$$\sqrt{n+1} + \sqrt{n} - \sqrt{2} > 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}}.$$

20. Find the greatest value of the functions

$$\frac{x^3}{x^4+5}, \quad x^6 - 0.6x^{10}.$$

Answer. $\frac{3}{4\sqrt[4]{15}}$; 0.4.

21. At what value of a is the least value of the function

$$\sqrt{x} + \frac{a}{x^2} \text{ equal to } 2.5?$$

Answer. $a = 8$.

2.2. The Hölder Inequality

In Theorem 7, by means of Theorems 5 and 6, the Hölder inequality is proved. This inequality will find application in solving problems.

Theorem 6. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $x > 0$, $y > 0$, *then*

$$xy \leqslant \frac{x^p}{p} + \frac{y^q}{q}. \qquad (20)$$

Proof. By virtue of Theorem 5, if $\alpha < 1$, $\alpha > 0$, $x \geqslant 0$, then

$$x^\alpha - ax \geqslant (1-\alpha)\frac{a}{\alpha}^{\frac{\alpha}{\alpha-1}}.$$

Assuming in this inequality that $\alpha = p$, $a = py$, we get

$$x^p - (py)\,x \geqslant (1-p)\left(\frac{py}{p}\right)^{\frac{p}{p-1}} = (1-p)\,y^{\frac{p}{p-1}}. \qquad (21)$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}, \quad q = \frac{p}{p-1}, \quad p - 1 = \frac{p}{q}.$$

Putting these values into the inequality (21), we get

$$x^p - pyx \geqslant -\frac{p}{q} y^q.$$

Dividing all the members of the latter inequality by p and transposing the negative members to the opposite side, we get the inequality (20).

Theorem 7. If $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ *are positive numbers, and p and q satisfy the conditions of Theorem 6, then*

$$a_1b_1 + a_2b_2 + \ldots + a_nb_n \leqslant$$
$$\leqslant (a_1^p + a_2^p + \ldots + a_n^p)^{\frac{1}{p}} (b_1^q + b_2^q + \ldots + b_n^q)^{\frac{1}{q}}. \qquad (22)$$

Proof. Suppose

$$a_1^p + a_2^p + \ldots + a_n^p = A^p, \quad b_1^q + b_2^q + \ldots + b_n^q = B^q.$$

Then the right member of the inequality (22) will be equal to

$$(A^p)^{\frac{1}{p}} (B^q)^{\frac{1}{q}} = AB.$$

Now suppose

$$a_1 = Ac_1, \quad a_2 = Ac_2, \ \ldots, \ a_n = Ac_n,$$
$$b_1 = Bd_1, \quad b_2 = Bd_2, \ \ldots, \ b_n + Bd_n.$$

Since

$$A^p = a_1^p + a_2^p + \ldots + a_n^p =$$
$$= A^pc_1^p + A^pc_2^p + \ldots + A^pc_n^p = A^p (c_1^p + c_2^p + \ldots + c_n^p),$$

then

$$c_1^p + c_2^p + \ldots + c_n^p = 1.$$

In a similar way, it is checked that

$$d_1^q + d_2^q + \ldots + d_n^q = 1.$$

Now using the inequality (20), we get

$$\left.\begin{aligned} a_1b_1 = AB\,(c_1d_1) &\leqslant AB\left(\frac{c_1^p}{p} + \frac{d_1^q}{q}\right), \\ a_2b_2 &\leqslant AB\left(\frac{c_2^p}{p} + \frac{d_2^q}{q}\right), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ a_nb_n &\leqslant AB\left(\frac{c_n^p}{p} + \frac{d_n^q}{q}\right). \end{aligned}\right\} \qquad (*)$$

From these inequalities it follows, that

$$a_1b_1 + a_2b_2 + \ldots + a_nb_n \leqslant$$

$$\leqslant AB\left(\frac{c_1^p + c_2^p + \ldots + c_n^p}{p} + \frac{d_1^q + d_2^q + \ldots + d_n^q}{q}\right) =$$

$$= AB\left(\frac{1}{p} + \frac{1}{q}\right) = AB$$

(let us recall that

$$\frac{1}{p} + \frac{1}{q} = 1, \quad c_1^p + c_2^p + \ldots + c_n^p = 1,$$

$$d_1^q + d_2^q + \ldots + d_n^q = 1).$$

Thus, it is proved that the left-hand member of the inequality (22) does not exceed AB, that is, does not exceed the right-hand member.

It is not difficult to mark the case when the sign of equality is valid in (22). Indeed, the sign of equality holds in (21) only when

$$x = \left(\frac{py}{p}\right)^{\frac{1}{p-1}} = y^{\frac{1}{p-1}} = y^{\frac{q}{p}}, \quad x^p = y^q$$

(refer to Theorem 6). Just in the same way, the equality sign will be valid in each line of (*) only when

$$c_1 = d_1^{\frac{q}{p}}, \quad c_2 = d_2^{\frac{q}{p}}, \quad \ldots, \quad c_n = d_n^{\frac{q}{p}},$$

i.e., when

$$c_1^p = d_1^q, \quad c_2^p = d_2^q, \quad \ldots, \quad c_n^p = d_n^q.$$

Finally, multiplying these equalities by A^pB^q, we get $B^q(Ac_1)^p = A^p(Bd_1)^q$, that is, $B^qa_1^p = A^pb_1^q$,

$$\frac{a_1^p}{b_1^q} = \frac{A^p}{B^q}, \quad \frac{a_2^p}{b_2^q} = \frac{A^p}{B^q}, \quad \ldots, \quad \frac{a_n^p}{b_n^q} = \frac{A^p}{B^q}.$$

Thus, in (22) the sign of equality is valid if

$$\frac{a_1^p}{b_1^q} = \frac{a_2^p}{b_2^q} = \ldots = \frac{a_n^p}{b_n^q}$$

Note. Taking in the inequality (22) $p = 2$, $q = 2$, we get the inequality (19) (refer to Exercise 16):

$$a_1b_1 + a_2b_2 + \ldots + a_nb_n \leqslant$$
$$\leqslant \sqrt{(a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2)}.$$

2.3. The Use of Inequalities for Calculation of Limits

In the following problems, the limits of quite complicated sequences are calculated by means of previously proved inequalities.

Problem 1. Prove the inequality

$$\frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}. \tag{23}$$

$\ln\left(1+\frac{1}{n}\right)$ denotes the logarithm from $\left(1+\frac{1}{n}\right)$ with base e (see pp. 21-22).

Solution. Combining the inequalities (8) and (9), we get

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}.$$

Finding the logarithm of these inequalities with base e, we finally get

$$n\ln\left(1+\frac{1}{n}\right) < \ln e = 1 < (n+1)\ln\left(1+\frac{1}{n}\right),$$
$$\frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}.$$

Problem 2. Assuming

$$z_1 = 1 + \frac{1}{2}, \quad z_2 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4},$$
$$z_3 = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6},$$
$$z_4 = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}, \ldots$$
$$\ldots, \quad z_n = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n},$$

find $\lim\limits_{n\to\infty} z_n$.

Solution. Substituting $n-1$ for n in the first member of the inequality (23), we get

$$\frac{1}{n} < \ln\left(1+\frac{1}{n-1}\right) = \ln\frac{n}{n-1}.$$

From this inequality and the second member of the inequality (23) it follows that

$$\ln \frac{n+1}{n} < \frac{1}{n} < \ln \frac{n}{n-1}. \qquad (24)$$

Now, using the inequality (24), we write the inequalities

$$\ln \frac{n+1}{n} < \frac{1}{n} < \ln \frac{n}{n-1},$$

$$\ln \frac{n+2}{n+1} < \frac{1}{n+1} < \ln \frac{n+1}{n},$$

$$\ln \frac{n+3}{n+2} < \frac{1}{n+2} < \ln \frac{n+2}{n+1},$$

$$\dots\dots\dots\dots\dots\dots$$

$$\ln \frac{2n+1}{2n} < \frac{1}{2n} < \ln \frac{2n}{2n-1}.$$

Adding them and taking into consideration that the sum of logarithms is equal to the logarithm of the product, we get

$$\ln \frac{(n+1)(n+2)(n+3)\dots(2n+1)}{n(n+1)(n+2)\dots 2n} < \frac{1}{n} + \frac{1}{n+1} + \dots$$

$$\dots + \frac{1}{2n} < \ln \frac{n(n+1)(n+2)\dots 2n}{(n-1)n(n+1)\dots(2n-1)},$$

that is

$$\ln \frac{2n+1}{n} < \frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n} < \ln \frac{2n}{n-1}. \qquad (25)$$

Since

$$\frac{2n+1}{n} = 2 + \frac{1}{n}, \text{ then}$$

$$\lim_{n\to\infty} \ln \frac{2n+1}{n} = \lim_{n\to\infty} \ln \left(2 + \frac{1}{n}\right) = \ln 2.$$

Exactly in the same way from $\frac{2n}{n-1} = 2 + \frac{2}{n-1}$ it follows, that

$$\lim_{n\to\infty} \ln \frac{2n}{n-1} = \ln 2.$$

Thus, the extreme terms of the inequalities (25) have the same limits. Hence, the mean term has also the same limit, that is

$$\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\ldots+\frac{1}{2n}\right)=\lim_{n\to\infty} z_n=\ln 2.$$

Problem 3. Taking $x_1=1, x_2=1-\frac{1}{2}, x_3=1-\frac{1}{2}+\frac{1}{3}, \ldots$ $\ldots, x_n=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots+(-1)^{n-1}\frac{1}{n}$, calculate $\lim\limits_{n\to\infty} x_n$.

