H. R. Dow.

KEY TO ALGEBRA.

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For the Use of Colleges and Schools.

BY

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THE Key to *Algebra for the Use of Colleges and Schools* has been published in consequence of applications from teachers and students. It is hoped that the Key will be acceptable to teachers by saving much of the time and trouble which they have to employ in correcting the mistakes of their pupils; and that it will be serviceable to those who enter on the study of Algebra without assistance, by affording them guidance and encouragement. The examples have been solved in the most simple and natural manner, in order to meet the difficulties which are most likely to occur; and the processes are given with sufficient detail to render them easily intelligible.

I. TODHUNTER.

ST JOHN'S COLLEGE,
November, 1870.

KEY

TO

ALGEBRA FOR COLLEGES AND SCHOOLS.

I.

1. $1+6+16=23.$ 2. $9+30-4=35.$

3. $3+24+36=63.$ 4. $4+96-12=88.$

5. $12+72+8-0=92.$ 6. $1+9+16+0=26.$

7. $\frac{24}{3}+\frac{24}{3}-\frac{24}{24}=8+8-1=15.$ 8. $256-256+12-6=6.$

9. $\frac{9+16}{8-3}=\frac{25}{5}=5.$ 10. $\frac{216-64}{36+24+16}=\frac{152}{76}=2.$

11. $\sqrt{81}-\sqrt[3]{8}+\sqrt{4}=9-2+2=9.$

12. $\sqrt{36}+\sqrt[3]{216}-\sqrt[3]{8}=6+6-2=10.$

13. $(9-5)(3+1)+(3+5)(5+7)-112=4\times4+8\times12-112$
$=16+96-112=0.$

14. $5\sqrt{(25-24)}+3\sqrt{(25+24)}=5\sqrt{1}+3\sqrt{49}=5+3\times7=5+21=26.$

15. $8\sqrt{(25-24)}+5\sqrt{(25+24)}=8\sqrt{1}+5\sqrt{49}=8+5\times7=8+35=43.$

16. $10+8\sqrt{(12+4)}-(10-8)\sqrt[3]{(12-4)}=10+8\sqrt{16}-2\sqrt[3]{8}$
$=10+8\times4-2\times2=10+32-4=38.$

17. $(10-5)(\sqrt{16}+10)+\sqrt{\{(16-10)(5+1)\}}=5(4+10)+\sqrt{\{6\times6\}}$
$=5\times14+6=70+6=76.$
$(16-1)\{\sqrt{100}+25\}+\sqrt{\{(16-5)(10+1)\}}$
$=15(10+25)+\sqrt{\{11\times11\}}=15\times35+11=525+11=536.$

18. $\sqrt[3]{\{(2+3)^2\times5\}}+\sqrt{\{(2+6)(5-4)\}}+\sqrt[3]{\{(5-3)^2\times2\}}$
$=\sqrt[3]{\{5^2\times5\}}+\sqrt[3]{8}+\sqrt[3]{\{2^2\times2\}}=5+2+2=9.$

II.

5. $4ab-x^2+3x^2-2ab+2ax+2bx=2ab+2x^2+2ax+2bx.$

6. $5a-3b+4c-7d-\{2a-2b+3c-d\}=5a-3b+4c-7d-2a+2b-3c+d$
$=3a-b+c-6d.$

7. $x^4+4x^3-2x^2+7x-1-\{x^4+2x^3-2x^2+6x-1\}$
$=x^4+4x^3-2x^2+7x-1-x^4-2x^3+2x^2-6x+1=2x^3+x.$

8. $3a^2-2ax+x^2-\{a^2-ax+x^2\}=3a^2-2ax+x^2-a^2+ax-x^2=2a^2-ax.$

9. $2(a-b)-c+d-\{a-b-2(c-d)\}=2a-2b-c+d-\{a-b-2c+2d\}$
$=2a-2b-c+d-a+b+2c-2d=a-b+c-d.$

10. $(a+b)x+(b+c)y-\{(a-b)x-(b-c)y\}$
$=ax+bx+by+cy-\{ax-bx-by+cy\}$
$=ax+bx+by+cy-ax+bx+by-cy=2bx+2by.$

11. $a-\{b-(c-d)\}=a-\{b-c+d\}=a-b+c-d.$

12. $a-\{(b-c)-d\}=a-\{b-c-d\}=a-b+c+d.$

13. $a+2b-6a-\{3b-(6a-6b)\}=a+2b-6a-\{3b-6a+6b\}$
$=a+2b-6a-3b+6a-6b=a-7b.$

14. $7a-\{3a-[4a-(5a-2a)]\}=7a-\{3a-[4a-(3a)]\}$
$=7a-\{3a-[4a-3a]\}=7a-\{3a-[a]\}=7a-\{3a-a\}$
$=7a-\{2a\}=7a-2a=5a.$

15. $3a-[a+b-\{a+b+c-(a+b+c+d)\}]$
$=3a-[a+b-\{a+b+c-a-b-c-d\}]$
$=3a-[a+b-\{-d\}]=3a-[a+b+d]$
$=3a-a-b-d=2a-b-d.$

16. $2x-[3y-\{4x-(5y-6x)\}]=2x-[3y-\{4x-5y+6x\}]$
$=2x-[3y-\{10x-5y\}]=2x-[3y-10x+5y]$
$=2x-[8y-10x]=2x-8y+10x=12x-8y.$

17. $a-[2b+\{3c-3a-(a+b)\}+2a-(b+3c)]$
$=a-[2b+\{3c-3a-a-b\}+2a-b-3c]$
$=a-[2b+3c-3a-a-b+2a-b-3c]$
$=a-2b-3c+3a+a+b-2a+b+3c=3a.$

18. $a-[5b-\{a-(3c-3b)+2c-(a-2b-c)\}]$
$=a-[5b-\{a-3c+3b+2c-a+2b+c\}]$
$=a-[5b-\{+5b\}]=a-[5b-5b]=a.$

19. $a+2x-\{b+y-[a-x-(b-2y)]\}$
$=a+2x-\{b+y-[a-x-b+2y]\}$
$=a+2x-\{b+y-a+x+b-2y\}$
$=a+2x-b-y+a-x-b+2y=2a+x-2b+y$
$=4+6-6+5=9.$

20. $4x^3-2x^2+x+1-(3x^3-x^2-x-7)-(x^3-4x^2+2x+8)$
$=4x^3-2x^2+x+1-3x^3+x^2+x+7-x^3+4x^2-2x-8$
$=3x^2.$

III.

29. Each expression will be found $=x^4+6x^3+11x^2+6x+1.$

30.
$$\begin{array}{l}
\quad a+x \\
\quad \underline{b+x} \\
ab+bx \\
\quad \underline{+ax+x^2} \\
ab+(a+b)x+x^2 \\
\quad \underline{c+x} \\
abc+(a+b)cx+cx^2 \\
\quad \underline{+\quad abx\quad +(a+b)x^2+x^3} \\
abc+(ab+bc+ca)x+(a+b+c)x^2+x^3.
\end{array}$$

31.

$$\begin{array}{l}
x-a \\
\underline{x-b} \\
x^2-ax \\
\underline{\quad -bx \qquad +ab} \\
x^2-(a+b)\,x+ab \\
\underline{x-c} \\
x^3-(a+b)\,x^2+abx \\
\underline{\quad -\quad cx^2 \quad +c\,(a+b)\,x-abc} \\
x^3-(a+b+c)\,x^2+(ab+bc+ca)\,x-abc \\
\underline{x-d} \\
x^4-(a+b+c)\,x^3+(ab+bc+ca)x^2-abcx \\
\underline{\qquad\qquad -dx^3+\ (a+b+c)\,dx^2\ -(ab+bc+ca)\,dx+abcd} \\
x^4-(a+b+c+d)\,x^3+x^2\,(ab+bc+ca+ad+bd+cd)-\&c.
\end{array}$$

32. We multiply together the first and second factors, then the third and fourth factors; and then multiply together the two results: and we use the method of Art. 56. Thus

$$\begin{aligned}
&(a+b-c)(a+c-b)=(a+b-c)\{a-(b-c)\}=a^2-(b-c)^2=a^2-b^2-c^2+2bc,\\
&(b+c-a)(a+b+c)= \qquad (b+c)^2-a^2=b^2+c^2+2bc-a^2\,;\\
&(b^2+c^2+2bc-a^2)(a^2-b^2-c^2+2bc)=(2bc+b^2+c^2-a^2)\{2bc-(b^2+c^2-a^2)\}\\
&\quad =4b^2c^2-(b^2+c^2-a^2)^2=4b^2c^2-(b^4+c^4+a^4+2b^2c^2-2c^2a^2-2a^2b^2)\\
&\quad =2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4.
\end{aligned}$$

33.

$$\begin{aligned}
(a+b)(b+c)&=b^2+ab+bc+ac,\\
(c+d)(d+a)&=d^2+cd+ac+ad,\\
(a+c)(b-d)&=ab+bc-ad-cd\,;
\end{aligned}$$

$$\begin{aligned}
&b^2+ab+bc+ac-(d^2+cd+ac+ad)-(ab+bc-ad-cd)\\
&=b^2+ab+bc+ac-d^2-cd-ac-ad-ab-bc+ad+cd\\
&=b^2-d^2.
\end{aligned}$$

34.

$$\begin{aligned}
(a+b+c+d)^2&=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd,\\
(a-b-c+d)^2&=a^2+b^2+c^2+d^2-2ab-2ac+2ad+2bc-2bd-2cd,\\
(a-b+c-d)^2&=a^2+b^2+c^2+d^2-2ab+2ac-2ad-2bc+2bd-2cd,\\
(a+b-c-d)^2&=a^2+b^2+c^2+d^2+2ab-2ac-2ad-2bc-2bd+2cd.
\end{aligned}$$

Thus the sum is $4\,(a^2+b^2+c^2+d^2)$.

35. See Art. 55.

36.

$$\begin{aligned}
&(a+b+c)^2-(ab+ac-a^2)-(ba+bc-b^2)-(ca+cb-c^2)\\
&=(a+b+c)^2-ab-ac+a^2-ba-bc+b^2-ca-cb+c^2=\&c.
\end{aligned}$$

37. $3\,(x-y)^2\,(x+y)=3\,(x^2-2xy+y^2)(x+y)=3(x^3-x^2y-xy^2+y^3)$,
$3\,(x+y)^2\,(x-y)=3\,(x^2+2xy+y^2)(x-y)=3\,(x^3+x^2y-xy^2-y^3)$;
and see Art. 55 for $(x-y)^3$ and $(x+y)^3$.

38. Use the result of Example 32.

39. Use the result of Example 32.

40. It will be found that

$$(x+y)^8=x^8+8x^7y+28x^6y^2+56x^5y^3+70x^4y^4+56x^3y^5+28x^2y^6+8xy^7+y^8.$$

41. $(x^2+xy+y^2)^2-(x^2-xy+y^2)^2=(2x^2+2y^2)\,(2xy)=4xy\,(x^2+y^2)$; see Arts. 55 and 56.

42. See the preceding solution.

43. It will be found that $(x^2-3x+2)^2=x^4-6x^3+13x^2-12x+4$.

44. When we arrange the expressions suitably we find we have to multiply $x^5-ax^4-a^4x+a^5$ by $x^3-ax^2-a^2x+a^3$.

45. $(a+b)^2=a^2+2ab+b^2$; $(a-b)^3=a^3-3a^2b+3ab^2-b^3$.

46.
$$s(s-2b)(s-2c)=s^3-2s^2(b+c)+4bcs,$$
$$s(s-2c)(s-2a)=s^3-2s^2(c+a)+4cas,$$
$$s(s-2a)(s-2b)=s^3-2s^2(a+b)+4abs;$$

by addition we obtain
$$3s^3-4s^2(a+b+c)+4s(bc+ca+ab),$$
that is
$$3s^3-4s^3+4s(bc+ca+ab),$$
that is
$$-s^3+4s(bc+ca+ab).$$
Again
$$(s-2a)(s-2b)(s-2c)+8abc$$
$$=s^3-2s^2(a+b+c)+4s(bc+ca+ab)-8abc+8abc$$
that is
$$-s^3+4s(bc+ca+ab).$$

IV.

18. The product is $x^6-24x^4+144x^2-256$.

19. The product is $x^6-5x^4+3x^3+6x^2-7x+2$.

20. The product is $2x^7-8x^6-3x^5+12x^4-7x^3+28x^2+3x-12$.

21. See Art. 70 for the factors of a^3+x^3 and of $a^4+a^2x^2+x^4$.

22. The product is $x^6-2x^5a-x^4a^2+4x^3a^3-x^2a^4-2xa^5+a^6$.

23.
$$\begin{array}{r|l|l} a^2-bc & a^3+a^2(b+c)-abc-b^2c-bc^2 & a+b+c \\ & \underline{a^3 \qquad\qquad -abc} & \\ & a^2(b+c) \qquad -b^2c-bc^2 & \\ & \underline{a^2(b+c) \qquad -bc(b+c)} & \end{array}$$

25. The product is $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

27.
$$\begin{array}{r|l|l} x+y-1 & x^3 \qquad\qquad +3yx+y^3-1 & x^2-x(y-1)+y^2+y+1 \\ & \underline{x^3+x^2(y-1)} & \\ & -x^2(y-1)+3yx+y^3-1 & \\ & \underline{-x^2(y-1)-x(y^2-2y+1)} & \\ & x(y^2+y+1)+y^3-1 & \\ & \underline{x(y^2+y+1)+y^3-1} & \end{array}$$

28.
$$\begin{array}{r|l|l} a+b-c & a^3 \qquad\qquad +3abc+b^3-c^3 & a^2-a(b-c)+b^2+bc+c^2 \\ & \underline{a^3+a^2(b-c)} & \\ & -a^2(b-c)+3abc+b^3-c^3 & \\ & \underline{-a^2(b-c)-a(b^2-2bc+c^2)} & \\ & a(b^2+bc+c^2)+b^3-c^3 & \\ & \underline{a(b^2+bc+c^2)+b^3-c^3} & \end{array}$$

30.
$$\begin{array}{r|l|l} a(b+c)-bc & a^2(b^2-c^2)+2abc^2-b^2c^2 & a(b-c)+bc \\ & \underline{a^2(b^2-c^2)-abc(b-c)} & \\ & a(bc^2+b^2c)-b^2c^2 & \\ & \underline{a(bc^2+b^2c)-b^2c^2} & \end{array}$$

31. $(a+b-c)(a-b+c)=a^2-b^2-c^2+2bc$

$$(a^2-b^2-c^2+2bc)(b+c-a)\div(a^2-b^2-c^2+2bc)=b+c-a.$$

32. The dividend arranged according to powers of a is

$$a^2(b+c)+a(b^2+2bc+c^2)+bc(b+c)$$

$$
\begin{array}{r|l|l}
a+b & a^2(b+c)+a(b^2+2bc+c^2)+bc(b+c) & a(b+c)+bc+c^2 \\
 & \underline{a^2(b+c)+a(b^2+bc)} & \\
 & a(bc+c^2)+bc(b+c) & \\
 & \underline{a(bc+c^2)+bc(b+c)} &
\end{array}
$$

33. The dividend arranged according to powers of a is

$$a^6-3a^4bc+3a^2b^2c^2+7b^3c^3$$

$$
\begin{array}{r|l|l}
a^2+bc & a^6-3a^4bc+3a^2b^2c^2+7b^3c^3 & a^4-4a^2bc+7b^2c^2 \\
 & \underline{a^6+\ a^4bc} & \\
 & -4a^4bc+3a^2b^2c^2 & \\
 & \underline{-4a^4bc-4a^2b^2c^2} & \\
 & 7a^2b^2c^2+7b^3c^3 & \\
 & \underline{7a^2b^2c^2+7b^3c^3} &
\end{array}
$$

34. It is convenient here first to divide by $x-a$, and then to divide the quotient by $a+b$.

Dividing by $x-a$ we have by Art. 70

$$b(x^2+ax+a^2)+ax(x+a)+a^3,$$

that is

$$a^3+a^2(b+x)+a(bx+x^2)+bx^2$$

$$
\begin{array}{r|l|l}
a+b & a^3+a^2(b+x)+a(bx+x^2)+bx^2 & a^2+ax+x^2 \\
 & \underline{a^3+a^2b} & \\
 & a^2x+a(bx+x^2)+bx^2 & \\
 & \underline{a^2x+abx} & \\
 & ax^2+bx^2 & \\
 & \underline{ax^2+bx^2} &
\end{array}
$$

35.

$$
\begin{array}{r|l|l}
y-x+z & y^3(x+2z)-y^2xz+y(-x^3+xz^2-2z^3)+x^3z-xz^3 & y^2(x+2z)+y(x^2-2z^2)-xz(x+z) \\
 & \underline{y^3(x+2z)+y^2(-x^2-xz+2z^2)} & \\
 & y^2(x^2-2z^2)+y(-x^3+xz^2-2z^3)+x^3z-xz^3 & \\
 & \underline{y^2(x^2-2z^2)+y(-x^3+x^2z+2xz^2-2z^3)} & \\
 & -y(x^2z+xz^2)\quad +x^3z-xz^3 & \\
 & \underline{-y(x^2z+xz^2)\quad +x^3z-xz^3} &
\end{array}
$$

36.