Solution. We have

$$x_{2n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}=$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots+\frac{1}{2n-1}+\frac{1}{2n}\right)-$$

$$-2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}\right)=$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots+\frac{1}{2n-1}+\frac{1}{2n}\right)-$$

$$-\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)=\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}.$$

In the previous problem, we have supposed that

$$z_n=\frac{1}{n}+\frac{1}{n+1}+\ldots+\frac{1}{2n}.$$

Therefore, $x_{2n}=z_n-\frac{1}{n}$. But $\lim\limits_{n\to\infty} z_n=\ln 2$ (refer to the previous problem). Thus,

$$\lim_{n\to\infty} x_{2n}=\lim_{n\to\infty}\left(z_n-\frac{1}{n}\right)=\ln 2.$$

It is necessary to note also, that $x_{2n+1}=x_{2n}+\frac{1}{2n+1}$, and, hence,

$$\lim_{n\to\infty} x_{2n+1}=\lim_{n\to\infty}\left(x_{2n}+\frac{1}{2n+1}\right)=\ln 2.$$

Thus,

$$\lim_{n\to\infty} x_n=\ln 2.$$

Note. The numbers $x_1=a_1$, $x_2=a_1+a_2$, $x_3=a_1+a_2+a_3, \ldots, x_n=a_1+a_2+\ldots+a_n$ are termed *partial sums* of the series

$$a_1 + a_2 + a_3 + \dots a_n + \dots.$$

The series is said to be *convergent*, if the sequence of its partial sums has a finite limit. In this case the number $S = \lim_{n\to\infty} x_n$ is called *the sum of the series.*

From Problem 3, it follows that the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots + \frac{1}{2n-1} - \frac{1}{2n} + \dots$$

converges and its sum equals ln 2.

Problem 4. The series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \dots$$

is called *harmonic series.* Prove that the harmonic series diverges.

Solution. According to the inequality (23)

$$\frac{1}{n} > \ln \frac{n+1}{n}.$$

Assuming $n = 1, 2, 3, \dots, n$, write n inequalities

$$1 > \ln \frac{2}{1},$$

$$\frac{1}{2} > \ln \frac{3}{2},$$

$$\frac{1}{3} > \ln \frac{4}{3},$$

$$\dots\dots\dots$$

$$\frac{1}{n} > \ln \frac{n+1}{n}.$$

Adding them, we get

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} > \ln \frac{2\cdot3\cdot4\dots(n+1)}{1\cdot2\cdot3\dots n} = \ln(n+1).$$

It follows from this inequality that

$$\lim_{n\to\infty} x_n \geqslant \lim_{n\to\infty} \ln(n+1) = \infty;$$

hence, the harmonic series diverges.

Problem 5. Prove that the series

$$1+\frac{1}{2^\alpha}+\frac{1}{3^\alpha}+\ldots+\frac{1}{n^\alpha}+\ldots \qquad (26)$$

converges at any $\alpha > 1$.

Solution. The sequence of partial sums of this series

$$x_1 = 1,$$

$$x_2 = 1+\frac{1}{2^\alpha},$$

$$x_3 = 1+\frac{1}{2^\alpha}+\frac{1}{3^\alpha},$$

$$x_4 = 1+\frac{1}{2^\alpha}+\frac{1}{3^\alpha}+\frac{1}{4^\alpha},$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$x_n = 1+\frac{1}{2^\alpha}+\frac{1}{3^\alpha}+\ldots+\frac{1}{n^\alpha}$$

is monotonically increasing, that is

$$x_1 < x_2 < x_3 < x_4 < \ldots < x_n < \ldots .$$

On the other hand, it is known that monotonically increasing limited sequence of numbers has a finite limit. Therefore, if we prove that the sequence of numbers x_n is limited, then the convergence of the series (26) will be proved as well. Suppose

$$y_{2n} = 1-\frac{1}{2^\alpha}+\frac{1}{3^\alpha}-\frac{1}{4^\alpha}+\frac{1}{5^\alpha}-\frac{1}{6^\alpha}+\ldots$$
$$\ldots+\frac{1}{(2n-1)^\alpha}-\frac{1}{(2n)^\alpha}.$$

Since

$$y_{2n} = 1-\left(\frac{1}{2^\alpha}-\frac{1}{3^\alpha}\right)-\left(\frac{1}{4^\alpha}-\frac{1}{5^\alpha}\right)-\ldots$$
$$\ldots-\left(\frac{1}{(2n-2)^\alpha}-\frac{1}{(2n-1)^\alpha}\right)-\frac{1}{(2n)^\alpha},$$

then (the numbers in each bracket are positive)

$$y_{2n} < 1.$$

On the other hand,

$$y_{2n} = 1-\frac{1}{2^\alpha}+\frac{1}{3^\alpha}-\frac{1}{4^\alpha}+\frac{1}{5^\alpha}-\frac{1}{6^\alpha}+\ldots+\frac{1}{(2n-1)^\alpha}-\frac{1}{(2n)^\alpha}=$$

$$=\left(1+\frac{1}{2^\alpha}+\frac{1}{3^\alpha}+\frac{1}{4^\alpha}+\frac{1}{5^\alpha}+\frac{1}{6^\alpha}+\ldots+\frac{1}{(2n-1)^\alpha}+\frac{1}{(2n)^\alpha}\right)-$$

$$-2\left(\frac{1}{2^\alpha}+\frac{1}{4^\alpha}+\frac{1}{6^\alpha}+\ldots+\frac{1}{(2n)^\alpha}\right)=$$

$$=\left(1+\frac{1}{2^\alpha}+\frac{1}{3^\alpha}+\frac{1}{4^\alpha}+\frac{1}{5^\alpha}+\frac{1}{6^\alpha}+\ldots+\frac{1}{(2n-1)^\alpha}+\frac{1}{(2n)^\alpha}\right)-$$

$$-\frac{2}{2^\alpha}\left(1+\frac{1}{2^\alpha}+\frac{1}{3^\alpha}+\ldots+\frac{1}{n^\alpha}\right).$$

Since $x_n=1+\frac{1}{2^\alpha}+\frac{1}{3^\alpha}+\ldots+\frac{1}{n^\alpha}$, then

$$y_{2n}=x_{2n}-\frac{2}{2^\alpha}x_n.$$

Now, since $x_{2n}>x_n$, $y_{2n}<1$, then

$$1>y_{2n}>x_n-\frac{2}{2^\alpha}x_n=\frac{2^\alpha-2}{2^\alpha}x_n.$$

Hence, it follows that

$$x_n<\frac{2^\alpha}{2^\alpha-2},$$

that is, the numbers x_n are limited when $\alpha>1$. Thus, it is proved that the series (26) converges and its sum is not greater than $\frac{2^\alpha}{2^\alpha-2}$.

For example, if $\alpha=2$, then

$$x_n=1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}<\frac{2^2}{2^2-2}=2,$$

$$S=\lim_{n\to\infty}x_n=1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots\leqslant 2.$$

In the course of higher mathematics it is proved that

$$S=1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots=\frac{\pi^2}{6}. \qquad (27)$$

Exercises

22. Find the sum of the series

$$S=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots+(-1)^{n-1}\frac{1}{n^2}+\ldots.$$

Indication. Use the equality (27).

Answer. $S=\frac{\pi^2}{12}$.

23. Prove the inequalities

$$\frac{n^{\alpha+1}}{\alpha+1} < 1+2^{\alpha}+3^{\alpha}+\ldots+n^{\alpha} < \frac{(n+1)^{\alpha+1}}{\alpha+1}, \quad \alpha>0.$$

24. Assuming

$$x_n = 1+2^{\alpha}+3^{\alpha}+\ldots+n^{\alpha},$$

prove that

$$\lim_{n\to\infty} \frac{x^n}{n^{\alpha+1}} = \frac{1}{\alpha+1}, \quad \alpha>0.$$

25. Prove the inequality

$$(a_1b_1c_1+a_2b_2c_2+\ldots+a_nb_nc_n)^3 \leqslant$$
$$\leqslant (a_1^3+a_2^3+\ldots+a_n^3)(b_1^3+b_2^3+\ldots+b_n^3)(c_1^3+c_2^3+\ldots+c_n^3),$$

if a_k, b_k, c_k are positive numbers.

Indication. Use the inequality (7) and the method given in (22).

26. Assuming $x_n=\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{kn}$, where k is a positive integral number, prove that

$$\lim_{n\to\infty} x_n = \ln k.$$

Indication. Use the method of solving Problem 2 of the present section.

2.4. The Use of Inequalities for Approximate Calculation of Quantities

At the very beginning of Chapter 1, we have paid attention to the fact that practical problems require, as a rule, an approximate calculation of quantities and, as well, an ability to treat such approximately calculated quantities. A more accurate estimation of such quantities will certainly permit to decrease errors in solving problems.

In the present section, we are going to return to an approximate calculation of numbers of the form

$$S_{n,k}=\frac{1}{k^{\alpha}}+\frac{1}{(k+1)^{\alpha}}+\ldots+\frac{1}{n^{\alpha}}, \quad 0<\alpha<1, \; k<n.$$

In Sec. 1.1 we have succeeded in finding the number $S_{n,k}$ with an accuracy of up to 0.4 for $k = 1$, $n = 1{,}000{,}000$ and $\alpha = \frac{1}{2}$ (refer to Problem 2). In the same section (see Exercises 2 and 3), for $n = 10^6$ and $k = 10{,}000$, we were able to find the number $S_{n,k}$ already with an accuracy of up to 0.01. The comparison of these two examples shows, that the indicated method of their solution yields much better results of calculation for greater values of k.

In Sec. 1.4 (Problem 3) we found the integral part of the number $S_{n,k}$ for $k = 4$, $n = 10^6$ and $\alpha = \frac{1}{3}$. Thus, this number was also calculated with an accuracy of up to 0.5. However, we could not find the integral part of the number $S_{n,1}$ for $\alpha = \frac{1}{3}$ and $n = 10^6$ because the method of calculation of such quantities, indicated in Chapter 1, did not permit doing it. In this section, we shall improve the method of calculation of the quantity $S_{n,1}$. This improvement will make it possible to find similar quantities with a higher degree of accuracy quite easily.

Lemma 1. *If* $x_1 > x_2 > x_3 > \ldots > x_n$, *then* $0 < A = x_1 - x_2 + x_3 - x_4 + \ldots + (-1)^{n-1} x_n < x_1$.

Proof. The number of positive terms in the written algebraic sum is not less than the number of negative terms. Besides this, the preceding positive terms are greater than the following negative term. This proves that their algebraic sum is positive, $A > 0$. On the other hand, since

$$A = x_1 - (x_2 - x_3 + x^{\alpha} - \ldots + (-1)^{n-2} x_n)$$

and the quantity in brackets is positive too, then $A < x_1$. Thus, the lemma is proved.

Lemma 2. *If* $0 < \alpha < 1$, then the *following inequalities are true*

$$\frac{(2n+1)^{1-\alpha}-(n+1)^{1-\alpha}}{1-\alpha} < \frac{1}{(n+1)^{\alpha}} + \frac{1}{(n+2)^{\alpha}} + \ldots$$

$$\ldots + \frac{1}{(2n)^{\alpha}} < \frac{(2n)^{1-\alpha}-n^{1-\alpha}}{1-\alpha}. \quad (28)$$

Proof. The inequality (28) follows from the inequality (14) (see Sec. 1.4, Problem 2) when substituting $n+1$ for m, $2n$ for n and $-\alpha$ for α.

Theorem 8. *True is the equality*

$$S_{n,1} = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \ldots + \frac{1}{n^\alpha} =$$
$$= \frac{2^\alpha}{2-2^\alpha}\left[\frac{1}{(n+1)^\alpha} + \frac{1}{(n+2)^\alpha} + \ldots + \frac{1}{(2n)^\alpha}\right] -$$
$$- \frac{2^\alpha}{2-2^\alpha}\left[1 - \frac{1}{2^\alpha} + \frac{1}{3^\alpha} - \ldots - \frac{1}{(2n)^\alpha}\right]. \qquad (29)$$

Proof. We have

$$S_{n,1} = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \ldots + \frac{1}{n^\alpha} + \frac{1}{(n+1)^\alpha} + \ldots$$
$$\ldots + \frac{1}{(2n)^\alpha} - \left[\frac{1}{(n+1)^\alpha} + \frac{1}{(n+2)^\alpha} + \ldots + \frac{1}{(2n)^\alpha}\right].$$

Adding and subtracting from the right-hand member of the equality the number

$$2\left[\frac{1}{2^\alpha} + \frac{1}{4^\alpha} + \frac{1}{6^\alpha} + \ldots + \frac{1}{(2n)^\alpha}\right],$$

we get

$$S_{n,1} = 1 - \frac{1}{2^\alpha} + \frac{1}{3^\alpha} - \frac{1}{4^\alpha} + \ldots - \frac{1}{(2n)^\alpha} +$$
$$+ 2\left[\frac{1}{2^\alpha} + \frac{1}{4^\alpha} + \frac{1}{6^\alpha} + \ldots + \frac{1}{(2n)^\alpha}\right] -$$
$$- \left[\frac{1}{(n+1)^\alpha} + \frac{1}{(n+2)^\alpha} + \ldots + \frac{1}{(m)^\alpha}\right].$$

The numbers of the first square brackets have a common factor $\frac{1}{2^\alpha}$. Taking it out of the brackets, we get

$$S_{n,1} = 1 - \frac{1}{2^\alpha} + \frac{1}{3^\alpha} - \ldots - \frac{1}{(2n)^\alpha} +$$
$$+ \frac{2}{2^\alpha}\left(1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \ldots + \frac{1}{n^\alpha}\right) -$$
$$- \left[\frac{1}{(n+1)^\alpha} + \frac{1}{(n+2)^\alpha} + \ldots + \frac{1}{(2n)^\alpha}\right].$$

Since in round brackets there is the number $S_{n,1}$, then

$$\frac{1}{(n+1)^\alpha}+\frac{1}{(n+2)^\alpha}+\ldots+\frac{1}{(2n)^\alpha}-$$
$$-\left[1-\frac{1}{2^\alpha}+\frac{1}{3^\alpha}-\ldots-\frac{1}{(2n)^\alpha}\right]=$$
$$=\left(\frac{2}{2^\alpha}-1\right)S_{n,1}=\frac{2-2^\alpha}{2^\alpha}S_{n,1}.$$

Hence, after multiplying by 2^α and dividing by $2-2^\alpha$, we get the equality (29).

The equality (29) is of interest because it brings the calculation of the quantity $S_{n,1}$ to the computation of the quantity $S_{2n,\,n+1}$ and the quantity $1-\frac{1}{2^\alpha}+\frac{1}{3^\alpha}-\ldots-\frac{1}{(2n)^\alpha}$. The first of these quantities for great n is calculated with a high degree of accuracy by means of the inequality (28). Concerning the second quantity, we know from Lemma 1, that it is less than zero and greater than $-\frac{2^\alpha}{2-2^\alpha}$. But if we find the sum of the first four summands of the latter quantity, then the remaining quantity (the error) will be less than zero and greater than $-\frac{1}{5\alpha}\cdot\frac{2^\alpha}{2-2^\alpha}$.

In the following problems we shall perform the calculation of this quantity with a higher degree of accuracy as well.

Problem 1. Find the sum

$$A=1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{10^6}}$$

accurate to 0.002.