$$
\begin{array}{r|l|l}
a-b+c & a^2(b+c)+a(-b^2+c^2+bc)-b^2c+c^2b & a(b+c)+bc \\
 & \underline{a^2(b+c)+a(-b^2+c^2)} & \\
 & abc-b^2c+c^2b & \\
 & \underline{abc-b^2c+c^2b} &
\end{array}
$$

37. The divisor is $(a-b)(x+a+b)$; so we first divide the dividend by $a-b$: this gives $x^3-x(a^2+ab+b^2)+ab(a+b)$

$$
\begin{array}{r|l|l}
x+a+b & x^3\qquad -x(a^2+ab+b^2)+ab(a+b) & x^2-x(a+b)+ab \\
 & \underline{x^3+x^2(a+b)} & \\
 & -x^2(a+b)-x(a^2+ab+b^2)+ab(a+b) & \\
 & \underline{-x^2(a+b)-x(a^2+2ab+b^2)} & \\
 & xab+ab(a+b) & \\
 & \underline{xab+ab(a+b)} &
\end{array}
$$

38. The dividend
$$=a(x^2-b^2)-x(x^2-b^2)=(a-x)(x^2-b^2)=(a-x)(x+b)(x-b).$$

39.
$$\begin{array}{r|l}
a^2-a(b+c)+bc & a^3(b-c) \qquad\qquad +a(-b^3+c^3)+b^3c-bc^3 \;\big(\; a(b-c)+b^2-c^2 \\
 & \underline{a^3(b-c)-a^2(b^2-c^2)+a(b^2c-bc^2)} \\
 & \quad a^2(b^2-c^2)+a(-b^3-b^2c+bc^2+c^3)+b^3c-bc^3 \\
 & \quad \underline{a^2(b^2-c^2)-a(b^3+b^2c-bc^2-c^3)+b^3c-bc^3}
\end{array}$$

40. The dividend $=(a^2+b^2+c^2)x^2+(a^2+b^2+c^2)y^2$.

41. The dividend
$$=b(a^2-x^2)+x(a^2-x^2)=(b+x)(a^2-x^2)=(b+x)(a-x)(a+x).$$

42. The expression $=(a-d)^2-(b-c)^2$. See Art. 70.

43. First divide by $x+a$: this gives
$$b(x^2-ax+a^2)+ax(x-a)+a^3,$$
that is
$$b(x^2-ax+a^2)+a(x^2-ax+a^2),$$
that is
$$(b+a)(x^2-ax+a^2).$$

44. The dividend $=2(x^6+6x^4y^2+6x^2y^4+y^6)$; or it may be put in the form $2(x^2+y^2)^3+6x^2y^2(x^2+y^2)$.

45. The dividend $=7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7xy^6$; the divisor $=x^4+2x^3y+3x^2y^2+2xy^3+y^4$.

47. $a^{16}-x^{16}=(a^8+x^8)(a^8-x^8)$; then resolve the second factor, and so on.

48. $4a^2b^2-(a^2+b^2-c^2)^2=(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)$
$$=\{(a+b)^2-c^2\}\{c^2-(a-b)^2\}=\&c.$$

49. $4(ad+bc)^2-(a^2-b^2-c^2+d^2)^2$
$$=\{2(ad+bc)+a^2-b^2-c^2+d^2\}\{2(ad+bc)-a^2+b^2+c^2-d^2\}=\&c.$$

50. It will be found that the proposed expression
$$=(a^2+b^2+c^2)(x^2+y^2+z^2).$$

V.

1. Each expression becomes $4(a+z)^2$.

2. $5\left(1-\frac{2}{3}\right)\sqrt[3]{\{8\times 8^2\}}-\frac{2}{3}\sqrt{\{8\times 8\}}+1=\frac{5}{3}\times 8-\frac{2}{3}\times 8+1=9.$

3. $$\left(\frac{50}{7}+10\right)\sqrt{\left\{\left(5-\frac{1}{2}\right)\frac{9}{2}\right\}}-\frac{15}{7}\sqrt[3]{\left\{\frac{9}{2}\times\frac{9}{2}\left(5-\frac{1}{2}\right)\right\}}+\frac{5}{2}$$
$$=\frac{120}{7}\times\frac{9}{2}-\frac{15}{7}\times\frac{9}{2}+\frac{5}{2}=70.$$

4. $$\left(\frac{4}{5}+2\right)\sqrt[3]{\left\{\left(\frac{10}{3}-2\right)\frac{4}{3}\times\frac{4}{3}\right\}}-\frac{4}{5}\sqrt{\left\{\frac{4}{3}\left(\frac{10}{3}-2\right)\right\}}+\frac{10}{3}$$
$$=\frac{14}{5}\times\frac{4}{3}-\frac{4}{5}\times\frac{4}{3}+\frac{10}{3}=6.$$

6. Each expression
$$=4\{a^4+b^4+c^4-2a^3(b+c)-2b^3(a+c)-2c^3(a+b)+3a^2b^2+3b^2c^2+3c^2a^2\}.$$

7. $(s-a)^2+(s-b)^2+(s-c)^2+s^2=4s^2-2s(a+b+c)+a^2+b^2+c^2$
$$=4s^2-4s^2+a^2+b^2+c^2=a^2+b^2+c^2.$$

8. $2(s-a)(s-b)+2(s-b)(s-c)+2(s-c)(s-a)$
$$=6s^2-4s(a+b+c)+2ab+2bc+2ca$$
$$=-2s^2+2ab+2bc+2ca=-\frac{1}{2}(a+b+c)^2+2ab+2bc+2ca$$
$$=-\frac{1}{2}(a^2+b^2+c^2)+ab+bc+ca.$$

And $2s^2-a^2-b^2-c^2=\frac{1}{2}(a+b+c)^2-a^2-b^2-c^2=$ &c.

9.
$$\begin{aligned} a(s-b)(s-c)&=a\{s^2-s(b+c)+bc\},\\ b(s-c)(s-a)&=b\{s^2-s(c+a)+ac\},\\ c(s-a)(s-b)&=c\{s^2-s(a+b)+ab\}.\end{aligned}$$

By addition we get
$$2s^3-2s(ab+bc+ca)+3abc.$$

And $2(s-a)(s-b)(s-c)=2\{s^3-s^2(a+b+c)+s(ab+bc+ca)-abc\}$
$$=2\{-s^3+s(ab+bc+ca)-abc\}.$$

The sum of these two results is abc.

10. See Art. 55.

11. $(s-a_1)^2=s^2-2sa_1+a_1^2$, $(s-a_2)^2=s^2-2sa_2+a_2^2$, &c.;
then by adding the n expressions we obtain
$$ns^2-2s\{a_1+a_2+\ldots+a_n\}+a_1^2+a_2^2+\ldots+a_n^2;$$
and
$$2s\{a_1+a_2+\ldots+a_n\}=ns^2:$$
so we have the required result.

12. $(\sigma^2-a^2)(\sigma^2-b^2)+(\sigma^2-b^2)(\sigma^2-c^2)+(\sigma^2-c^2)(\sigma^2-a^2)$
$$=3\sigma^4-2\sigma^2(a^2+b^2+c^2)+a^2b^2+b^2c^2+c^2a^2$$
$$=3\sigma^4-4\sigma^4+a^2b^2+b^2c^2+c^2a^2$$
$$=-\frac{(a^2+b^2+c^2)^2}{4}+a^2b^2+b^2c^2+c^2a^2$$
$$=\frac{1}{4}\left\{2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4\right\}.$$

And $4s(s-a)(s-b)(s-c)=\dfrac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4};$
then see Ex. 32 of Chapter III.

VI.

1.
$$x^2-x-2\,\Big)\,\begin{array}{r} x^2-3x+2 \\ \underline{x^2-x-2} \\ -2x+4 \end{array}\,\Big(\,1$$

Divide by 2 and change signs;
$$x-2\,\Big)\,\begin{array}{r} x^2-x-2 \\ \underline{x^2-2x} \\ x-2 \\ \underline{x-2} \end{array}\,\Big(\,x+1$$

2.
$$\begin{array}{r|l|l}
x^3+3x^2+4x+12 & x^3+4x^2+4x+3 & 1 \\
 & \underline{x^3+3x^2+4x+12} & \\
 & x^2 \qquad -9 & \\
\end{array}$$

$$\begin{array}{r|l|l}
x^2-9 & x^3+3x^2+4x+12 & x+3 \\
 & \underline{x^3 \qquad -9x} & \\
 & 3x^2+13x+12 & \\
 & \underline{3x^2 \qquad -27} & \\
 & 13x+39 & \\
\end{array}$$

Divide by 13;
$$\begin{array}{r|l|l}
x+3 & x^2 \qquad -9 & x-3 \\
 & \underline{x^2+3x} & \\
 & -3x-9 & \\
 & \underline{-3x-9} & \\
\end{array}$$

3.
$$\begin{array}{r|l|l}
x^3+x^2+x-3 & x^3+3x^2+5x+3 & 1 \\
 & \underline{x^3+x^2+x-3} & \\
 & 2x^2+4x+6 & \\
\end{array}$$

Divide by 2;
$$\begin{array}{r|l|l}
x^2+2x+3 & x^3+x^2+x-3 & x-1 \\
 & \underline{x^3+2x^2+3x} & \\
 & -x^2-2x-3 & \\
 & \underline{-x^2-2x-3} & \\
\end{array}$$

4.
$$\begin{array}{r|l|l}
x^3+1 & x^3+mx^2+mx+1 & 1 \\
 & \underline{x^3 \qquad +1} & \\
 & mx^2+mx & \\
\end{array}$$

Divide by mx;
$$\begin{array}{r|l|l}
x+1 & x^3 \qquad +1 & x^2-x+1 \\
 & \underline{x^3+x^2} & \\
 & -x^2 \qquad +1 & \\
 & \underline{-x^2-x} & \\
 & x+1 & \\
 & \underline{x+1} & \\
\end{array}$$

5. Divide the first expression by x;

$$\begin{array}{r|l|l}
3x^2+ax-4a^2 & 6x^2-7ax-20a^2 & 2 \\
 & \underline{6x^2+2ax-8a^2} & \\
 & -9ax-12a^2 & \\
\end{array}$$

Divide by $-3a$;
$$\begin{array}{r|l|l}
3x+4a & 3x^2+ax-4a^2 & x-a \\
 & \underline{3x^2+4ax} & \\
 & -3ax-4a^2 & \\
 & \underline{-3ax-4a^2} & \\
\end{array}$$

6.
$$\begin{array}{r|l|l}
x^2-y^2 & x^5 \qquad -y^5 & x^3+xy^2 \\
 & \underline{x^5-x^3y^2} & \\
 & x^3y^2 \qquad -y^5 & \\
 & \underline{x^3y^2-xy^4} & \\
 & xy^4-y^5 & \\
\end{array}$$

Divide by y^4;
$$\begin{array}{r|l|l}
x-y & x^2 \qquad -y^2 & x+y \\
 & \underline{x^2-xy} & \\
 & xy-y^2 & \\
 & \underline{xy-y^2} & \\
\end{array}$$

7.

$$\begin{array}{r|l|l} 3x^3-13x^2+23x-21 & 6x^3+x^2-44x+21 & 2 \\ & \underline{6x^3-26x^2+46x-42} & \\ & 27x^2-90x+63 & \end{array}$$

Divide by 9;

$$\begin{array}{r|l|l} 3x^2-10x+7 & 3x^3-13x^2+23x-21 & x-1 \\ & \underline{3x^3-10x^2+7x} & \\ & -3x^2+16x-21 & \\ & \underline{-3x^2+10x-7} & \\ & 6x-14 & \end{array}$$

Divide by 2;

$$\begin{array}{r|l|l} 3x-7 & 3x^2-10x+7 & x-1 \\ & \underline{3x^2-7x} & \\ & -3x+7 & \\ & \underline{-3x+7} & \end{array}$$

8.

$$\begin{array}{r|l|l} x^3-x^2-2x+2 & x^4-3x^3+2x^2+x-1 & x-2 \\ & \underline{x^4-x^3-2x^2+2x} & \\ & -2x^3+4x^2-x-1 & \\ & \underline{-2x^3+2x^2+4x-4} & \\ & 2x^2-5x+3 & \end{array}$$

Multiply the new dividend by 2;

$$\begin{array}{r|l|l} 2x^2-5x+3 & 2x^3-2x^2-4x+4 & x \\ & \underline{2x^3-5x^2+3x} & \\ & 3x^2-7x+4 & \end{array}$$

Multiply by 2, and continue the division;

$$\begin{array}{l|l} 6x^2-14x+8 & 3 \\ \underline{6x^2-15x+9} & \end{array}$$

$$\begin{array}{r|l|l} x-1 & 2x^2-5x+3 & 2x-3 \\ & \underline{2x^2-2x} & \\ & -3x+3 & \\ & \underline{-3x+3} & \end{array}$$

9.

$$\begin{array}{r|l|l} x^3-8x^2+19x-14 & x^4-7x^3+8x^2+28x-48 & x+1 \\ & \underline{x^4-8x^3+19x^2-14x} & \\ & x^3-11x^2+42x-48 & \\ & \underline{x^3-8x^2+19x-14} & \\ & -3x^2+23x-34 & \end{array}$$

Multiply the new dividend by 3, and change the signs of the divisor;

$$\begin{array}{r|l|l} 3x^2-23x+34 & 3x^3-24x^2+57x-42 & x \\ & \underline{3x^3-23x^2+34x} & \\ & -x^2+23x-42 & \end{array}$$

Multiply by -3, and continue the division;

$$\begin{array}{l|l} 3x^2-69x+126 & 1 \\ \underline{3x^2-23x+34} & \\ -46x+92 & \end{array}$$

Divide by -46;

$$\begin{array}{r|l|l} x-2 & 3x^2-23x+34 & 3x-17 \\ & \underline{3x^2-6x} & \\ & -17x+34 & \\ & \underline{-17x+34} & \end{array}$$

10.
$$\begin{array}{r|r|l} x^4+2x^3-x-2 & x^4-x^3+2x^2+x+3 & 1 \\ & \underline{x^4+2x^3 \qquad\quad -x-2} & \\ & -3x^3+2x^2+2x+5 & \end{array}$$

Multiply the new dividend by 3, and change the signs of the divisor;
$$\begin{array}{r|r|l} 3x^3-2x^2-2x-5 & 3x^4+6x^3 \qquad -3x-6 & x \\ & \underline{3x^4-2x^3-2x^2-5x} & \\ & 8x^3+2x^2+2x-6 & \end{array}$$

Multiply by 3, divide by 2, and continue the division;
$$\begin{array}{r|l} 12x^3+3x^2+3x-9 & 4 \\ \underline{12x^3-8x^2-8x-20} & \\ 11x^2+11x+11 & \end{array}$$

Divide by 11;
$$\begin{array}{r|r|l} x^2+x+1 & 3x^3-2x^2-2x-5 & 3x-5 \\ & \underline{3x^3+3x^2+3x} & \\ & -5x^2-5x-5 & \\ & \underline{-5x^2-5x-5} & \end{array}$$

11. Multiply the first expression by 3;
$$\begin{array}{r|r|l} 3x^3+5x^2-x+2 & 12x^4+27x^3+6x^2-6x-12 & 4x \\ & \underline{12x^4+20x^3-4x^2+8x} & \\ & 7x^3+10x^2-14x-12 & \end{array}$$

Multiply by 3, and continue the division;
$$\begin{array}{r|l} 21x^3+30x^2-42x-36 & 7 \\ \underline{21x^3+35x^2-7x+14} & \\ -5x^2-35x-50 & \end{array}$$

Divide by -5;
$$\begin{array}{r|r|l} x^2+7x+10 & 3x^3+5x^2-x+2 & 3x-16 \\ & \underline{3x^3+21x^2+30x} & \\ & -16x^2-31x+2 & \\ & \underline{-16x^2-112x-160} & \\ & 81x+162 & \end{array}$$

Divide by 81;
$$\begin{array}{r|r|l} x+2 & x^2+7x+10 & x+5 \\ & \underline{x^2+2x} & \\ & 5x+10 & \\ & \underline{5x+10} & \end{array}$$

12. Multiply the first expression by 2;
$$\begin{array}{r|r|l} 4x^3-18x^2+19x-3 & 4x^4-24x^3+38x^2-12x+18 & x \\ & \underline{4x^4-18x^3+19x^2-3x} & \\ & -6x^3+19x^2-9x+18 & \end{array}$$

Multiply by -2, and continue the division;
$$\begin{array}{r|l} 12x^3-38x^2+18x-36 & 3 \\ \underline{12x^3-54x^2+57x-9} & \\ 16x^2-39x-27 & \end{array}$$

Multiply the new dividend by 4;

$$\begin{array}{r}16x^2-39x-27\,\big)\,16x^3-72x^2+76x-12\,\big(\,x\\ \underline{16x^3-39x^2-27x}\\ -33x^2+103x-12\end{array}$$

Multiply by -16, and continue the division;

$$\begin{array}{r}528x^2-1648x+192\,\big(\,33\\ \underline{528x^2-1287x-891}\\ -361x+1083\end{array}$$