Solution. By virtue of Theorem 8

$$A=\frac{\sqrt{2}}{2-\sqrt{2}}\left(\frac{1}{\sqrt{10^6+1}}+\frac{1}{\sqrt{10^6+2}}+\ldots+\frac{1}{\sqrt{2\cdot 10^6}}\right)-$$
$$-\frac{\sqrt{2}}{2-\sqrt{2}}\left(1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\ldots-\frac{1}{\sqrt{2\cdot 10^6}}\right)=$$
$$=(\sqrt{2}+1)\left(\frac{1}{\sqrt{10^6+1}}+\ldots+\frac{1}{\sqrt{2\cdot 10^6}}\right)-$$
$$-(\sqrt{2}+1)\left(1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\ldots-\frac{1}{\sqrt{2\cdot 10^6}}\right)=$$
$$=(\sqrt{2}+1)(B-C),$$

where

$$B=\frac{1}{\sqrt{10^6+1}}+\frac{1}{\sqrt{10^6+2}}+\ldots+\frac{1}{\sqrt{2\cdot 10^6}},$$

$$C=1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\ldots-\frac{1}{\sqrt{2\cdot 10^6}}.$$

According to Lemma 2, the number B satisfies the inequalities

$$2\left(\sqrt{2\cdot 10^6+1}-\sqrt{10^6+1}\right)<B<2\left(\sqrt{2\cdot 10^6}-\sqrt{10^6}\right).$$

The extreme numbers of the inequalities differ from each other by less than $3\cdot 10^{-4}$. Indeed,

$$2\left(\sqrt{10^6+1}-\sqrt{10^6}\right)-2\left(\sqrt{2\cdot 10^6+1}-\sqrt{2\cdot 10^6}\right)=$$

$$=\frac{2}{\sqrt{10^6+1}+\sqrt{10^6}}-\frac{2}{\sqrt{2\cdot 10^6+1}+\sqrt{2\cdot 10^6}}\approx\frac{1}{\sqrt{10^6}}-$$

$$-\frac{1}{\sqrt{2\cdot 10^6}}=\frac{\sqrt{2}-1}{\sqrt{2}}\cdot\frac{1}{1,000}<3\cdot 10^{-4}.$$

Thus, the middle number will differ from the number B by less than $2\cdot 10^{-4}$. Calculating the first number and subtracting from it $2\cdot 10^{-4}$, we get

$$B=828.4269\pm\Delta_l,$$
$$|\Delta_l|<2\cdot 10^{-4}.$$

Now, proceed to calculating the number C. Let m be an odd number. Estimate the quantity

$$D=\frac{1}{\sqrt{m}}-\frac{1}{\sqrt{m+1}}+\frac{1}{\sqrt{m+2}}-\ldots-\frac{1}{\sqrt{2n}}.$$

For this reason, it is necessary to notice, that

$$\sqrt{k+1}-\sqrt{k-1}=\frac{2}{\sqrt{k+1}+\sqrt{k-1}}$$

and

$$E=\frac{2}{\sqrt{m+1}+\sqrt{m-1}}-\frac{2}{\sqrt{m+1}+\sqrt{m}}+$$

$$+\frac{2}{\sqrt{m+3}+\sqrt{m+1}}-\frac{2}{\sqrt{m+4}+\sqrt{m+2}}+\ldots-$$

$$-\frac{2}{\sqrt{2n+1}-\sqrt{2n-1}}=\sqrt{m+1}-\sqrt{m-1}-\sqrt{m+2}+$$
$$+\sqrt{m}+\sqrt{m+3}-\sqrt{m+1}-\sqrt{m+4}+$$
$$+\sqrt{m+2}+\ldots-\sqrt{2n+1}+\sqrt{2n-1}=\sqrt{m}-$$
$$-\sqrt{m-1}+\sqrt{2n}-\sqrt{2n+1}.$$

Thus, the number E is quite easily calculated. Subtracting the quantity D from the quantity E, we get

$$E-D=\left(\frac{2}{\sqrt{m+1}-\sqrt{m-1}}-\frac{1}{\sqrt{m}}\right)-$$
$$-\left(\frac{2}{\sqrt{m+2}+\sqrt{m}}-\frac{1}{\sqrt{m+1}}\right)+\ldots$$
$$\ldots-\left(\frac{2}{\sqrt{2n+1}-\sqrt{2n-1}}-\frac{1}{\sqrt{2n}}\right).$$

Demonstrate, that all the numbers in the brackets are positive and monotonically decreasing. Indeed,

$$\frac{2}{\sqrt{m+1}+\sqrt{m-1}}-\frac{1}{\sqrt{m}}=\frac{2\sqrt{m}-(\sqrt{m+1}+\sqrt{m-1})}{\sqrt{m}\,(\sqrt{m+1}+\sqrt{m-1})}=$$
$$=\frac{2m-2\sqrt{m^2-1}}{\sqrt{m}\,(\sqrt{m+1}+\sqrt{m-1})\,(2\sqrt{m}+\sqrt{m+1}+\sqrt{m-1})}=$$
$$=\frac{2}{\sqrt{m}\,(\sqrt{m+1}+\sqrt{m-1})\,(2\sqrt{m}+\sqrt{m+1}+\sqrt{m-1})\times(m+\sqrt{m^2-1})}.$$

Hence, it is proved, that such numbers are positive and monotonically decreasing with the increase of m. According to Lemma 1

$$0<E-D<$$
$$<\frac{2}{\sqrt{m}(\sqrt{m+1}+\sqrt{m-1})\,(2\sqrt{m}+\sqrt{m+1}+\sqrt{m-1})\times(m+\sqrt{m^2-1})}.$$

We shall not make a great mistake, substituting m for the numbers $m+1$ and $m-1$ in the denominator. Here, we get

$$0<E-D<\frac{2}{\sqrt{m}\cdot 2\sqrt{m}\cdot 4\sqrt{m}\cdot 2m}=\frac{1}{8\cdot m^{\frac{5}{2}}}.$$

Taking $m=9$ we get

$$0<E-D<\frac{1}{8\cdot 81\cdot 3}<0.0006.$$

This proves, that when $m=9$ and $n=10^6$

$$E-D=0.0003\pm\Delta_2, \quad |\Delta_2|<0.0003,$$

$$D=E-0.0003\pm\Delta_2=\sqrt{9}-\sqrt{8}+\sqrt{2\cdot 10^6}-$$
$$-\sqrt{2\cdot 10^6+1}-0.0003\pm\Delta_2=0.1710\pm\Delta_2,$$

Now let us return to the quantity C. We have

$$C=1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{6}}+\frac{1}{\sqrt{7}}-\frac{1}{\sqrt{8}}+D=$$
$$=1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{6}}+$$
$$+\frac{1}{\sqrt{7}}-\frac{1}{\sqrt{8}}+0.1710\pm\Delta_2=$$
$$=1-\frac{1}{2}-\frac{1}{\sqrt{2}}\left(1+\frac{1}{2}\right)+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{6}}+\frac{1}{\sqrt{7}}+$$
$$+0.1710\pm\Delta_2=\frac{1}{2}-\frac{3\sqrt{2}}{4}+\frac{\sqrt{3}}{3}+\frac{\sqrt{5}}{5}-\frac{\sqrt{6}}{6}+\frac{\sqrt{7}}{7}+$$
$$+0.1710\pm\Delta_2.$$

Thus, for the calculation of the number C with an accuracy of up to $3\cdot 10^{-4}$ it will be required to find only 5 roots and to produce a number of arithmetic operations. Using the tables and carrying out necessary calculations, we find

$$C=0.6035\pm\Delta_2.$$

Taking into consideration the found quantities B and C, and returning to the quantity A, we get

$$A=(\sqrt{2}+1)(B-C)=(\sqrt{2}+1)(827.8226\pm\Delta_3)=$$
$$=(\sqrt{2}+1)\cdot 827.8226\pm 2.5\Delta_3,$$

where

$$|2.5\Delta_3|\leqslant 2.5(|\Delta_1|+|\Delta_2|)<2.5\cdot 5\cdot 10^{-4}<2\cdot 10^{-3}.$$

Thus, the calculation with an accuracy of up to $2\cdot 10^{-3}$ will be

$$A=(\sqrt{2}+1)\,827.8226=1998.539.$$

Problem 2. Calculate the number

$$A = 1 + \frac{1}{\sqrt[4]{2}} + \frac{1}{\sqrt[4]{3}} + \ldots + \frac{1}{\sqrt[4]{10^{12}}}$$

with an accuracy of up to unity.

Solution. By virtue of Theorem 8

$$A = \frac{\sqrt[4]{2}}{2-\sqrt[4]{2}} \left(\frac{1}{\sqrt[4]{10^{12}+1}} + \frac{1}{\sqrt[4]{10^{12}+2}} + \ldots + \frac{1}{\sqrt[4]{2\cdot 10^{12}}} \right) -$$

$$- \frac{\sqrt[4]{2}}{2-\sqrt[4]{2}} \left(1 - \frac{1}{\sqrt[4]{2}} + \frac{1}{\sqrt[4]{3}} - \ldots - \frac{1}{\sqrt[4]{2\cdot 10^{12}}} \right).$$

The first term can be easily found and with a high degree of accuracy by means of the inequalities (28). By virtue of these inequalities the first term can be substituted by the number

$$\frac{\sqrt[4]{2}}{2-\sqrt[4]{2}} \frac{(2\cdot 10^{12})^{\frac{3}{4}} - (10^{12})^{\frac{3}{4}}}{1-\frac{1}{4}} = \frac{4}{3}\cdot 10^{9} (\sqrt[4]{8}-1) \frac{\sqrt[4]{2}}{2-\sqrt[4]{2}} = \frac{4}{3}\cdot 10^{9}.$$

By virtue of Lemma 1 the sum

$$\frac{\sqrt[4]{2}}{2-\sqrt[4]{2}} \left(1 - \frac{1}{\sqrt[4]{2}} + \frac{1}{\sqrt[4]{3}} - \ldots - \frac{1}{\sqrt[4]{2\cdot 10^{12}}} \right)$$

is positive and is not greater than the first term. Since the term is less than two, then

$$\frac{4}{3}\cdot 10^{9} - 2 < A < \frac{4}{3}\cdot 10^{9}.$$

The extreme numbers differ from each other by 2, and from the number A by less than 2. The middle number $\frac{4}{3}\cdot 10^{9} - 1$ differs from A by less than unity. Substituting this number, we get

$$A = 1333333332.3 \pm \Delta, \quad |\Delta| < 1.$$

Notice that the accuracy of calculating the number A, containing a trillion of addends, is extremely high. The relative error is less than

$$100 : 1333333332.3 < 0.0000001\,\%.$$

Exercises

27. Calculate (with an accuracy of up to unity) the sum

$$1+\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{3}}+\ldots+\frac{1}{\sqrt[3]{10^6}}.$$

Answer. 14,999.

28. Show that the equality

$$1+\frac{1}{2^\alpha}+\frac{1}{3^\alpha}+\ldots+\frac{1}{n^\alpha}=\frac{n^{1-\alpha}}{1-\alpha}-C+\beta_n$$

is true, where β_n is an infinitely small quantity, $\lim\limits_{n\to\infty}\beta_n=0$, and

$$C=\frac{2^\alpha}{2-2^\alpha}\left[1-\frac{1}{2^\alpha}+\frac{1}{3^\alpha}-\frac{1}{4^\alpha}+\ldots+(-1)^{n-1}\frac{1}{n^\alpha}+\ldots\right].$$

SOLUTIONS TO EXERCISES

1. Setting in the inequalities (1) (*p.* 9) $n = m, m+1, \ldots, n$:

$$2\sqrt{m+1}-2\sqrt{m}<\frac{1}{\sqrt{m}}<2\sqrt{m}-2\sqrt{m-1},$$

$$2\sqrt{m+2}-2\sqrt{m+1}<\frac{1}{\sqrt{m+1}}<2\sqrt{m+1}-2\sqrt{m},$$

$$2\sqrt{m+3}-2\sqrt{m+2}<\frac{1}{\sqrt{m+2}}<2\sqrt{m+2}-2\sqrt{m+1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$2\sqrt{n+1}-2\sqrt{n}<\frac{1}{\sqrt{n}}<2\sqrt{n}-2\sqrt{n-1}.$$

Adding these inequalities we get

$$2\sqrt{n+1}-2\sqrt{m}<\frac{1}{\sqrt{m}}+\frac{1}{\sqrt{m+1}}+\frac{1}{\sqrt{m+2}}+\ldots+\frac{1}{\sqrt{n}}<2\sqrt{n}-2\sqrt{m-1}.$$

2. Taking in the inequalities of Exercise 1 $m = 10{,}000$, $n = 1{,}000{,}000$, we obtain

$$2\sqrt{1{,}000{,}001}-2\sqrt{10{,}000}<\frac{1}{\sqrt{10{,}000}}+\frac{1}{\sqrt{10{,}001}}+\ldots+\frac{1}{\sqrt{1{,}000{,}000}}<2\sqrt{1{,}000{,}000}-2\sqrt{9{,}999}.$$

Since

$$2\sqrt{1{,}000{,}001}>2\sqrt{1{,}000{,}000}=2{,}000,\quad 2\sqrt{10{,}000}=200,$$
$$2\sqrt{9{,}999}=\sqrt{39{,}996}>199.98$$

(the last inequality can be easily checked, extracting the square root with an accuracy of up to 0.01), then

$$2{,}000-200=1{,}800<\frac{1}{\sqrt{10{,}000}}+\frac{1}{\sqrt{10{,}001}}+\ldots+\frac{1}{\sqrt{1{,}000{,}000}}<2{,}000-199.98=1800.02.$$

3. Multiplying the inequalities of Exercise 2 by 50, we shall get in our designation

$$90{,}000 < 50z < 90{,}001;$$

hence

$$[50z] = 90{,}000.$$

4. For $n = 1$, it is obivous, that the inequality is true

$$\frac{1}{2} \leqslant \frac{1}{\sqrt{3\cdot 1+1}} = \frac{1}{2}.$$

Assuming now that the inequality is true for $n = k$

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2k+1}{2k} \leqslant \frac{1}{\sqrt{3k+1}}, \qquad \text{(a)}$$

prove that it is true for $n = k + 1$, that is, prove that

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2k-1}{2k}\cdot\frac{2k+1}{2k+2} \leqslant \frac{1}{\sqrt{3k+4}}. \qquad \text{(b)}$$

Multiplying the inequality (a) by $\frac{2k+1}{2k+2}$, we get

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2k-1}{2k}\cdot\frac{2k+2}{2k+2} \leqslant \frac{1}{\sqrt{3k+1}}\cdot\frac{2k+1}{2k+2}.$$

What is left is to prove the inequality

$$\frac{1}{\sqrt{3k+1}}\cdot\frac{2k+1}{2k+2} < \frac{1}{\sqrt{3k+4}}.$$

Multiplying it by $(2k + 2)\sqrt{3k + 1}\sqrt{3k + 4}$ and squaring both parts of the obtained inequality, we get

$$(2k + 1)^2 (3k + 4) < (2k + 2)^2 (2k + 1),$$

or

$$12k^3 + 28k^2 + 19k + 4 < 12k^3 + 28k^2 + 20k + 4.$$

The latter inequality is obvious, since $k \geqslant 1$.

This proves that the inequality

$$\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n} \leqslant \frac{1}{\sqrt{3n+1}}$$

is true for all n.

5. Assuming in the inequality of Exercise 4 that $n = 50$, we get

$$\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{99}{100} < \frac{1}{\sqrt{3\cdot 50+1}} = \frac{1}{\sqrt{151}} < \frac{1}{\sqrt{144}} = \frac{1}{12}.$$

6. Assuming $y = 6 - x$, $x = 6 - y$, we shall bring the problem to finding the greatest value of the function

$$(6 - y)\, y^2 = 6y^2 - y^3$$

when $0 < y < 6$. Assuming then, that $y^2 = z$, we shall get the function

$$6z - z^{\frac{3}{2}},$$

whose greatest value (refer to note on p. 34) is equal to

$$\left(\frac{3}{2} - 1\right)\left(\frac{6}{\frac{3}{2}}\right)^{\frac{\frac{3}{2}}{\frac{3}{2}-1}} = 0.5 \cdot 4^3 = 32$$

and is obtained in the point

$$z = \left(\frac{6}{\frac{3}{2}}\right)^{\frac{1}{\frac{3}{2}-1}} = 4^2.$$

The function $6y^2 - y^3$ takes the greatest value in the point $y = \sqrt{z} = 4$, and this value equals 32.

The function $x\,(6 - x)^2$ attains the greatest value of 32 in the point $x = 6 - y = 6 - 4 = 2$.

7. The volume of a box (see Fig. 4, p. 38) equals

$$V = x\,(2a - 2x)^2 = 4x\,(a - x)^2, \quad 0 < x < a.$$

Assuming $y = a - x$, $y^2 = z$, we get

$$V = 4\,(az - z^{\frac{3}{2}}).$$

The greatest value of the function $az - z^{\frac{3}{2}}$ is obtained in the point

$$z = \left(\frac{a}{\frac{3}{2}}\right)^{\frac{1}{\frac{3}{2}-1}} = \left(\frac{2a}{3}\right)^2.$$

Therefore,

$$y = \sqrt{z} = \frac{2a}{3}, \quad x = a - y = a - \frac{2a}{3} = \frac{a}{3}.$$

Thus, the volume of a box will be the greatest, if the length of the side of the cut-out square is $\frac{1}{6}$ that of the side of the given square.

8. The least value of the function $x^6 + 8x^2 + 5$ equals 5 and is obtained when $x = 0$.

9. Assuming $y = x^2$, we shall bring the problem to finding the least value of the function

$$y^3 - 8y + 5$$

for positive values of y.