Divide by -361; $\quad x-3\,\big)\,16x^2-39x-27\,\big(\,16x+9$

$$\begin{array}{r}\underline{16x^2-48x}\\ 9x-27\\ \underline{9x-27}\end{array}$$

13. Divide the first expression by x, and multiply the second by 3;

$$\begin{array}{r}6x^3+x^2-1\,\big)\,12x^3-18x^2-12x+9\,\big(\,2\\ \underline{12x^3+2x^2-2}\\ -20x^2-12x+11\end{array}$$

Multiply the new dividend by 10, and change the signs of the divisor;

$$\begin{array}{r}20x^2+12x-11\,\big)\,60x^3+10x^2-10\,\big(\,3x\\ \underline{60x^3+36x^2-33x}\\ -26x^2+33x-10\end{array}$$

Multiply by -10, and continue the division;

$$\begin{array}{r}260x^2-330x+100\,\big(\,13\\ \underline{260x^2+156x-143}\\ -486x+243\end{array}$$

Divide by -243; $\quad 2x-1\,\big)\,20x^2+12x-11\,\big(\,10x+11$

$$\begin{array}{r}\underline{20x^2-10x}\\ 22x-11\\ \underline{22x-11}\end{array}$$

14. Multiply the second expression by 2;

$$\begin{array}{r}12x^2-15yx+3y^2\,\big)\,12x^3-12yx^2+4y^2x-4y^3\,\big(\,x\\ \underline{12x^3-15yx^2+3y^2x}\\ 3yx^2+y^2x-4y^3\end{array}$$

Divide by y; $\quad$

$$\begin{array}{r}3x^2+yx-4y^2\,\big)\,12x^2-15yx+3y^2\,\big(\,4\\ \underline{12x^2+4yx-16y^2}\\ -19yx+19y^2\end{array}$$

Divide by $-19y$; $\quad x-y\,\big)\,3x^2+yx-4y^2\,\big(\,3x+4y$

$$\begin{array}{r}\underline{3x^2-3yx}\\ 4yx-4y^2\\ \underline{4yx-4y^2}\end{array}$$

15.

$$\begin{array}{r|l|l}
2x^5-11x^2-9 & 4x^5+11x^4\qquad\qquad+81 & 2 \\
 & \underline{4x^5\qquad\qquad-22x^2-18} & \\
 & \qquad 11x^4+22x^2+99 &
\end{array}$$

Divide by 11;

$$\begin{array}{r|l|l}
x^4+2x^2+9 & 2x^5\qquad\qquad-11x^2\qquad-9 & 2x \\
 & \underline{2x^5+4x^3\qquad\qquad+18x} & \\
 & \quad -4x^3-11x^2-18x-9 &
\end{array}$$

Multiply the new dividend by 4, and change the signs of the divisor;

$$\begin{array}{r|l|l}
4x^3+11x^2+18x+9 & 4x^4\qquad+\;8x^2\qquad+36 & x \\
 & \underline{4x^4+11x^3+18x^2+9x} & \\
 & \quad -11x^3-10x^2-9x+36 &
\end{array}$$

Multiply by -4, and continue the division;

$$\begin{array}{l|l}
44x^3+\;\;40x^2+\;\;36x-144 & 11 \\
\underline{44x^3+121x^2+198x+\;\;99} & \\
\quad -\;81x^2-162x-243 &
\end{array}$$

Divide by -81;

$$\begin{array}{r|l|l}
x^2+2x+3 & 4x^3+11x^2+18x+9 & 4x+3 \\
 & \underline{4x^3+\;\;8x^2+12x} & \\
 & \qquad 3x^2+\;\;6x+9 & \\
 & \qquad \underline{3x^2+\;\;6x+9} &
\end{array}$$

16. Divide the first expression by a^2, and the second by ax; as the factor a is common to these it will be a factor of the G.C.M.;

$$\begin{array}{r|l|l}
2a^2+3ax-9x^2 & 6a^3-17a^2x+14ax^2-\quad 3x^3 & 3a-13x \\
 & \underline{6a^3+\;\;9a^2x-27ax^2} & \\
 & \quad -26a^2x+41ax^2-\quad 3x^3 & \\
 & \quad \underline{-26a^2x-39ax^2+117x^3} & \\
 & \qquad\qquad 80ax^2-120x^3 &
\end{array}$$

Divide by $40x^2$;

$$\begin{array}{r|l|l}
2a-3x & 2a^2+3ax-9x^2 & a+3x \\
 & \underline{2a^2-3ax} & \\
 & \qquad 6ax-9x^2 & \\
 & \qquad \underline{6ax-9x^2} &
\end{array}$$

17.

$$\begin{array}{r|l|l}
2x^2-13x+18 & 2x^3+(2a-\;\;9)x^2-(\;\;9a+\;\;6)\,x+27 & x+a+2 \\
 & \underline{2x^3\qquad\quad-13\,x^2\qquad\qquad\quad+18\;x} & \\
 & \qquad (2a+\;\;4)x^2-(\;\;9a+24)\,x+27 & \\
 & \qquad \underline{(2a+\;\;4)x^2-(13a+26)\,x+18a+36} & \\
 & \qquad\qquad\qquad\quad (\;\;4a+\;\;2)\,x-18a-9 &
\end{array}$$

Divide by $2a+1$;

$$\begin{array}{r|l|l}
2x-9 & 2x^2-13x+18 & x-2 \\
 & \underline{2x^2-\;\;9x} & \\
 & \quad -\;\;4x+18 & \\
 & \quad \underline{-\;\;4x+18} &
\end{array}$$

18. Multiply the first expression by 2, and divide the second by by;

$$\begin{array}{l|l|l}
2a^2x^2 - abxy - b^2y^2 & 2a^3x^3 - 2a^2bx^2y + 2ab^2xy^2 - 2b^3y^3 & ax \\
 & \underline{2a^3x^3 - a^2bx^2y - ab^2xy^2} & \\
 & \quad - a^2bx^2y + 3ab^2xy^2 - 2b^3y^3 &
\end{array}$$

Divide by by, multiply by -2, and continue the division;

$$\begin{array}{l|l}
2a^2x^2 - 6abxy + 4b^2y^2 & 1 \\
\underline{2a^2x^2 - abxy - b^2y^2} & \\
-5abxy + 5b^2y^2 &
\end{array}$$

Divide by $-5by$;

$$\begin{array}{l|l|l}
ax - by & 2a^2x^2 - abxy - b^2y^2 & 2ax + by \\
 & \underline{2a^2x^2 - 2abxy} & \\
 & \quad abxy - b^2y^2 & \\
 & \quad \underline{abxy - b^2y^2} &
\end{array}$$

19.

$$\begin{array}{l|l|l}
x^3 + ax^2 - axy - y^3 & x^4 + 2x^3y + x^2(y^2 - a^2) - 2axy^2 - y^4 & x + 2y - a \\
 & \underline{x^4 + ax^3 - ax^2y \qquad - xy^3} & \\
 & (2y-a)x^3 + x^2(y^2 + ay - a^2) + x(y^3 - 2ay^2) - y^4 & \\
 & \underline{(2y-a)x^3 + x^2(2ay - a^2) + x(-2ay^2 + a^2y) - y^3(2y-a)} & \\
 & \quad x^2(y^2 - ay) + x(y^3 - a^2y) + y^4 - ay^3 &
\end{array}$$

Divide by $y^2 - ay$;

$$\begin{array}{l|l|l}
x^2 + x(y+a) + y^2 & x^3 + ax^2 - axy - y^3 & x - y \\
 & \underline{x^3 + x^2(y+a) + xy^2} & \\
 & - x^2y - xy(a+y) - y^3 & \\
 & \underline{- x^2y - xy(a+y) - y^3} &
\end{array}$$

20.

$$\begin{array}{l|l|l}
x^5 + 3x^4 - 8x^2 - 9x - 3 & x^5 - 2x^4 - 6x^3 + 4x^2 + 13x + 6 & 1 \\
 & \underline{x^5 + 3x^4 \qquad - 8x^2 - 9x - 3} & \\
 & -5x^4 - 6x^3 + 12x^2 + 22x + 9 &
\end{array}$$

Multiply the new dividend by 5, and change the signs of the divisor;

$$\begin{array}{l|l|l}
5x^4 + 6x^3 - 12x^2 - 22x - 9 & 5x^5 + 15x^4 \qquad - 40x^2 - 45x - 15 & x \\
 & \underline{5x^5 + 6x^4 - 12x^3 - 22x^2 - 9x} & \\
 & 9x^4 + 12x^3 - 18x^2 - 36x - 15 &
\end{array}$$

Multiply by 5, and continue the division;

$$\begin{array}{l|l}
45x^4 + 60x^3 - 90x^2 - 180x - 75 & 9 \\
\underline{45x^4 + 54x^3 - 108x^2 - 198x - 81} & \\
6x^3 + 18x^2 + 18x + 6 &
\end{array}$$

Divide by 6;

$$\begin{array}{l|l|l}
x^3 + 3x^2 + 3x + 1 & 5x^4 + 6x^3 - 12x^2 - 22x - 9 & 5x - 9 \\
 & \underline{5x^4 + 15x^3 + 15x^2 + 5x} & \\
 & - 9x^3 - 27x^2 - 27x - 9 & \\
 & \underline{- 9x^3 - 27x^2 - 27x - 9} &
\end{array}$$

21. Multiply the first expression by 2;

$$4x^4+2x^3-18x^2+3x-5\ \big)\ 12x^5-8x^4-22x^3-6x^2-6x-2\ \big(\ 3x$$
$$\begin{array}{r} 12x^5+6x^4-54x^3+9x^2-15x \\ \hline -14x^4+32x^3-15x^2+9x-2 \end{array}$$

Multiply by -2, and continue the division;

$$\begin{array}{r} 28x^4-64x^3+30x^2-18x+4 \ \big(\ 7 \\ 28x^4+14x^3-126x^2+21x-35 \quad \\ \hline -78x^3+156x^2-39x+39 \quad \end{array}$$

Divide by -39; $2x^3-4x^2+x-1\ \big)\ 4x^4+2x^3-18x^2+3x-5\ \big(\ 2x+5$

$$\begin{array}{r} 4x^4-8x^3+2x^2-2x \\ \hline 10x^3-20x^2+5x-5 \\ 10x^3-20x^2+5x-5 \\ \hline \end{array}$$

22. Multiply the first expression by 3;

$$3x^3-7ax^2+3a^2x-2a^3\ \big)\ 3x^4-3ax^3-3a^2x^2-3a^3x-6a^4\ \big(\ x$$
$$\begin{array}{r} 3x^4-7ax^3+3a^2x^2-2a^3x \\ \hline 4ax^3-6a^2x^2-a^3x-6a^4 \end{array}$$

Divide by a, multiply by 3, and continue the division;

$$\begin{array}{r} 12x^3-18ax^2-3a^2x-18a^3\ \big(\ 4 \\ 12x^3-28ax^2+12a^2x-8a^3 \quad \\ \hline 10ax^2-15a^2x-10a^3 \quad \end{array}$$

Divide by $5a$, and multiply the new dividend by 2;

$$2x^2-3ax-2a^2\ \big)\ 6x^3-14ax^2+6a^2x-4a^3\ \big(\ 3x$$
$$\begin{array}{r} 6x^3-9ax^2-6a^2x \\ \hline -5ax^2+12a^2x-4a^3 \end{array}$$

Divide by a, multiply by -2, and continue the division;

$$\begin{array}{r} 10x^2-24ax+8a^2\ \big(\ 5 \\ 10x^2-15ax-10a^2 \quad \\ \hline -9ax+18a^2 \quad \end{array}$$

Divide by $-9a$; $x-2a\ \big)\ 2x^2-3ax-2a^2\ \big(\ 2x+a$

$$\begin{array}{r} 2x^2-4ax \\ \hline ax-2a^2 \\ ax-2a^2 \\ \hline \end{array}$$

23. $x^3-9x^2+26x-24\ \big)\ x^3-10x^2+31x-30\ \big(\ 1$

$$\begin{array}{r} x^3-9x^2+26x-24 \\ \hline -x^2+5x-6\ \big)\ x^3-9x^2+26x-24\ \big(\ -x+4 \\ x^3-5x^2+6x \quad\quad\quad\quad \\ \hline -4x^2+20x-24 \quad\quad\quad\quad \\ -4x^2+20x-24 \quad\quad\quad\quad \\ \hline \end{array}$$

Thus x^2-5x+6 is the G.C.M. of the first two expressions.

$$
\begin{array}{r|l|l}
x^2-5x+6 & x^3-11x^2+38x-40 & x-6 \\
 & \underline{x^3-\ \ 5x^2+\ \ 6x\qquad\quad} & \\
 & -\ \ 6x^2+32x-40 & \\
 & \underline{-\ \ 6x^2+30x-36} & \\
 & \qquad\qquad 2x-\ \ 4 &
\end{array}
$$

Divide by 2;

$$
\begin{array}{r|l|l}
x-2 & x^2-5x+6 & x-3 \\
 & \underline{x^2-2x\qquad} & \\
 & \quad -3x+6 & \\
 & \quad \underline{-3x+6} &
\end{array}
$$

24.

$$
\begin{array}{r|l|l}
x^4-10x^2+9 & x^4+10x^3+20x^2-10x-21 & 1 \\
 & \underline{x^4\qquad\quad -10x^2\qquad\quad +\ \ 9} & \\
 & \quad 10x^3+30x^2-10x-30 &
\end{array}
$$

Divide by 10;

$$
\begin{array}{r|l|l}
x^3+3x^2-x-3 & x^4\qquad\quad -10x^2\qquad +9 & x-3 \\
 & \underline{x^4+3x^3-\ \ \ x^2-3x\qquad} & \\
 & \quad -3x^3-\ \ 9x^2+3x+9 & \\
 & \quad \underline{-3x^3-\ \ 9x^2+3x+9} &
\end{array}
$$

Thus x^3+3x^2-x-3 is the G.C.M. of the first two expressions.

$$
\begin{array}{r|l|l}
x^3+3x^2-x-3 & x^4+4x^3-22x^2-4x+21 & x+1 \\
 & \underline{x^4+3x^3-\ \ \ x^2-3x\qquad\ \ } & \\
 & \quad x^3-21x^2-\ \ x+21 & \\
 & \quad \underline{x^3+\ \ 3x^2-\ \ x-\ \ 3} & \\
 & \quad -24x^2\qquad\quad +24 &
\end{array}
$$

Divide by -24;

$$
\begin{array}{r|l|l}
x^2-1 & x^3+3x^2-x-3 & x+3 \\
 & \underline{x^3\qquad\quad -x\qquad} & \\
 & \quad 3x^2\qquad -3 & \\
 & \quad \underline{3x^2\qquad -3} &
\end{array}
$$

VII.

1.

$$
\begin{array}{r|l|l}
2x^2+3x-2 & 6x^2-\ \ x-1 & 3 \\
 & \underline{6x^2+\ \ 9x-6} & \\
 & \quad -10x+5 &
\end{array}
$$

Divide by -5;

$$
\begin{array}{r|l|l}
2x-1 & 2x^2+3x-2 & x+2 \\
 & \underline{2x^2-\ \ x\qquad} & \\
 & \qquad 4x-2 & \\
 & \qquad \underline{4x-2} &
\end{array}
$$

Thus the G.C.M. is $2x-1$.

2.

$$
\begin{array}{r|l|l}
x^2+x-2 & x^3\qquad\qquad\quad -1 & x-1 \\
 & \underline{x^3+x^2-2x\qquad} & \\
 & -x^2+2x-1 & \\
 & \underline{-x^2-\ \ x+2} & \\
 & \qquad\quad 3x-3 &
\end{array}
$$

Divide by 3;

$$
\begin{array}{r|l|l}
x-1 & x^2+\ \ x-2 & x+2 \\
 & \underline{x^2-\ \ x\qquad} & \\
 & \qquad 2x-2 & \\
 & \qquad \underline{2x-2} &
\end{array}
$$

Thus the G.C.M. is $x-1$.

3. $$x^2-8x+7\,\big)\,x^3-9x^2+23x-15\,\big(\,x-1$$
$$\begin{array}{r} x^3-8x^2+7x \\ \hline -x^2+16x-15 \\ -x^2+8x-7 \\ \hline 8x-8 \end{array}$$

Divide by 8;
$$x-1\,\big)\,x^2-8x+7\,\big(\,x-7$$
$$\begin{array}{r} x^2-x \\ \hline -7x+7 \\ -7x+7 \\ \hline \end{array}$$

Thus the G.C.M. is $x-1$.

4. Multiply the second expression by 3;
$$3x^2-5x+2\,\big)\,12x^3-12x^2-3x+3\,\big(\,4x$$
$$\begin{array}{r} 12x^3-20x^2+8x \\ \hline 8x^2-11x+3 \end{array}$$

Multiply by 3, and continue the division;
$$\begin{array}{r} 24x^2-33x+9 \\ 24x^2-40x+16 \\ \hline 7x-7 \end{array}\,\Big(\,8$$

Divide by 7;
$$x-1\,\big)\,3x^2-5x+2\,\big(\,3x-2$$
$$\begin{array}{r} 3x^2-3x \\ \hline -2x+2 \\ -2x+2 \\ \hline \end{array}$$

Thus the G.C.M. is $x-1$.

5. $$x^3-1\,\big)\,x^3+x^2-x-1\,\big(\,1$$
$$\begin{array}{r} x^3-1 \\ \hline x^2-x \end{array}$$

Divide by x;
$$x-1\,\big)\,x^3-1\,\big(\,x^2+x+1$$
$$\begin{array}{r} x^3-x^2 \\ \hline x^2-1 \\ x^2-x \\ \hline x-1 \\ x-1 \\ \hline \end{array}$$

Thus the G.C.M. is $x-1$.

6. $$x^3-2x^2y-xy^2+2y^3\,\big)\,x^3+2x^2y-xy^2-2y^3\,\big(\,1$$
$$\begin{array}{r} x^3-2x^2y-xy^2+2y^3 \\ \hline 4x^2y-4y^3 \end{array}$$

Divide by $4y$;
$$x^2-y^2\,\big)\,x^3-2x^2y-xy^2+2y^3\,\big(\,x-2y$$
$$\begin{array}{r} x^3-xy^2 \\ \hline -2x^2y+2y^3 \\ -2x^2y+2y^3 \\ \hline \end{array}$$

Thus the G.C.M. is x^2-y^2.

7. $2x-1$ divides $4x^2-1$, so that $4x^2-1$ is the L.C.M. of the first two expressions. Then it will be found that $4x^2-1$ and $4x^2+1$ have no common measure greater than unity; so their L.C.M. is their product.

8. The L.C.M. of the second and third expressions is their product $(x^3+1)(x^3-1)$ that is x^6-1; the first expression $=x(x^2-1)=x(x+1)(x-1)$: now $x+1$ divides x^3+1 and $x-1$ divides x^3-1: thus x^2-1 divides x^6-1, and is the G.C.M. of $x(x^2-1)$ and x^6-1; and therefore their L.C.M. is $x(x^6-1)$.

9. The L.C.M. of the second and third expressions is their product; and this is divisible by the first expression.

10.

$$\begin{array}{r|l|l}
x^3-6x^2+11x-6 & x^3-9x^2+26x-24 & 1 \\
 & \underline{x^3-6x^2+11x-6} & \\
 & -3x^2+15x-18 &
\end{array}$$

Divide by -3;

$$\begin{array}{r|l|l}
x^2-5x+6 & x^3-6x^2+11x-6 & x-1 \\
 & \underline{x^3-5x^2+6x} & \\
 & -x^2+5x-6 & \\
 & \underline{-x^2+5x-6} &
\end{array}$$

Thus x^2-5x+6 is the G.C.M. of the first two expressions; and their L.C.M. is $(x^3-9x^2+26x-24)(x-1)$, that is $x^4-10x^3+35x^2-50x+24$.

$$\begin{array}{r|l|l}
x^3-8x^2+19x-12 & x^4-10x^3+35x^2-50x+24 & x-2 \\
 & \underline{x^4-8x^3+19x^2-12x} & \\
 & -2x^3+16x^2-38x+24 & \\
 & \underline{-2x^3+16x^2-38x+24} &
\end{array}$$

Thus $x^3-8x^2+19x-12$ is the G.C.M.; the L.C.M. is $(x^3-9x^2+26x-24)(x-1)$: it will be found that this $=(x-1)(x-2)(x-3)(x-4)$.

11. The G.C.M. of the first two expressions is x^2-5x+6, by Ex. 23 of Chapter VI.; and their L.C.M. is $(x^3-10x^2+31x-30)(x-4)$, that is $x^4-14x^3+71x^2-154x+120$: it will be found that this is divisible by the third expression, and so it is the L.C.M. of the three. And it may be shewn to be $=(x-2)(x-3)(x-4)(x-5)$.

12. The G.C.M. of the first two expressions is x^3+3x^2-x-3 by Ex. 24 of Chapter VI.; and their L.C.M. is $(x^4+10x^3+20x^2-10x-21)(x-3)$, that is $x^5+7x^4-10x^3-70x^2+9x+63$: it will be found that this is divisible by the third expression, and so it is the L.C.M. of the three. And it may be shewn to be $=(x^2-1)(x^2-9)(x+7)$.

13.

$$\begin{array}{r|l|l}
x^2-4a^2 & x^3+2ax^2+4a^2x+8a^3 & x+2a \\
 & \underline{x^3-4a^2x\phantom{+8a^3}} & \\
 & 2ax^2+8a^2x+8a^3 & \\
 & \underline{2ax^2-8a^3} & \\
 & 8a^2x+16a^3 &
\end{array}$$

Divide by $8a^2$;

$$\begin{array}{r|l|l}
x+2a & x^2-4a^2 & x-2a \\
 & \underline{x^2+2ax} & \\
 & -2ax-4a^2 & \\
 & \underline{-2ax-4a^2} &
\end{array}$$

Thus $x+2a$ is the G.C.M. of the first two expressions; and their L.C.M. is $(x^3+2ax^2+4a^2x+8a^3)(x-2a)$, that is x^4-16a^4. Then it will be found that this is divisible by the third expression, and so it is the L.C.M. of the three.

14. The three expressions are obviously

$$(x-a)(x-b), \qquad (x-b)(x-c), \qquad (x-c)(x-a);$$

and their L.C.M. is $\qquad (x-a)(x-b)(x-c)$.

15.
$$
2x^2-(3b-2c)x-3bc\;\Big)\;
\begin{array}{r}
2x^3+(2a-3b)x^2-(2b^2+3ab)x+3b^3\\
\underline{2x^3-(3b-2c)x^2-3bcx\qquad\qquad\qquad}\\
2(a-c)x^2-(2b^2+3ab-3bc)x+3b^3\\
\underline{2(a-c)x^2-(3ab-3bc-2ac+2c^2)x-3bc(a-c)}\\
-2(b^2+ac-c^2)x+3b(b^2+ac-c^2)
\end{array}
\;\Big(\;x+a-c
$$

Divide by $-(b^2+ac-c^2)$;
$$
2x-3b\;\Big)\;
\begin{array}{r}
2x^2-(3b-2c)x-3bc\\
\underline{2x^2-3bx\qquad\qquad}\\
2cx-3bc\\
\underline{2cx-3bc}
\end{array}
\;\Big(\;x+c
$$

Thus the G.C.M. is $2x-3b$; and it will be found that the first expression $=(2x-3b)(x^2+ax-b^2)$, and that the second expression $=(2x-3b)(x+c)$.

16. Resolving into algebraical factors the three expressions become by Art. 70,

$$6(a^2+ab+b^2)(a-b)^4,\quad 9(a^2+b^2)(a+b)(a-b)^3,\quad 12(a+b)^3(a-b)^3;$$

hence we infer that the L.C.M. of the three is

$$36(a-b)^4(a+b)^3(a^2+b^2)(a^2+ab+b^2);$$

this may be put in the forms

$$36(a^2-b^2)^3(a^2+b^2)(a^3-b^3)\quad\text{and}\quad 36(a^2-b^2)^2(a^4-b^4)(a^3-b^3).$$

VIII.

1.
$$
x^2+2x-3\;\Big)\;
\begin{array}{r}
x^2+6x-7\\
\underline{x^2+2x-3}\\
4x-4
\end{array}
\;\Big(\;1
$$

Divide by 4;
$$
x-1\;\Big)\;
\begin{array}{r}
x^2+2x-3\\
\underline{x^2-\;\;x\qquad}\\
3x-3\\
\underline{3x-3}
\end{array}
\;\Big(\;x+3
$$

Thus $x-1$ is the G.C.M.

2.
$$
x^2-3x-4\;\Big)\;
\begin{array}{r}
x^2-4x-5\\
\underline{x^2-3x-4}\\
-\;x-1
\end{array}
\;\Big(\;1
$$

Change the signs;
$$
x+1\;\Big)\;
\begin{array}{r}
x^2-3x-4\\
\underline{x^2+\;\;x\qquad}\\
-4x-4\\
\underline{-4x-4}
\end{array}
\;\Big(\;x-4
$$

Thus $x+1$ is the G.C.M.

3.
$$
x^2-3x+2\;\Big)\;
\begin{array}{r}
x^3-6x^2+11x-6\\
\underline{x^3-3x^2+\;\;2x\qquad}\\
-3x^2+\;\;9x-6\\
\underline{-3x^2+\;\;9x-6}
\end{array}
\;\Big(\;x-3
$$

Thus x^2-3x+2 is the G.C.M.

4. The numerator $=(a+b)^3$, and the denominator $=(a+b)$.

5.
$$
\begin{array}{r|l|l}
x^3+9x^2+26x+24 & x^4+10x^3+35x^2+50x+24 & x+1 \\
 & \underline{x^4+\ \ 9x^3+26x^2+24x} & \\
 & \qquad x^3+\ \ 9x^2+26x+24 & \\
 & \qquad \underline{x^3+\ \ 9x^2+26x+24} &
\end{array}
$$

Thus $x^3+9x^2+26x+24$ is the G. C. M.

6. Multiply the numerator by 2;
$$
\begin{array}{r|l|l}
2x^3-11x^2+17x-6 & 6x^3-32x^2+46x-12 & 3 \\
 & \underline{6x^3-33x^2+51x-18} & \\
 & \qquad x^2-\ \ 5x+\ \ 6 &
\end{array}
$$

$$
\begin{array}{r|l|l}
x^2-5x+6 & 2x^3-11x^2+17x-6 & 2x-1 \\
 & \underline{2x^3-10x^2+12x} & \\
 & \quad -\ \ x^2+\ \ 5x-6 & \\
 & \quad \underline{-\ \ x^2+\ \ 5x-6} &
\end{array}
$$

Thus x^2-5x+6 is the G. C. M.

7.
$$
\begin{array}{r|l|l}
2x^3-x^2-x+2 & 6x^3-5x^2\qquad\ +4 & 3 \\
 & \underline{6x^3-3x^2-3x+6} & \\
 & \quad -2x^2+3x-2 &
\end{array}
$$

Change the signs;
$$
\begin{array}{r|l|l}
2x^2-3x+2 & 2x^3-\ \ x^2-\ \ x+2 & x+1 \\
 & \underline{2x^3-3x^2+2x} & \\
 & \quad 2x^2-3x+2 & \\
 & \quad \underline{2x^2-3x+2} &
\end{array}
$$

Thus $2x^2-3x+2$ is the G. C. M.

8. Multiply the numerator by 3;
$$
\begin{array}{r|l|l}
3x^3+5x^2-15x+4 & 6x^3+27x^2+21x-\ \ 9 & 2 \\
 & \underline{6x^3+10x^2-30x+\ \ 8} & \\
 & \qquad 17x^2+51x-17 &
\end{array}
$$

$$
\text{Divide by 17;}\quad
\begin{array}{r|l|l}
x^2+3x-1 & 3x^3+5x^2-15x+4 & 3x-4 \\
 & \underline{3x^3+9x^2-\ \ 3x} & \\
 & \quad -4x^2-12x+4 & \\
 & \quad \underline{-4x^2-12x+4} &
\end{array}
$$

Thus x^2+3x-1 is the G. C. M.

9. Divide the numerator by 3;
$$
\begin{array}{r|l|l}
x^2+4x+3 & x^5\qquad\quad +\ \ 5x^3+\ \ 6 & x^3-4x^2+18x-60 \\
 & \underline{x^5+4x^4+\ \ 3x^3} & \\
 & \quad -4x^4+\ \ 2x^3\qquad\quad +6 & \\
 & \quad \underline{-4x^4-16x^3-12x^2} & \\
 & \qquad\qquad 18x^3+12x^2\qquad\ +6 & \\
 & \qquad\qquad \underline{18x^3+72x^2+\ \ 54x} & \\
 & \qquad\qquad\quad -60x^2-\ \ 54x+\ \ 6 & \\
 & \qquad\qquad\quad \underline{-60x^2-240x-180} & \\
 & \qquad\qquad\qquad\qquad 186x+186 &
\end{array}
$$

Divide by 186;

$$\begin{array}{r|l|l} x+1 & x^2+4x+3 & x+3 \\ & \underline{x^2+\ \ x} & \\ & \ \ \ 3x+3 & \\ & \ \ \ \underline{3x+3} & \end{array}$$

Thus $x+1$ is the G. C. M.

10.

$$\begin{array}{r|l|l} x^3-6x^2-37x+210 & x^3+\ 4x^2-47x-210 & 1 \\ & \underline{x^3-\ 6x^2-37x+210} & \\ & \ \ \ 10x^2-10x-420 & \end{array}$$

Divide by 10;

$$\begin{array}{r|l|l} x^2-x-42 & x^3-6x^2-37x+210 & x-5 \\ & \underline{x^3-\ x^2-42x} & \\ & \ \ -5x^2+\ 5x+210 & \\ & \ \ \underline{-5x^2+\ 5x+210} & \end{array}$$

Thus x^2-x-42 is the G. C. M.

11.

$$\begin{array}{r|l|l} x^4+2x^2+9 & x^4-4x^3+4x^2-\ 9 & 1 \\ & \underline{x^4\qquad\quad +2x^2+\ 9} & \\ & \ \ -4x^3+2x^2-18 & \end{array}$$

Divide by -2, and multiply the new dividend by 2;

$$\begin{array}{r|l|l} 2x^3-x^2+9 & 2x^4\qquad +4x^2\qquad +18 & x \\ & \underline{2x^4-x^3\qquad +9x} & \\ & \ \ \ x^3+4x^2-9x+18 & \end{array}$$

Multiply by 2, and continue the division;

$$\begin{array}{l|l} 2x^3+8x^2-18x+36 & 1 \\ \underline{2x^3-\ x^2\qquad +\ 9} & \\ \ \ \ 9x^2-18x+27 & \end{array}$$

Divide by 9;

$$\begin{array}{r|l|l} x^2-2x+3 & 2x^3-\ x^2\qquad +9 & 2x+3 \\ & \underline{2x^3-4x^2+6x} & \\ & \ \ \ 3x^2-6x+9 & \\ & \ \ \ \underline{3x^2-6x+9} & \end{array}$$

Thus x^2-2x+3 is the G. C. M.

12. Divide both numerator and denominator by x, so that x is a factor of the G. C. M.;

$$\begin{array}{r|l|l} x^2+2x+2 & x^4\qquad\qquad\qquad +4 & x^2-2x+2 \\ & \underline{x^4+2x^3+2x^2} & \\ & -2x^3-2x^2\qquad +4 & \\ & \underline{-2x^3-4x^2-4x} & \\ & \ \ \ 2x^2+4x+4 & \\ & \ \ \ \underline{2x^2+4x+4} & \end{array}$$

Thus $x(x^2+2x+2)$ is the G. C. M.

13.
$$\begin{array}{r} x^4-x^3-x+1\,\big)\,x^4-2x^3-x^2-2x+1\,\big(\,1 \\ \underline{x^4-x^3-x+1} \\ -x^3-x^2-x \end{array}$$

Divide by $-x$;
$$\begin{array}{r} x^2+x+1\,\big)\,x^4-x^3-x+1\,\big(\,x^2-2x+1 \\ \underline{x^4+x^3+x^2} \\ -2x^3-x^2-x+1 \\ \underline{-2x^3-2x^2-2x} \\ x^2+x+1 \\ \underline{x^2+x+1} \end{array}$$

Thus x^2+x+1 is the G.C.M.

14. Divide the denominator by a;
$$\begin{array}{r} a^3-a^2b-ab^2+b^3\,\big)\,a^5-a^4b-ab^4+b^5\,\big(\,a^2+b^2 \\ \underline{a^5-a^4b-a^3b^2+a^2b^3} \\ a^3b^2-a^2b^3-ab^4+b^5 \\ \underline{a^3b^2-a^2b^3-ab^4+b^5} \end{array}$$

Thus $a^3-a^2b-ab^2+b^3$ is the G.C.M.

15.
$$\begin{array}{r} bx+2\,\big)\,b^2x+2b(1-x^2)-4x\,\big(\,b-2x \\ \underline{b^2x+2b} \\ -2bx^2-4x \\ \underline{-2bx^2-4x} \end{array}$$

Thus $bx+2$ is the G.C.M.

16. The numerator is
$$7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7xy^6,$$
and the denominator is
$$5x^4y+10x^3y^2+10x^2y^3+5xy^4.$$

Divide the numerator by $7xy$ and the denominator by $5xy$, so that xy is a factor of the G.C.M.
$$\begin{array}{r} x^3+2x^2y+2xy^2+y^3\,\big)\,x^5+3x^4y+5x^3y^2+5x^2y^3+3xy^4+y^5\,\big(\,x^2+xy+y^2 \\ \underline{x^5+2x^4y+2x^3y^2+x^2y^3} \\ x^4y+3x^3y^2+4x^2y^3+3xy^4+y^5 \\ \underline{x^4y+2x^3y^2+2x^2y^3+xy^4} \\ x^3y^2+2x^2y^3+2xy^4+y^5 \\ \underline{x^3y^2+2x^2y^3+2xy^4+y^5} \end{array}$$

Thus $xy(x^3+2x^2y+2xy^2+y^3)$ is the G.C.M.

17. $\dfrac{a(a-b)+b(a+b)}{a^2-b^2}=$ &c.   18. $\dfrac{a}{2a-2b}-\dfrac{b}{2a-2b}=\dfrac{a-b}{2(a-b)}=\dfrac{1}{2}.$

19. $\dfrac{2(4x^2-1)-3(2x+1)x-(2x-3)x}{x(4x^2-1)}=-\dfrac{2}{x(4x^2-1)}.$

20. $\dfrac{(m+n)(a+b)-n(a+b)+m(a-b)}{mn}=\dfrac{2ma}{mn}=\dfrac{2a}{n}$.