In Theorem 5, we have proved that the least value of the function $y^3 - 8y$ is equal to

$$(1-3)\left(\frac{8}{3}\right)^{\frac{3}{3-1}} = -2\,\frac{8^{\frac{3}{2}}}{3^{\frac{3}{2}}} = \frac{32\sqrt{6}}{9}.$$

The least value of the function $y^3 - 8y + 5$ is equal to

$$-\frac{32\sqrt{6}}{9} + 5 = -3.6\ldots.$$

10. Assuming $y = x^\alpha$ we get the function

$$y - ay^{\frac{1}{\alpha}} = a\left(\frac{1}{a}y - y^{\frac{1}{\alpha}}\right), \quad a > 0, \quad \frac{1}{\alpha} > 1.$$

By virtue of Theorem 5, the greatest value of the function $\frac{1}{a}y - y^{\frac{1}{\alpha}}$ is

$$\left(\frac{1}{\alpha} - 1\right)\left(\frac{\frac{1}{a}}{\frac{1}{\alpha}}\right)^{\frac{\frac{1}{\alpha}}{\frac{1}{\alpha}-1}} = \left(\frac{1}{\alpha} - 1\right)\left(\frac{\alpha}{a}\right)^{\frac{1}{1-\alpha}} =$$

$$= \frac{1-\alpha}{\alpha}\left(\frac{a}{\alpha}\right)^{\frac{1}{\alpha-1}}.$$

Multiplying the last quantity by a, we shall find the greatest value of the function $a\left(\frac{1}{a}y - y^{\frac{1}{\alpha}}\right)$ which is, hence, equal to

$$(1-\alpha)\frac{a}{\alpha}\cdot\left(\frac{a}{\alpha}\right)^{\frac{1}{\alpha-1}} = (1-\alpha)\left(\frac{a}{\alpha}\right)^{1+\frac{1}{\alpha-1}} =$$

$$= (1-\alpha)\left(\frac{a}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}.$$

11. The function $\sqrt[4]{x}-2x$, $x\geqslant 0$, $\alpha=\frac{1}{4}$, $a=2$, has the greatest value, equal to

$$\left(1-\frac{1}{4}\right)\left(\frac{2}{\frac{1}{4}}\right)^{\frac{\frac{1}{4}}{\frac{1}{4}-1}}=\frac{3}{4}\cdot 8^{-\frac{1}{3}}=\frac{3}{8}.$$

Therefore, for all $x\geqslant 0$ the following inequality is true

$$\sqrt[4]{x}-2x\leqslant\frac{3}{8}, \text{ or } \sqrt[4]{x}\leqslant\frac{3}{8}+2x.$$

12. Write down the inequality (8) in the form of

$$\left(\frac{n+1}{n}\right)^n<e, \quad (n+1)^n<en^n.$$

If $n\geqslant 3>e$, then

$$(n+1)^n<en^n<3n^n\leqslant nn^n=n^{n+1}.$$

Raising both members of the latter inequality to the power of $\frac{1}{n(n+1)}$, we get

$$\sqrt[n+1]{n+1}<\sqrt[n]{n}.$$

13. Since $1<\sqrt{2}=\sqrt[6]{8}<\sqrt[6]{9}=\sqrt[3]{3}$, then $\sqrt[3]{3}$ is the greatest of the numbers $1, \sqrt{2}, \sqrt[3]{3}$. On the other hand, in the previous problem we have shown that the sequence of the numbers $\sqrt[3]{3}, \sqrt[4]{4}, \ldots, \sqrt[n]{n}, \ldots$ decreases. Hence, $\sqrt[3]{3}$ is the greatest of the numbers $1, \sqrt{2}, \sqrt[3]{3}, \ldots$ $\ldots, \sqrt[n]{n}, \ldots$.

14. Suppose $\sqrt[n]{n}=1+\alpha_n$, $\alpha_n>0$. Raising to a power of n we get

$$n=(1+\alpha_n)^n=\left[(1+\alpha_n)^{\frac{n}{2}}\right]^2.$$

Assuming that $n\geqslant 2$, $\frac{n}{2}\geqslant 1$, taking Theorem 3 as the basis, we get

$$(1+\alpha_n)^{\frac{n}{2}}>1-\frac{n}{2}\alpha_n, \quad n>\left(1+\frac{n}{2}\alpha_n\right)^2=1+n\alpha_n+\frac{n^2}{4}\alpha_n^2.$$

Hence, it follows that

$$n > \frac{n^2}{4}\alpha_n^2,\quad \alpha_n^2 < \frac{4}{n},\quad \alpha_n < \frac{2}{\sqrt{n}},\quad \sqrt[n]{n} = 1 + \alpha_n < 1 + \frac{2}{\sqrt{n}}.$$

Note. Using Newton's binomial, it is easy to check that

$$\sqrt[n]{n} < 1 + \sqrt{\frac{2}{n}}.$$

Indeed,

$$\left(1 + \sqrt{\frac{2}{n}}\right)^n = 1 + n\sqrt{\frac{2}{n}} + \frac{n(n-1)}{2}\frac{2}{n} + \ldots > 1 + \\ + \frac{n(n-1)}{2}\frac{2}{n} = n.$$

Hence, it follows that

$$\sqrt[n]{n} < 1 + \sqrt{\frac{2}{n}}.$$

15. When $n = 1$ and $a_1 > -1$, the inequality is obvious

$$1 + a_1 \geqslant 1 + a_1.$$

Let us assume, that the inequality is true for $n = k$, that is

$$(1 + a_1)(1 + a_2)\ldots(1 + a_k) \geqslant \\ \geqslant 1 + a_1 + a_2 + \ldots + a_k.$$

Multiplying both members of the inequality by $(1 + a_{k+1})$, we get

$$(1 + a_1)(1 + a_2)\ldots(1 + a_k)(1 + a_{k+1}) \geqslant \\ \geqslant (1 + a_1 + a_2 + \ldots + a_k)(1 + a_{k+1}) = \\ = 1 + a_1 + \ldots + a_k + a_{k+1} + a_1a_{k+1} + \\ + a_2a_{k+1} + \ldots + a_ka_{k+1}.$$

Since the numbers $a_1, a_2, \ldots, a_k, a_{k+1}$ are of the same sign, then

$$a_1a_{k+1} + a_2a_{k+1} + \ldots + a_ka_{k+1} \geqslant 0$$

and, therefore,

$$(1 + a_1)(1 + a_2)\ldots(1 + a_k)(1 + a_{k+1}) \geqslant \\ \geqslant 1 + a_1 + a_2 + \ldots + a_k + a_{k+1},$$

that is, the inequality is proved also for $n = k + 1$.

This finally proves the inequality to be true

$$(1 + a_1)(1 + a_2) \ldots (1 + a_n) \geqslant$$
$$\geqslant 1 + a_1 + a_2 + \ldots + a_n$$

for all n.

16. If the polynomial $(a_1x - b_1)^2 + (a_2x - b_2)^2 + \ldots$ $\ldots + (a_nx - b_n)^2$ has a true root $x = x_1$, that is

$$(a_1x_1 - b_1)^2 + (a_2x_1 - b_2)^2 + \ldots + (a_nx_1 - b_n)^2 = 0,$$

then every number $a_1x_1 - b_1$, $a_2x_1 - b_2$, $\ldots$, $a_nx_1 - b_n$ is equal to zero, that is,

$$0 = a_1x_1 - b_1 = a_2x_1 - b_2 = \ldots = a_nx_1 - b_n,$$
$$x_1 = \frac{b_1}{a_1} = \frac{b_2}{a_2} = \ldots = \frac{b_n}{a_n}.$$

Thus we proved that the polynomial

$$(a_1x - b_1)^2 + (a_2x - b_2)^2 + \ldots + (a_nx - b_n)^2 =$$
$$= x^2 (a_1^2 + a_2^2 + \ldots + a_n^2) -$$
$$- 2x (a_1b_1 + a_2b_2 + \ldots + a_nb_n) +$$
$$+ (b_1^2 + b_2^2 + \ldots + b_n^2)$$

cannot have two different true roots and, therefore,

$$(a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 -$$
$$- (a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) \leqslant 0.$$

From this follows the inequality (19)

$$(a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 \leqslant$$
$$\leqslant (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2).$$

Notice, that the sign of equality holds only when the polynomial under consideration has a true root, i.e. when

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n}.$$

17. Using the inequality (19), we get

$$c_1^2 = \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^2 =$$
$$= \left(\frac{a_1}{\sqrt{n}}\frac{1}{\sqrt{n}} + \ldots + \frac{a_n}{\sqrt{n}}\frac{1}{\sqrt{n}}\right)^2 \leqslant$$
$$\leqslant \left(\frac{a_1^2}{n} + \frac{a_2^2}{n} + \ldots + \frac{a_n^2}{n}\right)\underbrace{\left(\frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n}\right)}_{n} =$$
$$= \frac{a_1^2 + a_2^2 + \ldots + a_n^2}{n} = c_2^2.$$

Hence, it follows that $c_1 \leqslant c_2$ (the arithmetic mean does not exceed the root-mean-square).