21. $\dfrac{(x+2)^2-(x+2)(x-1)-3(x-1)}{(x-1)(x+2)^2}=\dfrac{9}{(x-1)(x+2)^2}$.

22. $\dfrac{25(x-1)(2x+3)-(x+1)(2x+3)-48(x^2-1)}{10(x^2-1)(2x+3)}=\dfrac{20x-30}{10(x^2-1)(2x+3)}=$ &c.

23. $\dfrac{(b-a)(x+b)-(a-2b)(x-b)+3x(a-b)}{x^2-b^2}=\dfrac{ax-b^2}{x^2-b^2}$.

24. $\dfrac{(3+2x)(2+x)-(2-3x)(2-x)-(16x-x^2)}{4-x^2}=\dfrac{2-x}{4-x^2}=\dfrac{1}{2+x}$.

25. $\dfrac{3(1+2x)-7(1-2x)+4-20x}{1-4x^2}=0$.

26. $\dfrac{(a-b)(a^2+b^2)+b(a^2+b^2)-a(a^2-b^2)}{a^4-b^4}=\dfrac{2ab^2}{a^4-b^4}$.

27. $\dfrac{x^2-y^2+(x-y)^2-(x+y)^2}{(x^2-y^2)^2}=\dfrac{x^2-4xy-y^2}{(x^2-y^2)^2}$.

28. $\dfrac{(a^2+b^2)^2-a^2(a-b)^2-b^2(a-b)^2-2ab(a-b)^2}{ab(a-b)^2}=\dfrac{4a^2b^2}{ab(a-b)^2}=$ &c.

29. $\dfrac{a(a+x)+3a(a-x)-2ax}{a^2-x^2}=\dfrac{4a^2-4ax}{a^2-x^2}=\dfrac{4a}{a+x}$.

30. $\dfrac{12(3a-4b)-28(2a-b-c)+7(15a-4c)-4(a-4b)}{84}=\dfrac{81a-4b}{84}$.

31. $\dfrac{a^2-b^2+b^2-c^2+c^2-a^2}{(a-b)(b-c)(c-a)}=0$.

32. $\dfrac{(b+c)(a^2-bc)+(c+a)(b^2-ca)+(a+b)(c^2-ab)}{(a+b)(b+c)(c+a)}=$ &c.

33. $\dfrac{(c+b)(a^2-bc)-(a-c)(b^2+ca)-(a-b)(c^2+ab)}{(a-b)(a-c)(b+c)}=$ &c.

34. $\dfrac{(b-c)bc+(c-a)ca+(a-b)ab}{(a-b)(b-c)(c-a)}=$ &c.

35. $\dfrac{bc(c-b)+ca(a-c)+ab(b-a)}{abc(a-b)(b-c)(c-a)}=$ &c.

36. Here the common denominator is $(a+b)(b+c)(c+a)$: the numerator is

$(a-b)(b+c)(c+a)+(b-c)(c+a)(a+b)+(c-a)(a+b)(b+c)+(a-b)(b-c)(c-a)$;

this will be found to be zero.

37. Here the common denominator is $(a-b)(b-c)(c-a)$: the numerator is

$$2(b-c)(c-a)+2(a-b)(c-a)+2(a-b)(b-c)+(a-b)^2+(b-c)^2+(c-a)^2;$$

this will be found to be zero.

40. $\dfrac{3ax}{4by} \times \dfrac{(a+x)(a-x)}{(c+x)(c-x)} \times \dfrac{b(c+x)}{a(a+x)} \times \dfrac{c-x}{a-x} = \&c.$

41. Each expression $=6+\dfrac{b^2}{c^2}+\dfrac{c^2}{b^2}+\dfrac{c^2}{a^2}+\dfrac{a^2}{c^2}+\dfrac{a^2}{b^2}+\dfrac{b^2}{a^2}$.

42. $1+\dfrac{x}{1-x}=\dfrac{1-x+x}{1-x}=\dfrac{1}{1-x}$.

45. $\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}-\dfrac{4y^2}{x^2-y^2}=\dfrac{(x+y)^2-(x-y)^2-4y^2}{x^2-y^2}=\dfrac{4xy-4y^2}{x^2-y^2}=\dfrac{4y}{x+y}$.

48. Multiply x^2-x+1 by $1+\dfrac{1}{x}+\dfrac{1}{x^2}$ by ordinary work. Or thus:

$$(x^2-x+1)\times\left(\frac{1}{x^2}+\frac{1}{x}+1\right)=(x^2-x+1)\times\frac{1+x+x^2}{x^2}=\frac{(x^2+1)^2-x^2}{x^2}=\&c.$$

53. $\dfrac{2x+y}{x+y}+\dfrac{2y-x}{x-y}-\dfrac{x^2}{x^2-y^2}=\dfrac{(2x+y)(x-y)+(2y-x)(x+y)-x^2}{x^2-y^2}=\dfrac{y^2}{x^2-y^2}$.

54. $\dfrac{x^2}{y^3}+\dfrac{1}{x}=\dfrac{x^3+y^3}{xy^3}$, $\dfrac{x}{y^2}-\dfrac{1}{y}+\dfrac{1}{x}=\dfrac{x^2-xy+y^2}{xy^2}$.

56. $\dfrac{x+2y}{x+y}+\dfrac{x}{y}=\dfrac{x^2+2xy+2y^2}{(x+y)y}$; $\dfrac{x+2y}{y}-\dfrac{x}{x+y}=\dfrac{x^2+2xy+2y^2}{y(x+y)}$.

58. Divide $x^2+2+\dfrac{1}{x^2}$ by $x+\dfrac{1}{x}$ by ordinary work. Or thus:

$$x^2+2+\frac{1}{x^2}=\frac{x^4+2x^2+1}{x^2}=\frac{(x^2+1)^2}{x^2};\quad x+\frac{1}{x}=\frac{x^2+1}{x}.$$

59. $\left(x^2+1+\dfrac{1}{x^2}\right)\div\left(x-1+\dfrac{1}{x}\right)=\dfrac{x^4+x^2+1}{x(x^2-x+1)}=\&c.$

60. $a^2-b^2-c^2+2bc=a^2-(b-c)^2=(a+b-c)(a-b+c)$.

62. $a^2-b^2-c^2-2bc=a^2-(b+c)^2=(a+b+c)(a-b-c)$.

63. $x^2-3ax-2a^2+\dfrac{12a^3}{x+3a}=\dfrac{(x+3a)(x^2-3ax-2a^2)+12a^3}{x+3a}$

$$=\frac{x^3-11a^2x+6a^3}{x+3a};\quad 3x-6a-\frac{2x^2}{x+3a}=\frac{x^2+3ax-18a^2}{x+3a}.$$

65. $\dfrac{a+b}{c+d}+\dfrac{a-b}{c-d}=\dfrac{2(ac-bd)}{c^2-d^2}$; $\dfrac{a+b}{c-d}+\dfrac{a-b}{c+d}=\dfrac{2(ac+bd)}{c^2-d^2}$.

67. The second fraction $=\dfrac{bc(a-1)+ca(b-1)+ab(c-1)}{bc+ca-ab}\doteq$ &c.

68. $\dfrac{a+b}{a-b}+\dfrac{a^2+b^2}{a^2-b^2}=\dfrac{(a+b)^2+a^2+b^2}{a^2-b^2}=\dfrac{2(a^2+ab+b^2)}{a^2-b^2}$;

$\dfrac{a-b}{a+b}-\dfrac{a^3-b^3}{a^3+b^3}=\dfrac{a-b}{a+b}-\dfrac{(a-b)(a^2+ab+b^2)}{(a+b)(a^2-ab+b^2)}=\dfrac{a-b}{a+b}\left(1-\dfrac{a^2+ab+b^2}{a^2-ab+b^2}\right)$

$=\dfrac{a-b}{a+b}\times\dfrac{-2ab}{a^2-ab+b^2}=\dfrac{-2ab(a-b)}{(a+b)(a^2-ab+b^2)}$;

$\dfrac{2(a^2+ab+b^2)}{a^2-b^2}\times\dfrac{(a+b)(a^2-ab+b^2)}{-2ab(a-b)}=-\dfrac{(a^2+ab+b^2)(a^2-ab+b^2)}{ab(a-b)^2}=$ &c.

72. $\dfrac{\dfrac{m^2+n^2}{n}-m}{\dfrac{1}{n}-\dfrac{1}{m}}=\dfrac{\dfrac{m^2-mn+n^2}{n}}{\dfrac{m-n}{mn}}=\dfrac{m(m^2-mn+n^2)}{m-n}$;

$\dfrac{m^2-n^2}{m^3+n^3}=\dfrac{(m+n)(m-n)}{(m+n)(m^2-mn+n^2)}=\dfrac{m-n}{m^2-mn+n^2}$.

73. $\dfrac{x+a}{x-a}-\dfrac{x-a}{x+a}=\dfrac{(x+a)^2-(x-a)^2}{x^2-a^2}=\dfrac{4ax}{x^2-a^2}$;

$\dfrac{x+a}{x-a}+\dfrac{x-a}{x+a}=\dfrac{(x+a)^2+(x-a)^2}{x^2-a^2}=\dfrac{2(x^2+a^2)}{x^2-a^2}$;

$\dfrac{4ax}{x^2-a^2}\div\dfrac{2(x^2+a^2)}{x^2-a^2}=\dfrac{4ax}{2(x^2+a^2)}=\dfrac{2ax}{x^2+a^2}$;

$\dfrac{x}{x-a}-\dfrac{x}{x+a}-\dfrac{2ax}{x^2+a^2}=\dfrac{2ax}{x^2-a^2}-\dfrac{2ax}{x^2+a^2}=\dfrac{4a^3x}{x^4-a^4}$.

74. $\dfrac{1}{a}+\dfrac{1}{b+c}=\dfrac{a+b+c}{a(b+c)}$; $\dfrac{1}{a}-\dfrac{1}{b+c}=\dfrac{b+c-a}{a(b+c)}$;

$\dfrac{a+b+c}{a(b+c)}\div\dfrac{b+c-a}{a(b+c)}=\dfrac{a+b+c}{b+c-a}$;

$1+\dfrac{b^2+c^2-a^2}{2bc}=\dfrac{(b+c)^2-a^2}{2bc}=\dfrac{(b+c+a)(b+c-a)}{2bc}$.

75. $1+\dfrac{x+1}{3-x}=\dfrac{3-x+x+1}{3-x}=\dfrac{4}{3-x}$;

$1\div\dfrac{4}{3-x}=\dfrac{3-x}{4}$; $x+\dfrac{3-x}{4}=\dfrac{4x+3-x}{4}=\dfrac{3x+3}{4}$;

$1\div\dfrac{3x+3}{4}=\dfrac{4}{3x+3}$.

76. $d+\dfrac{e}{f}=\dfrac{df+e}{f}$; $c\div\dfrac{df+e}{f}=\dfrac{cf}{df+e}$;

$b+\dfrac{cf}{df+e}=\dfrac{bdf+be+cf}{df+e}$; $a\div\dfrac{bdf+be+cf}{df+e}=\dfrac{adf+ae}{bdf+be+cf}$.

IX.

1. Multiply by 8; $4(2x+1)=7x+5$, &c.
2. Multiply by 20; $10x-40=5x+4x-20$, &c.
3. Multiply by 40; $20(x+1)+8(3x-4)+5=5(6x+7)$, &c.
4. Multiply by 60; $15(5x-11)-6(x-1)=5(11x-1)$, &c.
5. Multiply by 12; $6x+4x-3x=6$, &c.
6. Multiply by 12; $6(x+1)+4(x+2)=192-3(x+3)$, &c.
7. Multiply by 6; $6x+2(11-x)=3(26-x)$, &c.
8. Multiply by 2; $38x+7x-2=8x+35$, &c.
9. Multiply by 24; $6(x-3)+8(x-4)=12(x-5)+3(x+1)$, &c.
10. Multiply by 6; $3(5x-7)-2(2x+7)=6(3x-14)$, &c.
11. Multiply by 60; $15(x-3)-10(2x-5)=41+12(3x-8)-4(5x+6)$, &c.
12. Multiply by 6; $2(5x+3)-3(3x-7)=6(5x-10)$, &c.
13. Multiply by 6; $8-x+6x-10=3(x+6)-2x$, &c.
14. Multiply by 12; $6(x+3)-4(x-2)=3x-5+3$, &c.
15. Multiply by 30; $6(3x-1)-15(13-x)=70x-55(x+3)$, &c.
16. Multiply by 42; $6(5x-3)-14(9-x)=105x+133(x-4)$, &c.
17. Multiply by $5\times 7\times 11$; $55(5x-1)+35(9x-5)=77(9x-7)$, &c.
18. Multiply by $3\times 5\times 7$; $15(3x+5)-35(2x+7)+1050-63x=0$, &c.
19. Multiply by 12; $3x-2(5x+8)=4(2x-9)$, &c.
20. Multiply by 6; $12x-3(19-2x)=2(2x-11)$, &c.
21. $\dfrac{7x+9}{4}-x+\dfrac{2x-1}{9}=7$; multiply by 36, &c.
22. $\dfrac{7+9x}{4}-1+\dfrac{2-x}{9}=7x$; multiply by 36, &c.
23. Multiply by 12; $6(x+1)-3(5-x)=168-4(x+2)$, &c.
24. Multiply by 22; $2(7x-8)+\dfrac{22(15x+8)}{13}=66x-11(31-x)$;

that is $$14x-16+\frac{22(15x+8)}{13}=66x-341+11x;$$

therefore $$\frac{22(15x+8)}{13}=63x-325;$$ multiply by 13, &c.

25. Multiply by 8; $2(3x-11)-28+9x=32x-118$, &c.
26. Multiply by 12; $4(2x-1)-3(3x-2)=2(5x-4)-7x-6$, &c.
27. Multiply by 108; $4(2x-9)+6x-27(x-3)=900-108x$, &c.
28. Multiply by 120;

$$40(x-1)+24\left(4x-\frac{3}{4}\right)-15(7x-6)=240+60(x-2)+12(3x-9), \text{ \&c.}$$

29. Multiply by $5 \times 9 \times 13$; $117(2x-6)-65(x-4)-135x=0$, &c.

30. Multiply by 6; $6x=18x-3(4-x)+2$, &c.

31. Multiply by 45; $9(3x-7)+5(25-4x)=15(5x-14)$, &c.

32. Multiply by $8 \times 13 \times 19$; $152(2x+5)+247(40-x)=104(10x-427)$, &c.

33. $\frac{x}{7}-\frac{x-5}{11}+5=x-\frac{2x}{77}-1$; multiply by 77, &c.

34. Multiply by 12; $6(x-1)+4(x-2)=3(x+3)+2(x+4)+12$, &c.

35. $$\frac{(x-1)(x-3)-(x-2)^2}{(x-2)(x-3)}=\frac{(x-5)(x-7)-(x-6)^2}{(x-6)(x-7)};$$

that is $$\frac{1}{(x-2)(x-3)}=\frac{1}{(x-6)(x-7)};$$ clear of fractions;

$(x-6)(x-7)=(x-2)(x-3)$; $x^2-13x+42=x^2-5x+6$, &c.

36. $x^2-7x+10-(2x^2-15x+25)+x^2+5x-14=0$, &c.

37. $3-x-2(x^2+x-2)=-2x^2+11x-15$, &c.

38. $x-3-(3+2x-x^2)=x^2-2x-3+3-x$, &c.

39. Multiply by 30; $10(x+10)-18(3x-4)+5(3x-2)(2x-3)=30x^2-16$;
$10x+100-54x+72+5(6x^2-13x+6)=30x^2-16$, &c.

40. $x^2+x-\frac{15}{4}-(x^2+2x-15)+\frac{3}{4}=0$;

therefore $$x^2+x-(x^2+2x-15)=3, \text{ \&c.}$$

41. $x^2-x-\frac{15}{4}-(x^2-2x-15)-\frac{93}{4}=0$;

therefore $$x^2-x-(x^2-2x-15)=27, \text{ \&c.}$$

42. Multiply by 56; $4(9x+5)+\frac{28(8x-7)}{3x+1}=36x+15+41$;

therefore $\frac{28(8x-7)}{3x+1}=36$; therefore $\frac{7(8x-7)}{3x+1}=9$;

therefore $$7(8x-7)=9(3x+1), \text{ \&c.}$$

43. Multiply by 15; $6x+7-\frac{15(2x-2)}{7x-6}=3(2x+1)$;

therefore $4=\frac{15(2x-2)}{7x-6}$; therefore $28x-24=30x-30$, &c.

44. Multiply by 15; $6x+1-\frac{15(2x-4)}{7x-16}=3(2x-1)$;

therefore $4=\frac{15(2x-4)}{7x-16}$; therefore $28x-64=30x-60$, &c.

45. $\frac{4(x+3)+7(x+2)}{x^2+5x+6}=\frac{37}{x^2+5x+6}$; therefore $11x+26=37$, &c.

46. $x^2+2x+1=(6-1+x)x-2$; therefore $x^2+2x+1=5x+x^2-2$, &c.