18. From the inequality

$$(\sqrt{n+1}+\sqrt{n-1})^2 = n+1+2\sqrt{n^2-1}+n-1 =$$
$$= 2n+2\sqrt{n^2-1} < 2n+2\sqrt{n^2} = 4n$$

it follows that

$$\sqrt{n+1}+\sqrt{n-1} < 2\sqrt{n},$$

$$\frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{n+1}+\sqrt{n-1}} =$$
$$= \frac{\sqrt{n+1}-\sqrt{n-1}}{(\sqrt{n+1}+\sqrt{n-1})(\sqrt{n+1}-\sqrt{n-1})} = \frac{\sqrt{n+1}-\sqrt{n-1}}{2}.$$

Multiplying by 2, we get

$$\frac{1}{\sqrt{n}} < \sqrt{n+1}-\sqrt{n-1}.$$

19. Setting in the inequality of Exercise 18$n = 2, 3, \ldots, \sqrt{n}$

$$\frac{1}{\sqrt{2}} < \sqrt{3}-1,$$
$$\frac{1}{\sqrt{3}} < \sqrt{4}-\sqrt{2},$$
$$\frac{1}{\sqrt{4}} < \sqrt{5}-\sqrt{3},$$
$$\frac{1}{\sqrt{5}} < \sqrt{6}-\sqrt{4},$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\frac{1}{\sqrt{n}} < \sqrt{n+1}-\sqrt{n-1}.$$

Combining the written inequalities, we get

$$\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}} < \sqrt{n+1}+\sqrt{n}-\sqrt{2}-1.$$

Adding 1 to both parts of the inequality, we finally get

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}+\ldots+\frac{1}{\sqrt{n}} <$$
$$< \sqrt{n+1}+\sqrt{n}-\sqrt{2}.$$

Note. It was proved in Sec. 2.1 that

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}>2\sqrt{n+1}-2\sqrt{2}+1.$$

The numbers $\sqrt{n+1}+\sqrt{n}-\sqrt{2}$ and $2\sqrt{n+1}-2\sqrt{2}+1$ differ from each other less than by 0.42. Each of these numbers could be taken for an approximate value of the sum

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}=z_n.$$

Let us notice without proving, that the number $\sqrt{n+1}+\sqrt{n}-\sqrt{2}$ differs less from the number z_n, than the number $2\sqrt{n+1}-2\sqrt{2}+1$.

20. The function $\frac{x^3}{x^4+5}$ takes a negative value when $x<0$. Therefore, the greatest value of the function is obtained for positive values of x.

Since

$$\frac{x^3}{x^4+5}=\frac{1}{5\left(\frac{1}{5}x+x^{-3}\right)},$$

then the greatest value of the function is reached in the same point in which the function $\frac{1}{5}x+x^{-3}$ takes the least value. It follows from Problem 4 Sec. 2.1 that the least value of this function is equal to

$$(1+3)\left(\frac{\frac{1}{5}}{3}\right)^{\frac{-3}{-3-1}}=4\left(\frac{1}{15}\right)^{\frac{3}{4}}.$$

The greatest value of the function $\frac{x^3}{x^4+5}$ is equal to

$$\frac{1}{5\cdot4\cdot\left(\frac{1}{15}\right)^{\frac{3}{4}}}=\frac{15^{\frac{3}{4}}}{20}=\frac{15}{20\sqrt[4]{15}}=\frac{3}{4\sqrt[4]{15}}.$$

To find the greatest value of the function $x^6-0.6x^{10}$, we get $y=x^6$. It is clear that $y\geqslant0$. The function

$$y-0.6y^{\frac{10}{6}}=0.6\left(\frac{10}{6}y-y^{\frac{10}{6}}\right)$$

takes the greatest value (see the note on p. 34) equal to

$$0.6\left(\frac{10}{6}-1\right)\left(\frac{\frac{10}{6}}{\frac{10}{6}}\right)^{\frac{\frac{10}{6}}{\frac{10}{6}-1}}=0.4.$$

21 Assuming in this exercise that $y=\frac{1}{x^2}$, we get

$$\sqrt[4]{x}+\frac{a}{x^2}=y^{-\frac{1}{4}}+ay.$$

The least value of the function $y^{-\frac{1}{4}}+ay$, as it follows from Problem 4 Sec. 2.1, is equal to

$$\left(1+\frac{1}{4}\right)(4a)^{\frac{1}{5}}=\frac{5}{4}(4a)^{\frac{1}{5}}.$$

Assuming $\frac{5}{4}(4a)^{\frac{1}{5}}=2.5$, we get

$$(4a)^{\frac{1}{5}}=2,\quad 4a=32,\quad a=8.$$

22. $S=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\ldots=$

$$=\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\ldots\right)-$$
$$-2\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\ldots\right)=$$
$$=\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\ldots\right)-$$
$$-\frac{2}{2^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)=$$
$$=\frac{1}{2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)=\frac{1}{2}\cdot\frac{\pi^2}{6}=\frac{\pi^2}{12}$$

(we have used the equality (27)).

23. Since $\alpha>0$, then $\alpha+1>1$ and, hence,

$$\left(1+\frac{1}{n}\right)^{1+\alpha}>1+\frac{1+\alpha}{n},$$
$$\left(1-\frac{1}{n}\right)^{1+\alpha}>1-\frac{1+\alpha}{n}.$$

Multiplying these inequalities by $n^{1+\alpha}$, we get

$$(n+1)^{1+\alpha} > n^{1+\alpha} + (1+\alpha)\, n^{\alpha},$$

$$(n-1)^{1+\alpha} > n^{1+\alpha} - (1+\alpha)\, n^{\alpha}.$$

From these inequalities it follows that

$$\frac{n^{1+\alpha}-(n-1)^{1+\alpha}}{1+\alpha} < n^{\alpha} < \frac{(n+1)^{1+\alpha}-n^{1+\alpha}}{1+\alpha}.$$

Write these inequalities for the values $n = 1, 2, 3, \ldots, n$:

$$\frac{1}{1+\alpha} < 1 < \frac{2^{1+\alpha}-1}{1+\alpha},$$

$$\frac{2^{1+\alpha}-1}{1+\alpha} < 2^{\alpha} < \frac{3^{1+\alpha}-2^{1+\alpha}}{1+\alpha},$$

. . . .

$$\frac{n^{1+\alpha}-(n-1)^{1+\alpha}}{1+\alpha} < n^{\alpha} < \frac{(n+1)^{1+\alpha}-n^{1+\alpha}}{1+\alpha}.$$

Adding them, we get

$$\frac{n^{1+\alpha}}{1+\alpha} < 1+2^{\alpha}+3^{\alpha}+\ \ldots\ +n^{\alpha} < \frac{(n+1)^{1+\alpha}-1}{1+\alpha} < \frac{(n+1)^{1+\alpha}}{1+\alpha}.$$

24. It follows from the inequalities of Exercise 23 that

$$\frac{1}{1+\alpha} < \frac{1+2^{\alpha}+3^{\alpha}+\ldots+n^{\alpha}}{n^{1+\alpha}} < \frac{\left(1+\frac{1}{n}\right)^{1+\alpha}}{1+\alpha}.$$

The left-hand member of the latter inequalities is a constant number $\frac{1}{1+\alpha}$, and the right-hand member tends to a limit equal to $\frac{1}{1+\alpha}$, when n tends to infinity. Hence, the mean member of the inequalities tends to the same limit as well, that is

$$\lim_{n\to\infty} \frac{1+2^{\alpha}+3^{\alpha}+\ldots+n^{\alpha}}{n^{1+\alpha}} = \frac{1}{1+\alpha}.$$

25. Let us introduce the designations

$$A^3 = a_1^3 + a_2^3 + \ldots + a_n^3,$$

$$B^3 = b_1^3 + b_2^3 + \ldots + b_n^3,$$

$$C^3 = c_1^3 + c_2^3 + \ldots + c_n^3,$$

$$x_1=\frac{a_1}{A},\quad x_2=\frac{a_2}{A},\quad \ldots,\quad x_n=\frac{a_n}{A},$$

$$y_1=\frac{b_1}{B},\quad y_2=\frac{b_2}{B},\quad \ldots,\quad y_n=\frac{b_n}{B},$$

$$z_1=\frac{c_1}{C},\quad z_2=\frac{c_2}{C},\quad \ldots,\quad z_n=\frac{c_n}{C}.$$