47. $\dfrac{x-4-(x-2)}{(x-2)(x-4)}=\dfrac{x-8-(x-6)}{(x-6)(x-8)}$; $\dfrac{-2}{(x-2)(x-4)}=\dfrac{-2}{(x-6)(x-8)}$;
therefore $(x-6)(x-8)=(x-2)(x-4)$; $x^2-14x+48=x^2-6x+8$, &c.

48. Clear of fractions;

$$2(x-3)(3x-1)+(2x-5)(3x-1)=6(2x-5)(x-3);$$
$$2(3x^2-10x+3)+6x^2-17x+5=6(2x^2-11x+15), \text{ \&c.}$$

49. Multiply by $x+1$; $25-\dfrac{1}{3}x+\dfrac{(x+1)\left(16x+\frac{21}{5}\right)}{3x+2}=23+5x+5$;

$$\frac{(x+1)\left(16x+\frac{21}{5}\right)}{3x+2}=3+\frac{16x}{3};$$

$$(x+1)\left(16x+\frac{21}{5}\right)=(3x+2)\left(3+\frac{16x}{3}\right);$$

$$16x^2+\frac{21x}{5}+16x+\frac{21}{5}=16x^2+\frac{32x}{3}+9x+6, \text{ \&c.}$$

50. Multiply by 60; $80\left(x-\dfrac{a}{8}\right)-20\left(x-\dfrac{a}{4}\right)+15\left(x-\dfrac{a}{5}\right)=0$;

$$80x-10a-20x+5a+15x-3a=0;\ 25x=8a, \text{ \&c.}$$

51. $x^2+(a+b)x+ab=x^2+(c+d)x+cd$, &c.

52. Multiply by $a(b^2-a^2)$; $x(b^2-a^2)+ax(b+a)=a^2(b-a)$, &c.

53. Multiply by ab; $a^2bx+ab^2=bx+a$, &c.

54. Multiply by abc; $ac(x-a)+ab(x-b)+bc(x-c)=x-a-b-c$, &c.

55. $x^2+(a+b)x+ab-ab-ac=\dfrac{a^2c}{b}+x^2$;

$$(a+b)x=\frac{a^2c}{b}+ac=\frac{ac(a+b)}{b};\ \text{divide by } a+b.$$

56. Clear of fractions;

$$(a+b)(x-a)(x-b)=a(x-b)(x-c)+b(x-c)(x-a);$$
$$(a+b)x^2-(a+b)^2x+ab(a+b)=(a+b)x^2-\{c(a+b)+2ab\}x+2abc;$$

therefore $$x\{a^2+b^2-c(a+b)\}=ab(a+b)-2abc.$$

57. Clear of fractions; $(ax^2+bx+c)(px+q)=(px^2+qx+r)(ax+b)$;

$$pax^3+(pb+qa)x^2+(pc+qb)x+qc=pax^3+(pb+qa)x^2+(qb+ra)x+rb;$$

therefore $$(pc-ra)x=rb-qc.$$

58. Multiply by $a(a+b)^2$;

$$3a^2bc(a+b)+\frac{a^3b^2}{a+b}+(2a+b)b^2x=3ac(a+b)^2x+b(a+b)^2x;$$

therefore $$3a^2bc(a+b)+\frac{a^3b^2}{a+b}=3ac(a+b)^2x+a^2bx;$$

therefore $$a^2b\,\frac{3c(a+b)^2+ab}{a+b}=ax\left\{3c(a+b)^2+ab\right\},\text{ \&c.}$$

59. Clear of fractions; $m(x+a)^2+n(x+b)^2=(m+n)(x+a)(x+b)$;

$(m+n)x^2+2(ma+nb)x+ma^2+nb^2=(m+n)x^2+(m+n)(a+b)x+(m+n)ab$;

therefore $(m-n)(a-b)x=m(ab-a^2)+n(ab-b^2)=(nb-ma)(a-b)$, &c.

60. Clear of fractions; $(x-a)^3(x+a+2b)=(x+b)^3(x-2a-b)$;

$$x^4+(2b-2a)x^3-6abx^2+2(a^3+3a^2b)x-a^3(a+2b)$$
$$=x^4+(2b-2a)x^3-6abx^2-2(b^3+3ab^2)x-b^3(b+2a);$$

therefore $$2(a+b)^3x=a^4-b^4+2ab(a^2-b^2)=(a^2-b^2)(a+b)^2,\text{ \&c.}$$

61. $$3x^3-3(a+b+c)x^2+3(a^2+b^2+c^2)x-a^3-b^3-c^3$$
$$=3\{x^3-(a+b+c)x^2+(ab+bc+ca)x-abc\};$$

therefore $$3(a^2+b^2+c^2-ab-bc-ca)x=a^3+b^3+c^3-3abc$$
$$=(a^2+b^2+c^2-ab-bc-ca)(a+b+c),\text{ \&c.}$$

62. $\frac{15x}{100}+\frac{1575}{1000}-\frac{875x}{1000}=\frac{625x}{10000}$; $1500x+15750-8750x=625x$, &c.

63. $\frac{12x}{10}-\frac{10}{5}\left(\frac{18x}{100}-\frac{5}{100}\right)=\frac{4x}{10}+\frac{89}{10}$; $\frac{12x}{10}-\frac{36x}{100}+\frac{1}{10}=\frac{4x}{10}+\frac{89}{10}$, &c.

64. $\frac{48x}{10}-\frac{10}{5}\left(\frac{72x}{100}-\frac{5}{100}\right)=\frac{16x}{10}+\frac{89}{10}$;

$$\frac{48x}{10}-\frac{144x}{100}+\frac{1}{10}=\frac{16x}{10}+\frac{89}{10},\text{ \&c.}$$

X.

1. Suppose the property of the poorer person to be x pounds, then that of the richer person is $2x$ pounds: thus $x+2x=3870$.

2. Let x denote the number of shillings which one receives, and the number of half-crowns which the other receives: thus $\frac{x}{20}+\frac{x}{8}=420$.

3. Let x denote the number of shillings in the money of the purse: thus $\frac{x}{4}+\frac{x}{5}=45$.

4. Let x denote the number of pounds in the amount of the bill: thus $x-\frac{x}{7}-\frac{x}{5}=92$.

5. Let x denote the first part; then $46-x$ denotes the other part: thus $\frac{x}{7}+\frac{46-x}{3}=10$.

6. Let x denote the number of children; then $4x$ denotes the number of men, and $2x$ the number of women: thus $x+4x+2x=266$.

7. Let x denote the number of pounds in the person's income: thus

$$\frac{x}{3}+\frac{x}{8}+\frac{x}{10}+318=x.$$

8. Let x denote the number of pounds which B contributes; then A contributes $\frac{3x}{5}$ pounds, and C contributes $\frac{7x}{8}$ pounds: thus $x+\frac{3x}{5}+\frac{7x}{8}=594$.

9. Let x denote the number of pounds which A has; then B has $x+100$ pounds, and C has $x+100+270$ pounds: thus

$$x+x+100+x+100+270=1520.$$

10. Let x denote the number of pounds in the sum; then A has $\frac{x}{2}-30$ pounds, B has $\frac{x}{3}-10$ pounds, and C has $\frac{x}{4}+8$ pounds: thus

$$\frac{x}{2}-30+\frac{x}{3}-10+\frac{x}{4}+8=x.$$

11. Let x denote the greater number, then $5760-x$ denotes the less number: thus $x-(5760-x)=\frac{x}{3}$.

12. Let x denote the number of quarts which each cask originally contained: thus $x-34=2(x-80)$.

13. Let x denote the number of shillings in the cost of the print: thus $x-20=\frac{x+15}{2}$.

14. Let x denote the number of pounds in the value of a sheep: thus $72x+35=92x-35$.

15. Let x denote the number of pounds in the price of the house; then $850-x$ denotes the number of pounds in the price of the garden: thus

$$5x=12(850-x).$$

16. Let x denote the number of inches in the length of the rod: thus

$$\frac{x}{10}+\frac{x}{20}+\frac{x}{30}+\frac{x}{40}+\frac{x}{50}+\frac{x}{60}+302=x.$$

17. Let x denote the number of persons; then $\frac{2x}{3}$ persons received $\frac{3}{2}$ shillings each, and $\frac{x}{3}$ persons received $\frac{5}{2}$ shillings each: thus

$$\frac{2x}{3} \times \frac{3}{2} + \frac{x}{3} \times \frac{5}{2} = 55.$$

18. Let x denote the number: thus $\frac{x}{3} + \frac{x}{7} = 20.$

19. Let x denote the less number, then $x+1$ denotes the other: thus

$$(x+1)^2 - x^2 = 15.$$

20. Let x denote the whole number of kings: thus

$$\frac{x}{3} + \frac{x}{4} + \frac{x}{8} + \frac{x}{12} + 5 = x.$$

21. Let x denote the number of miles per hour at which the stream flows: then *with* the stream the boat moves over $9+x$ miles an hour, and *against* the stream the boat moves over $9-x$ miles an hour: thus $9+x=2(9-x)$.

22. Let x denote the number of shillings each had at the commencement. After the first game A has $x+\frac{x}{2}+1$ shillings, and B has $x-\frac{x}{2}-1$ shillings, that is A has $\frac{3x}{2}+1$ shillings and B has $\frac{x}{2}-1$. After the second game A has $\frac{3x}{2}+1-\frac{1}{2}\left(\frac{3x}{2}+1\right)-1$, and B has $\frac{x}{2}-1+\frac{1}{2}\left(\frac{3x}{2}+1\right)+1$: thus $\frac{x}{2}-1+\frac{1}{2}\left(\frac{3x}{2}+1\right)+1=3x+2-\left(\frac{3x}{2}+1\right)-2.$

23. Let x denote the number of pounds in the cost of the house; then $12000-x$ denotes the remainder: thus

$$\frac{12000-x}{3} \times \frac{4}{100} + \frac{2(12000-x)}{3} \times \frac{5}{100} = 392.$$

24. Let x denote the number of oxen, and therefore $35-x$ the number of sheep: thus $\frac{25x}{2} + \frac{9(35-x)}{4} = 191\frac{1}{2}.$

25. Let x denote the number of shillings in the purse: then A takes $2+\frac{x-2}{6}$, and B takes $3+\frac{1}{6}\left\{x-2-\frac{x-2}{6}-3\right\}$: thus

$$2+\frac{x-2}{6} = 3+\frac{x-5}{6} - \frac{x-2}{36}.$$

26. Let x denote the number of leaps the hare takes; then the greyhound takes $\frac{2x}{3}$ leaps, which are equivalent to $\frac{4x}{3}$ leaps of the hare: thus

$$80+x=\frac{4x}{3}.$$

27. Let x denote the number of yards in the breadth of the first field, and therefore $2x$ the number of yards in the length; then this field contains $2x^2$ square yards. Similarly the second field contains $(2x+50)(x+10)$ square yards. Thus $(2x+50)(x+10)=2x^2+6800$.

28. Let x denote the number of minutes. Then, as in Art. 171,

$$\frac{x}{80}+\frac{x}{200}+\frac{x}{300}=1.$$

29. Let x denote the number of pounds in the sum of the incomes below £100 a year, and therefore $500000-x$ the number in the sum of the incomes above £100 a year. Then $\frac{7x}{240}$ is the number of pounds raised by the tax on the former incomes, and $\frac{500000-x}{20}$ is the number of pounds raised by the tax on the latter incomes: thus $\frac{7x}{240}+\frac{500000-x}{20}=18750$.

30. Let x denote the number of pounds of the inferior tea; the cost in shillings of the tea which is mixed is $3x+5$, and it is sold for $\frac{44}{12}(x+1)$ shillings: thus $\frac{11}{3}(x+1)=3x+5+\frac{3x+5}{10}$.

31. Let x denote the number of oranges, and therefore $x+180$ the number of apples. Each apple is sold for $\frac{3}{5}$ of a penny; thus 35 apples are sold for 21 pence, and therefore 15 oranges are sold for $22\frac{1}{2}$ pence. Therefore each orange is sold for $\frac{45}{30}$ pence, that is for $\frac{3}{2}$ pence. Thus

$$\frac{3x}{2}+\frac{3}{5}(x+180)=234.$$

32. Let x denote the number of gallons drawn from A, and therefore $14-x$ the number drawn from B. Two-fifths of what is drawn from A consists of wine, and three-fourths of what is drawn from B: thus

$$\frac{2x}{5}+\frac{3(14-x)}{4}=7.$$

33. Let $3x$ denote the number of days in which C could dig the trench; then B could dig it in $2x$ days, and A in x days; therefore in 6 days C digs $\frac{6}{3x}$ of the trench, B digs $\frac{6}{2x}$ of it, and A digs $\frac{6}{x}$ of it: thus $\frac{6}{3x}+\frac{6}{2x}+\frac{6}{x}=1$.

34. Let x denote the number of pounds the person had at first: thus

$$x-\frac{7x}{240}=408\tfrac{113}{480}.$$

35. Let x denote the number of minutes after one o'clock. In x minutes the long hand will move over x divisions, and as the long hand moves twelve times faster than the short hand, the short hand will move over $\frac{x}{12}$ divisions in x minutes. At one o'clock the short hand is 5 divisions in advance of the long hand. Thus $5+\frac{x}{12}=x-1$.

36. Let x denote the number of miles the person can ride in the coach; then this takes $\frac{x}{b}$ hours; and to walk back takes $\frac{x}{c}$ hours: thus $\frac{x}{b}+\frac{x}{c}=a$.

37. Let x shillings per cwt. denote the duty after reduction. Since the consumption increases one-half $\frac{3x}{2}$ shillings is now obtained for every 6 shillings formerly obtained. As the revenue falls one-third what is now obtained is two-thirds of what was formerly obtained. Thus $\frac{3x}{2}=\frac{2}{3}\times 6$.

38. Let x denote the number of men. Then the ship sails with $60x$ lbs. of biscuit. In 20 days $20x$ lbs. are consumed. Then in the remaining 64 days $\frac{5}{7}(x-5)\,64$ lbs. are consumed. Thus $60x=20x+\frac{5(x-5)\,64}{7}$.

XI.

15. Simplify the equations; thus $x+5y=48$, $7x+y=132$.

16. From the first equation $143x=91y$; therefore $11x=7y$.

17. Multiply the first equation by 12;

$$8x-48+6y+12x=96-9y+1, \text{ that is } 20x+15y=145;$$

therefore $4x+3y=29$. Multiply the second equation by 6;

$$y-3x+12=1-12x+36, \text{ that is } 9x+y=25.$$

Then multiply the latter by 3, and subtract the former.

18. $4x+8y=\frac{24}{10}$, $\frac{102x}{10}-6y=\frac{348}{100}$: multiply the first equation by 3, and the second by 4, and add.

19. Substitute the value of x from the first equation in the second: thus

$$\frac{1}{5}(8y+7y)-1=\frac{2}{3}(8y-6y+1),$$

that is $$3y-1=\frac{4y}{3}+\frac{2}{3}.$$ Then multiply by 3.

20. Multiply the first equation by 4;

$$4x+6x-2y-2=1+3y-3, \text{ that is } 10x-5y=0; \text{ therefore } y=2x.$$

Multiply the second equation by 10;

$$8x+6y=7y+20, \text{ that is } y=8x-20.$$

21. Multiply the first equation by 10;

$$15x-25y+30=4x+2y, \text{ that is } 11x-27y+30=0.$$

Multiply the second equation by 12;

$$96-3x+6y=6x+4y, \text{ that is } 9x-2y=96.$$

22. Multiply the first equation by 180;

$$54x-12y-80=15x-10y, \text{ that is } 39x-2y=80.$$

Multiply the second equation by 60;

$$120x-160=5x-4y+66, \text{ that is } 115x+4y=226.$$

Then multiply the former by 2 and add to the latter.

23. Multiply the first equation by 30;

$$24x-18y-42=9x-4y-25, \text{ that is } 15x-14y=17.$$

Multiply the second equation by 60;

$$20y-20+30x-9y=4y-4x+10x+66, \text{ that is } 24x+7y=86.$$

Then multiply the latter by 2, and add to the former.

24. The first equation is

$$\frac{4}{7}\left(\frac{2x}{3}-\frac{5y}{12}\right)-\frac{2}{23}\left(\frac{3x}{2}-\frac{y}{3}\right)=2,$$

that is

$$\frac{8x}{21}-\frac{5y}{21}-\frac{3x}{23}+\frac{2y}{69}=2.$$

Multiply by $3\times21\times23$: thus $552x-345y-189x+42y=2898$, that is $363x-303y=2898$; divide by 3: thus $121x-101y=966$. The second equation gives $5(x-y)=x+y$, that is $4x=6y$; therefore $2x=3y$.

25. Multiply the first equation by $3\times5\times7\times8$;

$$840x-560y+840+1155y-1050=480x-360y+600+7560-168x,$$

that is $528x+955y=8370$. Multiply the second equation by 18;

$$810-24x+12=55x+71y+1, \text{ that is } 79x+71y=821.$$

Then multiply the former by 71, and the latter by 955, and subtract.

26. The first equation is

$$\frac{24x}{10}+\frac{32y}{100}-\frac{10}{5}\left(\frac{36x}{100}-\frac{5}{100}\right)=\frac{8x}{10}+\frac{100}{25}\left(\frac{26}{10}+\frac{5y}{1000}\right),$$

that is

$$\frac{24x}{10}+\frac{32y}{100}-\frac{72x}{100}+\frac{10}{100}=\frac{8x}{10}+\frac{104}{10}+\frac{20y}{1000}:$$

multiply by 100; $240x+32y-72x+10=80x+1040+2y$,

that is $88x+30y=1030$. The second equation is

$$\frac{10}{3}\left(\frac{4y}{100}+\frac{1}{10}\right)=\frac{10}{6}\left(\frac{7x}{100}-\frac{1}{10}\right); \text{ therefore } 2\left(\frac{4y}{100}+\frac{1}{10}\right)=\frac{7x}{100}-\frac{1}{10}.$$

Multiply by 100; $8y+20=7x-10$, that is $7x-8y=30$. Then multiply the former by 4, and the latter by 15, and add.

27. Multiply the first equation by 6, and the second by 11, and add.

28. Multiply the first equation by m, and the second by n, and subtract: thus $\frac{m^2-n^2}{x}=m-n$; therefore $x=\frac{m^2-n^2}{m-n}=m+n$. Again multiply the first equation by n, and the second by m, and subtract: thus we obtain $y=m+n$.

29. Multiply the second equation by 6; $\frac{2x}{a}+\frac{y}{b}=4$; then subtract the first equation.

30. Multiply the first equation by n, and the second by b, and add: thus $(na+mb)x=nc+bd$. Again multiply the first equation by m, and the second by a, and subtract: thus $(na+mb)y=mc-ad$.

31. $x(a+c)+y(b+c)=2(a+c)(b+c)$, $ax-by=(a-b)c$. Multiply the first equation by b, and the second by $(b+c)$ and add: thus

$$x\{c(a+b)+2ab\}=c^2(a+b)+(3ab+b^2)c+2ab^2;$$

hence, by division, $x=c+b$. Substitute this value in the first equation: thus $y(b+c)=(a+c)(b+c)$; therefore $y=a+c$.

32. $x(a-b)+y(a+b)=2a(a^2-b^2)$, $x-y=4ab$. Multiply the second equation by $a+b$, and add it to the first; thus

$$2ax=2a(a^2-b^2)+4ab(a+b);$$

therefore $x=a^2-b^2+2ab+2b^2=(a+b)^2$. Substitute this value in the second equation: thus $y=(a+b)^2-4ab=(a-b)^2$.

XII.

10. Add the first and second equations; $\frac{2}{x}+\frac{1}{y}+\frac{1}{z}=3$: subtract the third; $\frac{2}{x}=\frac{3}{2}$: therefore $x=\frac{4}{3}$. Substitute the value of x in the first and second equations.

11. Multiply the first equation by 2, and add to the second;

$$\frac{4}{x}+\frac{3}{z}=\frac{6}{z}+2, \text{ that is } \frac{4}{x}-\frac{3}{z}=2:$$

multiply the third equation by 3 and add to this; $\frac{7}{x}=6$: therefore $x=\frac{7}{6}$. Then find z from the third equation, and y from the second.

12. Add the second and third equations; $\frac{17}{15x}+\frac{6}{z}=\frac{788}{30}$: therefore $\frac{17}{x}+\frac{90}{z}=394$. Multiply the first equation by 5, and the second by 8, and add; $\frac{53}{3x}+\frac{21}{z}=\frac{358}{3}$: therefore $\frac{53}{x}+\frac{63}{z}=358$. Multiply the former result by 7 and the latter by 10, and subtract; $\frac{411}{x}=822$: therefore $x=\frac{1}{2}$.

13. Multiply the first equation by 20, the second by 12, the third by 42, and simplify;

$$10x+15y-24z=41, \quad 15x-12y+16z=10, \quad 18x-14y-7z=-13.$$

14. Clear of fractions, and simplify;

$$35x+3y-15z=0, \quad 3x-y-z=-4, \quad 125x+129y-105z=0.$$

15. Multiply the second equation by 3, and the fourth by 7, and subtract; $133x-33z=4$. Multiply the first equation by 7, and the third by 3, and subtract; $49x-12z=4$. Hence find x and z.

16. Find x, y, and z from the last three equations; and then find u from the first.

17. Multiply the third equation by 3, and the fourth by 4, and subtract; $6x+20y=38$: use this and the first two equations.

18. Find u, x, and z from the first, third, and fourth equations; and then find y from the second.

19. Multiply the second equation by 2, and subtract the fourth; $4y-4z+3u=13$. Multiply the first equation by 3, and the third by 7, and add; $35y-6z-5u=107$. From these two and the fifth equation find y, z, and u.

20. Find v from the third equation; substitute in the others; clear of fractions and simplify: thus we eliminate v and obtain

$$\begin{aligned} 6x+22y-3z-3u&=28, \\ 3x-5y+2z-4u&=11, \\ -2x+10y+2z+7u&=5, \\ 8x-34y+9z+3u&=6. \end{aligned}$$

Next we eliminate u from these equations; we find u from the first equation, substitute in the others, and simplify; thus we obtain

$$15x+103y-18z=79, \quad 36x+184y-15z=211, \quad 7x-6y+3z=17.$$

21. Add the first and second equations, and subtract the third.

22. From the first equation $y=\dfrac{c-bx}{a}$; from the second $z=\dfrac{b-cx}{a}$; substitute in the third; $\dfrac{b(b-cx)}{a}+\dfrac{c(c-bx)}{a}=a$: multiply by a and simplify.

23. Add the first and second equations, and subtract the third.

24. Substitute the value of z from the first equation in the second;

$$(b+c)x+(c+a)y-(a+b)(x+y)=0, \text{ that is } (c-a)x=(b-c)y.$$

Again substitute the value of y from the first equation in the second: thus we get $(b-a)x=(c-b)z$. Then from the third equation

$$bcx+\frac{ca(c-a)x}{b-c}+\frac{ab(b-a)x}{c-b}=1,$$

that is $$bcx + \frac{ca(c-a) - ab(b-a)}{b-c}x = 1,$$

that is $x\{bc + a^2 - a(b+c)\} = 1$, that is $x(a-b)(a-c) = 1$.

25. Multiply the first equation by c and subtract the second;
$$a(c-a)x + b(c-b)y = A(c-A).$$
Multiply the second equation by c and subtract the third;
$$a^2(c-a)x + b^2(c-b)y = A^2(c-A).$$
Multiply the former by b and subtract the latter;
$$a(b-a)(c-a)x = A(b-A)(c-A).$$
Similarly for y and z.

26. $xyz = a(yz - zx - xy)$. Divide by $axyz$: thus $\frac{1}{a} = \frac{1}{x} - \frac{1}{y} - \frac{1}{z}$. Similarly we get $\frac{1}{b} = \frac{1}{y} - \frac{1}{x} - \frac{1}{z}$, and $\frac{1}{c} = \frac{1}{z} - \frac{1}{x} - \frac{1}{y}$. Add the second and third of these.

27.
$$x + y + z = a + b + c,$$
$$bx + cy + az = a^2 + b^2 + c^2,$$
$$cx + ay + bz = a^2 + b^2 + c^2.$$
Substitute the value of z from the first in the others;
$$(b-a)x + (c-a)y = b^2 + c^2 - a(b+c),$$
$$(c-b)x + (a-b)y = a^2 + c^2 - b(a+c).$$
Multiply the former by $b-a$, and the latter by $c-a$, and add;
$$x\{(b-a)^2 + (c-a)(c-b)\} = (b-a)(b^2 + c^2 - ab - ac) + (c-a)(a^2 + c^2 - ba - bc),$$
that is
$$x(a^2 + b^2 + c^2 - ab - bc - ca)$$
$$= -a^3 + b^3 + c^3 + 2a^2b + 2a^2c - 2b^2a - 2c^2a - abc,$$
therefore by division $x = b + c - a$. Similarly for y and z.

28. Subtract the second equation from the first,
$$-(a-b)y + (a^2 - b^2)z = a^3 - b^3;$$
divide by $a-b$, $\quad -y + (a+b)z = a^2 + ab + b^2.$

Subtract the third equation from the second,
$$-(b-c)y + (b^2 - c^2)z = b^3 - c^3;$$
divide by $b-c$, $\quad -y + (b+c)z = b^2 + bc + c^2.$

Subtract the latter from the former,
$$(a-c)z = a^2 - c^2 + b(a-c);$$
divide by $a-c$, $\quad z = a + c + b.$

Hence $y = (b+c)(a+c+b) - b^2 - bc - c^2$; from which we get $y = ab + bc + ca$. Substitute the values of y and z in any of the three given equations, and we obtain the value of x.

XIII.

1. Let x denote the numerator, and y the denominator of the fraction: thus $\frac{x+3}{y}=1$, $\frac{x}{y+2}=\frac{1}{2}$.

2. Let x denote the number of pounds in A's money, and y the number in B's: thus $x+y=570$, $3x+5y=2350$.

3. Let x denote the numerator, and y the denominator of the fraction: thus $\frac{x+1}{y}=\frac{1}{3}$, $\frac{x}{y+1}=\frac{1}{4}$.

4. Let x denote the first number, and y the second: thus

$$x+4y=29, \quad 6x+y=36.$$

5. Let x denote the number of shillings in A's money, and y the number in B's: thus $x+36=3y$, $y-5=\frac{x}{2}$.

6. Let x denote the number of shillings in A's money, and y the number in B's: thus $x-10=2(y+10)-25$, $y-10=\frac{5}{17}(x+10)$.

7. Let x denote the first number, and y the second: thus

$$2x+y=17, \quad 2y+x=19.$$

8. Let x denote the first number, and y the second: thus

$$\frac{x}{2}+\frac{3y}{4}=3x-y, \quad 3x-y=11.$$

9. Suppose that tea costs x shillings a pound and coffee y shillings a pound: thus $32x+15y=105$, $36x+9y=105$.

10. Let x, y, and z denote the first, second, and third numbers respectively: thus $x+y+z=9$, $x+2y+3z=22$, $x+4y+9z=58$.

11. Let x shillings denote the price of a pound of tea, and y shillings the price of a pound of sugar: thus

$$x+3y=6, \quad \frac{11x}{10}+\frac{3}{2}\times 3y=7.$$

12. Let x denote the number of pounds invested in consols, and y the number in railway shares: thus $x+y=2550$. The annual income from consols is $\frac{3x}{81}$; and that from the railway shares is $\frac{y}{24}$: thus $\frac{3x}{81}=\frac{y}{24}$. We shall find that $y=1200$, so that the number of railway shares $=50$.

13. Let x denote the number of pounds in the capital of the first person, and y the rate per cent. at which it is invested: thus

$$\frac{y+1}{100}(x+1000)=\frac{yx}{100}+80, \quad \frac{y+2}{100}(x+1500)=\frac{yx}{100}+150;$$

therefore $\quad x+1000y+1000=8000, \quad 2x+1500y+3000=15000.$

14. Suppose that there are x persons and each receives y shillings: thus $(x+4)(y-1)=xy$, $(x-5)(y+2)=xy$.

15. Let x denote the number of gallons which the first plug hole would discharge in an hour, and y the number the second would discharge: thus $3x+3y+11y=192$, $6x+6y+6y=192$.

16. Suppose the original income to be x pounds, and the poor-rate to be y pence per pound: thus

$$x-\frac{7x}{240}-\frac{yx}{240}=486, \quad \frac{yx}{240}=\frac{7x}{240}+22\tfrac{1}{3};$$

therefore by addition, $x-\dfrac{7x}{240}=\dfrac{7x}{240}+508\tfrac{1}{3}$; this finds x, and by substituting the value of x in either equation we obtain y.

17. Let x, y, and z denote the numbers in the first, second, and third classes respectively: thus

$$x+y-z=4(y+z-x)-10,$$
$$x+30=y+z-29+1,$$
$$x+y+z=8(z-y)+34.$$

18. Suppose that the farmer has x pounds, that an ox costs y pounds, and a lamb z pounds. Thus $x=4y+32z$, $x=4y+16z+\dfrac{6}{20}(4+16)+9$. Each ox cost y shillings for conveyance, and each lamb $\dfrac{16z}{3}$ shillings; so that the whole cost of conveyance was $4y+\dfrac{256z}{3}$ shillings, and as 20 animals were conveyed at an average cost of 6 shillings per head we have $4y+\dfrac{256z}{3}=120$.

19. Suppose that A won x games and B won y games. Then A has received $2x$ shillings and paid $3y$ shillings. Thus $2x-3y=3$. Similarly we obtain $2(x-1)-5(y+1)=-30$.

20. Suppose that originally A, B, C, D, and E had x, y, z, u, v shillings respectively: thus $x+\dfrac{1}{2}y=30$, $y-\dfrac{1}{2}y+\dfrac{1}{3}z=30$, $z-\dfrac{1}{3}z+\dfrac{1}{4}u=30$, $u-\dfrac{1}{4}u+\dfrac{1}{6}v=30$, $v-\dfrac{1}{6}v=30$. From the last equation find v; then u from the preceding equation; and so on.

21. Suppose that the distance is x miles, and that the coach goes at the rate of y miles an hour; then the railway train goes at the rate of $2y$ miles an hour. Thus $\dfrac{x-15}{y}=\dfrac{x}{2y}+3$, $\dfrac{\frac{2x}{3}}{y}=\dfrac{\frac{2x}{3}}{2y}+3$.

22. Suppose that A could do the work in x days, and B in y days: thus

$$\frac{30}{x}+\frac{30}{y}=1, \quad \frac{30+5\frac{1}{3}-8}{x}+\frac{30+5\frac{1}{3}-4}{y}=1.$$

23. Suppose that A could run a mile in x minutes and that B could run a mile in y minutes. At first B runs $1760-44$ yards in 51 seconds more time than A takes to run a mile: thus $\frac{1760-44}{1760}y=x+\frac{51}{60}$. At the second heat A runs $1760-88$ yards in 1 minute 15 seconds less time than B takes to run a mile: thus $\frac{1760-88}{1760}x=y-1\frac{1}{4}$.

24. Suppose that the distance from the foot to the summit of the mountain is x miles, that A walks y miles per hour, and B walks z miles per hour. Therefore $\frac{x}{y}$ is the time in which A would reach the summit: thus $\frac{x}{y}=\frac{x}{z}-\frac{1}{2}$. And $\frac{2}{2y}$ is the time A takes over the needless mile and back: thus $\frac{x}{y}+\frac{2}{2y}=\frac{x}{z}-\frac{6}{60}$. Also $\frac{x}{2\frac{1}{7}}=\frac{x}{z}+\frac{10}{60}-\frac{20}{60}$. Subtract the second equation from the first and we find y; then subtract the third from the first and we find x.

25. Suppose that the length of the line is x miles, and that the train originally goes y miles an hour. When the accident happens the train has $x-y$ miles still to go, and at the diminished rate this will take $\frac{x-y}{\frac{3}{5}y}$ hours.

Then $2+\frac{x-y}{\frac{3}{5}y}=\frac{x}{y}+3$. If the accident had happened 50 miles further on the time taken before the accident would have been $1+\frac{50}{y}$ hours; and there would have been $x-50-y$ miles still to go. Thus $2+\frac{50}{y}+\frac{x-50-y}{\frac{3}{5}y}=\frac{x}{y}+3-1\frac{1}{3}$.

Multiply each equation by $3y$;

$$6y+5(x-y)=3x+9y, \quad 6y+150+5(x-50-y)=3x+5y.$$

26. Suppose that originally A has x shillings, B has y shillings, and C has z shillings. After the first game A has $x-y-z$ shillings; B has $2y$; and C has $2z$. After the second game A has $2x-2y-2z$ shillings; B has $2y-(x-y-z)-2z$, that is $3y-x-z$; and C has $4z$. After the third game A has $4x-4y-4z$ shillings; B has $6y-2x-2z$; and C has $4z-(2x-2y-2z)-(3y-x-z)$, that is $7z-y-x$. Thus $4x-4y-4z=16$, $6y-2x-2z=16$, $7z-y-x=16$.

27. Suppose that A could do it in x days, and B in y days: thus

$$\frac{m}{x}+\frac{m}{y}=1, \quad \frac{n}{x}+\frac{n+p}{y}=1.$$

28. Suppose that the original rate of the train is x miles per hour, and that the accident happens y miles from Cambridge. By the diminishing of the speed the time taken over these y miles is $\frac{y}{\frac{1}{n}x}$ hours instead of $\frac{y}{x}$ hours: thus $\frac{y}{\frac{1}{n}x}=\frac{y}{x}+a$; therefore $ny=y+ax$. Similarly $\frac{y-b}{\frac{1}{n}x}=\frac{y-b}{x}+c$; therefore $n(y-b)=y-b+cx$. Subtract the second equation from the first, and we find x.

29. Suppose the circumference of the fore-wheel to be x yards, and that of the hind-wheel to be y yards: thus $\frac{120}{x}-\frac{120}{y}=6$, $\frac{120}{\frac{5x}{4}}-\frac{120}{\frac{6y}{5}}=4$.

30. Let x denote the digit in the tens' place, and y the digit in the units' place; then the number is $10x+y$: thus

$$10x+y=3(x+y),\quad 10x+y+45=10y+x.$$

31. Let x denote the digit in the tens' place, and y the digit in the units' place; then the number is $10x+y$: thus $10x+y=7(x+y)$, $10x+y-27=10y+x$.