On the basis of the inequalities (7) we have

$$a_1b_1c_1=ABCx_1y_1z_1\leqslant ABC\,\frac{x_1^3+y_1^3+z_1^3}{3},$$

$$a_2b_2c_2=ABCx_2y_2z_2\leqslant ABC\,\frac{x_2^3+y_2^3+z_2^3}{3},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_nb_nc_n=ABCx_ny_nz_n\leqslant ABC\,\frac{x_n^3+y_n^3+z_n^3}{3}.$$

Adding the written inequalities we get

$$a_1b_1c_1+a_2b_2c_2+\ldots+a_nb_nc_n\leqslant$$
$$\leqslant ABC\left(\frac{x_1^3+x_2^3+\ldots+x_n^3}{3}+\frac{y_1^3+y_2^3+\ldots+y_n^3}{3}+\frac{z_1^3+z_2^3+\ldots+z_n^3}{3}\right).$$

Taking into consideration the introduced designations, it is easy to calculate that

$$x_1^3+x_2^3+\ldots+x_n^3=\frac{a_1^3+a_2^3+\ldots+a_n^3}{A^3}=\frac{A^3}{A^3}=1,$$

$$y_1^3+y_2^3+\ldots+y_n^3=1,\quad z_1^3+z_2^3+\ldots+z_n^3=1.$$

Hence,

$$a_1b_1c_1+a_2b_2c_2+\ldots+a_nb_nc_n\leqslant ABC\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)=ABC.$$

Raising both members of the inequality to a cube, we finally obtain

$$(a_1b_1c_1+a_2b_2c_2+\ldots+a_nb_nc_n)^3\leqslant A^3B^3C^3=$$
$$=(a_1^3+a_2^3+\ldots+a_n^3)(b_1^3+b_2^3+\ldots+b_n^3)(c_1^3+c_2^3+\ldots+c_n^3).$$

26. Write down the inequalities (24) for different values of n

$$\ln\frac{n+1}{n}<\frac{1}{n}<\ln\frac{n}{n-1},$$

$$\ln\frac{n+2}{n+1}<\frac{1}{n+1}<\ln\frac{n+1}{n},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ln\frac{kn+1}{kn}<\frac{1}{kn}<\ln\frac{kn}{kn-1}.$$

Adding these inequalities, we get

$$\ln \frac{(n+1)(n+2)\dots(kn+1)}{n(n+1)\dots kn} < \frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{kn} < \ln\left[\frac{n}{n-1}\cdot\frac{n+1}{n}\dots\frac{kn}{kn-1}\right],$$

that is

$$\ln \frac{kn+1}{n} = \ln\left(k+\frac{1}{n}\right) < \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{kn} < \ln \frac{kn}{n-1} = \ln\left(k+\frac{k}{n-1}\right).$$

If n tends to infinity, then $\ln\left(k+\frac{1}{n}\right)$ tends to $\ln k$ and $\ln\left(k+\frac{k}{n-1}\right)$ tends to the same limit as well. Therefore,

$$\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\dots+\frac{1}{kn}\right) = \ln k.$$

27. By virtue of Theorem 6

$$1+\frac{1}{\sqrt[3]{2}}+\dots+\frac{1}{\sqrt[3]{10^6}} = \frac{\sqrt[3]{2}}{2-\sqrt[3]{2}}\left(\frac{1}{\sqrt[3]{10^6+1}}+\frac{1}{\sqrt[3]{10^6+2}}+\dots+\frac{1}{\sqrt[3]{2\cdot 10^6}}\right) - \frac{\sqrt[3]{2}}{2-\sqrt[3]{2}}\left(1-\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{3}}-\dots-\frac{1}{\sqrt[3]{2\cdot 10^6}}\right).$$

The second addend is negative but greater than $-\frac{\sqrt[3]{2}}{2-\sqrt[3]{2}} > -1.9$. The first addend, according to the inequalities (28), satisfies the inequalities

$$\frac{3}{2}\left(\sqrt[3]{2\cdot 10^6+1}-\sqrt[3]{10^6+1}\right)\frac{\sqrt[3]{2}}{2-\sqrt[3]{2}} < \frac{\sqrt[3]{2}}{2-\sqrt[3]{2}} - \frac{1}{\sqrt[3]{10^6+1}}+\frac{1}{\sqrt[3]{10^6+2}}+\dots+\frac{1}{\sqrt[3]{2\cdot 10^6}} < \frac{\sqrt[3]{2}}{2-\sqrt[3]{2}}\left(\sqrt[3]{2\cdot 10^6}-\sqrt[3]{10^6}\right)\frac{3}{2} = 15{,}000.$$

Since the extreme terms of the latter inequalities differ from each other very slightly (less than 0.1), then

$$15{,}000-2<1+\frac{1}{\sqrt[3]{2}}+\ldots+\frac{1}{\sqrt[3]{10^6}}<15{,}000.$$

The mean number 14,999 differs from $\sum_{k=1}^{10^6}\frac{1}{\sqrt[3]{k}}$ less than by 1.

28. By virtue of Theorem 6

$$1+\frac{1}{2^\alpha}+\ldots+\frac{1}{n^\alpha}=$$

$$=\frac{2^\alpha}{2-2^\alpha}\left[\frac{1}{(n+1)^\alpha}+\frac{1}{(n+2)^\alpha}+\ldots+\frac{1}{(2n)^\alpha}\right]-$$

$$-\frac{2^\alpha}{2-2^\alpha}\left[1-\frac{1}{2^\alpha}+\frac{1}{3^\alpha}-\ldots-\frac{1}{(2n)^\alpha}\right]=A_n-B_n,$$

where

$$A_n=\frac{2^\alpha}{2-2^\alpha}\left[\frac{1}{(n+1)^\alpha}+\frac{1}{(n+2)^\alpha}+\ldots+\frac{1}{(2n)^\alpha}\right],$$

$$B_n=\frac{2^\alpha}{2-2^\alpha}\left[1-\frac{1}{2^\alpha}+\frac{1}{3^\alpha}-\ldots-\frac{1}{(2n)^\alpha}\right].$$

The number B_n is a partial sum of the series

$$\sum_{k=1}^{\infty}\frac{2^\alpha}{2-2^\alpha}(-1)^{k-1}\frac{1}{k^\alpha}.$$

This series is sign-alternating with monotonically decreasing (by absolute value) terms. Its remainder (by absolute value) is not greater than the absolute value of the first term of the remainder, that is, the number $\frac{2^\alpha}{2-2^\alpha}\cdot\frac{1}{n^\alpha}$.

Since this number tends to zero when $n\to\infty$, then the series converges and

$$\lim_{n\to\infty}B_n=\sum_{k=1}^{\infty}\frac{2^\alpha}{2-2^\alpha}(-1)^k\frac{1}{k^\alpha}=C,$$

that is $\gamma_n = B_n - C$ is an infinitesimally small value. Now, using the inequalities (28), we get

$$\frac{2^\alpha}{2-2^\alpha}[(2n-1)^{1-\alpha} - (n+1)^{1-\alpha}] < A_n <$$

$$< \frac{2^\alpha}{2-2^\alpha}[(2n)^{1-\alpha} - n^{1-\alpha}] = \frac{n^{1-\alpha}}{1-\alpha}.$$

Since the difference between the extreme terms of the inequalities tends to zero when $n \to \infty$, then $\delta_n = A_n - \frac{n^{1-\alpha}}{1-\alpha}$ is an infinitesimally small value.

Thus,

$$1 + \frac{1}{2^\alpha} + \ldots + \frac{1}{n^\alpha} = A_n - B_n =$$

$$= \frac{n^{1-\alpha}}{1-\alpha} - \delta_n - C + \gamma_n = \frac{n^{1-\alpha}}{1-\alpha} - C + \beta_n,$$

where $\beta_n = \delta_n + \gamma_n$ is an infinitesimally small value.

TO THE READER

Mir Publishers welcome your comments on the content, translation and design of this book.

We would also be pleased to receive any proposals you care to make about our future publications.

Our address is:

USSR, 129820, Moscow I-110, GSP Pervy Rizhsky Pereulok, 2

Mir Publishers

Printed in the Union of Soviet Socialist Republics

The booklet contains some particularly interesting inequalities playing an important role in various sections of higher mathematics. These inequalities are used for finding the greatest and least values as well as for calculating the limits. The booklet contains 63 problems and most of them are provided with detailed solutions.

The book is intented for students of senior classes of secondary schools.

Mir Publishers
Moscow