32. Suppose that the distance from A to B is x miles, from A to C is y miles, and from C to B is z miles; and that the coach goes u miles an hour, and therefore the train $3u$ miles. Then

$$\frac{x}{u}=\frac{y}{3u}+\frac{z}{3u}-\frac{1}{3};\text{ therefore } 3x=y+z-u;$$

$$\frac{2x}{u}=\frac{y}{3u}+\frac{z}{3u}+\frac{z}{3u};\text{ therefore } 6x=y+2z;$$

$$\frac{x}{u}+1=\frac{y}{3u}+\frac{z+\frac{1}{2}z}{3u};\text{ therefore } 6x+6u=2y+3z;$$

$$x+y+z=76\tfrac{2}{3}.$$

33. Suppose that the course is x yards, that A's original rate is y yards per minute and B's original rate z yards per minute. A goes twice round the course and 150 yards more while B goes twice round: thus $\frac{2x+150}{y}=\frac{2x}{z}$. In the second race A goes round four times at the rate of $\frac{4y}{3}$ yards per minute; B goes round once at the rate of $\frac{9z}{8}$ yards per minute, once at the rate of z yards per minute, and once round all but 180 yards at the rate of $\frac{9z}{10}$ yards per minute: thus

$$\frac{4x}{\frac{4y}{3}}=\frac{x}{\frac{9z}{8}}+\frac{x}{z}+\frac{x-180}{\frac{9z}{10}},\ \text{ that is } \frac{3x}{y}=\frac{1}{z}\left\{\frac{8x}{9}+x+\frac{10(x-180)}{9}\right\}.$$

Divide the terms of this equation by the terms of the first: thus

$$\frac{3x}{2x+150}=\frac{8x}{18x}+\frac{x}{2x}+\frac{10(x-180)}{18x},$$

that is
$$\frac{3x}{2x+150}=\frac{4}{9}+\frac{1}{2}+\frac{5}{9}-\frac{100}{x};$$

therefore
$$\frac{100}{x}=\frac{3}{2}-\frac{3x}{2x+150}=\frac{450}{2(2x+150)};$$

therefore $400x+30000=450x$; therefore $x=600$.

34. Suppose the original rate of the coach to be x miles per hour, and that of the man y miles. Suppose that the coach goes z hours before it overtakes the man; then as the man started p hours before the coach,

$$(p+z)\,y=zx\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

At the increased rates the coach goes $\frac{6x}{5}(z+q)$ miles in $z+q$ hours, and the man goes $\frac{5y}{4}(z+q)$ miles: thus $\frac{6x}{5}(z+q)-\frac{5y}{4}(z+q)=92$,

that is
$$\left(\frac{6x}{5}-\frac{5y}{4}\right)(z+q)=92\ldots\ldots\ldots\ldots\ldots(2).$$

Also
$$(x-y)(z+q)=80\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Divide (2) by (3);
$$\frac{\frac{6x}{5}-\frac{5y}{4}}{x-y}=\frac{92}{80};$$

from this we get
$$x=2y\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Then from (1) we get $z=p$. Therefore from (3) if $p+q=16$, we get $x-y=\frac{80}{16}=5$: from this and (4) find x and y.

XIV.

1. $3a-[b+\{2a-(b-c)\}]=3a-[b+\{2a-b+c\}]=3a-[b+2a-b+c]$
$=3a-[2a+c]=3a-2a-c=a-c,$

$$\frac{2c^2-\frac{1}{2}}{2c+1}=\frac{4c^2-1}{2(2c+1)}=\frac{2c-1}{2},$$

$$a-c+\frac{1}{2}+\frac{2c-1}{2}=a-c+\frac{1}{2}+c-\frac{1}{2}=a.$$

2.
$$\begin{array}{r|l|l}
3x^3+14x^2+22x+21 & 6x^4+10x^3+2x^2-20x-28 & 2x-6 \\
 & 6x^4+28x^3+44x^2+42x & \\ \hline
 & -18x^3-42x^2-62x-28 & \\
 & -18x^3-84x^2-132x-126 & \\ \hline
 & 42x^2+70x+98 &
\end{array}$$

Divide by 14;

$$3x^2+5x+7\ \Big)\ \begin{array}{r} 3x^3+14x^2+22x+21 \\ \underline{3x^3+\ \,5x^2+\ \ 7x\quad\quad\ } \\ 9x^2+15x+21 \\ \underline{9x^2+15x+21} \end{array}\ \Big(\ x+3$$

Thus $3x^2+5x+7$ is the G.C.M.

3. $$\frac{\dfrac{a^3}{a-b}-a}{b}-\frac{\dfrac{a^2}{a-b}-b}{a}=\frac{a^3-a^2+ab}{b(a-b)}-\frac{a^2-ab+b^2}{a(a-b)}$$

$$=\frac{ab}{b(a-b)}-\frac{a^2-ab+b^2}{a(a-b)}=\frac{a^2-(a^2-ab+b^2)}{a(a-b)}=\frac{ab-b^2}{a(a-b)}=\frac{b}{a}.$$

4. The expression $=\dfrac{c-b+a-c+b-a}{(a-b)(b-c)(c-a)}=0.$

5. First take $m=1$. Multiply out the factors in the numerator and in the denominator: some of the terms cancel, and we shall then have

$$\frac{(b-d)(ac-bd)}{(a-c)(ac-bd)}.$$

Next take $m=2$. Multiply out, and we shall then have

$$\frac{(b-d)(abc+acd-abd-bcd)}{(a-c)(abc+acd-abd-bcd)}.$$

6. $$\frac{a^3+b^3+c^3-3abc}{(a-b)^2+(b-c)^2+(c-a)^2}=\frac{(a^2+b^2+c^2-ab-bc-ca)(a+b+c)}{2(a^2+b^2+c^2-ab-bc-ca)}.$$

7. $$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}$$

$$=\frac{x(1-y^2)(1-z^2)+y(1-x^2)(1-z^2)+z(1-x^2)(1-y^2)}{(1-x^2)(1-y^2)(1-z^2)}.$$

The numerator of the last expression

$$=x+y+z-x(y^2+z^2)-y(z^2+x^2)-z(x^2+y^2)+xyz(yz+zx+xy);$$

and since $yz+zx+xy=1$, we may write this

$$(x+y+z)(yz+zx+xy)-x(y^2+z^2)-y(z^2+x^2)-z(x^2+y^2)+xyz,$$

which gives by multiplying out $3xyz+xyz$, that is $4xyz$.

8. $$x^3-6ax^2+12a^2x-8a^3+x^3-6bx^2+12b^2x-8b^3$$
$$=2\{x^3-3(a+b)x^2+3(a+b)^2x-(a+b)^3\},$$

therefore $\quad x\{12a^2+12b^2-6(a+b)^2\}=8a^3+8b^3-2(a+b)^3,$

that is $\quad 6x(a^2-2ab+b^2)=6a^3+6b^3-6a^2b-6ab^2,$

therefore $\quad x(a^2-2ab+b^2)=(a+b)(a^2-2ab+b^2),$

therefore $\quad x=a+b.$

9.
$$x+y+z=a+b+c,$$
$$bx+cy+az=ab+bc+ca,$$
$$cx+ay+bz=ab+bc+ca.$$

Multiply the first equation by a, and subtract the second;
$$(a-b)\,x+(a-c)\,y=a^2-bc.$$

Multiply the first equation by b, and subtract the third;
$$(b-c)\,x+(b-a)\,y=b^2-ac.$$

Multiply the former by $a-b$, and the latter by $a-c$, and add;
$$x\{(a-b)^2+(a-c)(b-c)\}=(a-b)(a^2-bc)+(a-c)(b^2-ac);$$

that is $x(a^2+b^2+c^2-ab-bc-ca)=a(a^2+b^2+c^2-ab-bc-ca)$,

therefore $$x=a.$$

10.
$$\begin{array}{r|l|l}
x^3+6x^2+11x+6 & x^3+7x^2+14x+8 & 1 \\
 & \underline{x^3+6x^2+11x+6} & \\
 & \qquad x^2+\ 3x+2 &
\end{array}$$

$$\begin{array}{r|l|l}
x^2+3x+2 & x^3+6x^2+11x+6 & x+3 \\
 & \underline{x^3+3x^2+\ 2x} & \\
 & \quad 3x^2+\ 9x+6 & \\
 & \quad \underline{3x^2+\ 9x+6} &
\end{array}$$

Thus the G.C.M. of the first two expressions is x^2+3x+2; and their L.C.M. is $(x+3)(x^3+7x^2+14x+8)$, that is $x^4+10x^3+35x^2+50x+24$.

It will be found that this is divisible by the other two expressions, and is therefore the L.C.M. of the four. It may be shewn that it

$$=(x+1)(x+2)(x+3)(x+4).$$

XV.

1.
$$\begin{array}{r|l|l}
x^4+2x^3-10x^2-11x-12 & x^4+3x^3-\ 7x^2-21x-36 & 1 \\
 & \underline{x^4+2x^3-10x^2-11x-12} & \\
 & \qquad x^3+\ 3x^2-10x-24 &
\end{array}$$

$$\begin{array}{r|l|l}
x^3+3x^2-10x-24 & x^4+2x^3-10x^2-11x-12 & x-1 \\
 & \underline{x^4+3x^3-10x^2-24x} & \\
 & \quad -\ x^3 \qquad\quad +13x-12 & \\
 & \quad \underline{-\ x^3-\ 3x^2+10x+24} & \\
 & \qquad\quad 3x^2+\ 3x-36 &
\end{array}$$

Divide by 3;
$$\begin{array}{r|l|l}
x^2+x-12 & x^3+3x^2-10x-24 & x+2 \\
 & \underline{x^3+\ x^2-12x} & \\
 & \quad 2x^2+\ 2x-24 & \\
 & \quad \underline{2x^2+\ 2x-24} &
\end{array}$$

Thus x^2+x-12 is the G.C.M.

3. $$\frac{2}{2-y}=\frac{2}{2-\dfrac{2}{2-x}}=\frac{2(2-x)}{4-2x-2}=\frac{2-x}{1-x},$$

$$\frac{2}{2-z}=\frac{2}{2-\dfrac{2-x}{1-x}}=\frac{2(1-x)}{2-2x-2+x}=\frac{2(1-x)}{-x},$$

$$\frac{2}{2-w}=\frac{2}{2+\dfrac{2(1-x)}{x}}=\frac{2x}{2x+2-2x}=\frac{2x}{2}=x.$$

4. Reduce the expression to a common denominator; then the numerator
$=s(s-b)(s-c)+s(s-a)(s-c)+s(s-a)(s-b)-(s-a)(s-b)(s-c)$
$=s\{3s^2-2s(a+b+c)+bc+ca+ab\}-\{s^3-s^2(a+b+c)+s(ab+bc+ca)-abc\}$
$=2s^3-s^2(a+b+c)+abc=abc.$

5. It is shewn in Art. 110 that every expression which is a measure of A and B divides D: thus D is a common multiple of all the common measures. But no expression lower than D can be divided by D. Thus D is the least common multiple of all the divisors.

6. Multiply out; thus $x^4-22x^3+164x^2-458x+315$
$=x^4-22x^3+164x^2-488x+480.$

7. Multiply the first equation by c, and subtract the second;

$$(c-a)x+(c-b)y=0;$$

therefore $y=\dfrac{(c-a)x}{b-c}$. Multiply the first equation by b, and subtract the second; $(b-a)x+(b-c)z=0$; therefore $z=\dfrac{(a-b)x}{b-c}$. Substitute the values of y and z in the third equation;

$$x\left\{bc+\frac{ca(c-a)}{b-c}+\frac{ab(a-b)}{b-c}\right\}+(a-b)(b-c)(c-a)=0.$$

Thus we get $$\frac{x(a-b)(b-c)(c-a)}{b-c}=(a-b)(b-c)(c-a).$$

9. Suppose that x pounds are left to a child, and y pounds to a brother;

thus $$5x+3y=12670,\quad x-\frac{x}{100}=2\left(y-\frac{3y}{100}\right).$$

10. Clear of fractions;

$$(x-3a)(x+2a)(x+a)+2(x+6a)(x+2a)(x+a)+3(x+6a)(x-3a)(x+a)$$
$$=6(x+6a)(x-3a)(x+2a);$$

$$x^3-7a^2x-6a^3+2(x^3+9ax^2+20a^2x+12a^3)+3(x^3+4ax^2-15a^2x-18a^3)$$
$$=6(x^3+5ax^2-12a^2x-36a^3);$$

therefore $-12a^2x-36a^3=-72a^2x-216a^3;$

therefore $60x=-180a;$ therefore $x=-3a.$

XVI.

All these examples may be worked by ordinary multiplication.

XVII.

1.

$$\begin{array}{r|l}
 & x^4-2x^3+3x^2-2x+1\ (x^2-x+1 \\
 & x^4 \\ \hline
2x^2-x & -2x^3+3x^2-2x+1 \\
 & -2x^3+\ x^2 \\ \hline
2x^2-2x+1 & 2x^2-2x+1 \\
 & 2x^2-2x+1 \\ \hline
\end{array}$$

2.

$$\begin{array}{r|l}
 & x^4-4x^3\qquad+8x+4\ (x^2-2x-2 \\
 & x^4 \\ \hline
2x^2-2x & -4x^3\qquad+8x+4 \\
 & -4x^3+4x^2 \\ \hline
2x^2-4x-2 & -4x^2+8x+4 \\
 & -4x^2+8x+4 \\ \hline
\end{array}$$

3.

$$\begin{array}{r|l}
 & 4x^4+12x^3+5x^2-6x+1\ (2x^2+3x-1 \\
 & 4x^4 \\ \hline
4x^2+3x & 12x^3+5x^2-6x+1 \\
 & 12x^3+9x^2 \\ \hline
4x^2+6x-1 & -4x^2-6x+1 \\
 & -4x^2-6x+1 \\ \hline
\end{array}$$

4.

$$\begin{array}{r|l}
 & 4x^4-4x^3+5x^2-2x+1\ (2x^2-x+1 \\
 & 4x^4 \\ \hline
4x^2-x & -4x^3+5x^2-2x+1 \\
 & -4x^3+\ x^2 \\ \hline
4x^2-2x+1 & 4x^2-2x+1 \\
 & 4x^2-2x+1 \\ \hline
\end{array}$$

5.

$$\begin{array}{r|l}
 & 4x^4-12ax^3+25a^2x^2-24a^3x+16a^4\ (2x^2-3ax+4a^2 \\
 & 4x^4 \\ \hline
4x^2-3ax & -12ax^3+25a^2x^2-24a^3x+16a^4 \\
 & -12ax^3+\ 9a^2x^2 \\ \hline
4x^2-6ax+4a^2 & 16a^2x^2-24a^3x+16a^4 \\
 & 16a^2x^2-24a^3x+16a^4 \\ \hline
\end{array}$$

6.

$$\begin{array}{r|l}
 & 25x^4-30ax^3+49a^2x^2-24a^3x+16a^4\ (5x^2-3ax+4a^2 \\
 & 25x^4 \\ \hline
10x^2-3ax & -30ax^3+49a^2x^2-24a^3x+16a^4 \\
 & -30ax^3+\ 9a^2x^2 \\ \hline
10x^2-6ax+4a^2 & 40a^2x^2-24a^3x+16a^4 \\
 & 40a^2x^2-24a^3x+16a^4 \\ \hline
\end{array}$$

7.

$$
\begin{array}{r|l}
 & x^6 - 6ax^5 + 15a^2x^4 - 20a^3x^3 + 15a^4x^2 - 6a^5x + a^6 \;\big(\; x^3 - 3ax^2 + 3a^2x - a^3 \\
 & x^6 \\
2x^3 - 3ax^2 & - 6ax^5 + 15a^2x^4 - 20a^3x^3 + 15a^4x^2 - 6a^5x + a^6 \\
 & - 6ax^5 + 9a^2x^4 \\
2x^3 - 6ax^2 + 3a^2x & 6a^2x^4 - 20a^3x^3 + 15a^4x^2 - 6a^5x + a^6 \\
 & 6a^2x^4 - 18a^3x^3 + 9a^4x^2 \\
2x^3 - 6ax^2 + 6a^2x - a^3 & - 2a^3x^3 + 6a^4x^2 - 6a^5x + a^6 \\
 & - 2a^3x^3 + 6a^4x^2 - 6a^5x + a^6
\end{array}
$$

8. Multiplying out we find that the proposed expression becomes $a^4 + 2a^2b^2 + b^4$: the square root is $a^2 + b^2$.

9. Multiplying out we find that the proposed expression becomes

$$a^4(c^2+d^2)^2 + 2a^2b^2(c^2+d^2)^2 + b^4(c^2+d^2)^2:$$

the square root is $\quad a^2(c^2+d^2) + b^2(c^2+d^2)$.

10.

$$
\begin{array}{r|l}
 & a^4 - 2a^2(b^2-c^2+d^2) + b^4 - 2b^2(c^2-d^2) + c^4 - 2c^2d^2 + d^4 \;\big(\; a^2 - (b^2-c^2+d^2) \\
 & a^4 \\
2a^2 - (b^2-c^2+d^2) & - 2a^2(b^2-c^2+d^2) + b^4 - 2b^2(c^2-d^2) + c^4 - 2c^2d^2 + d^4 \\
 & - 2a^2(b^2-c^2+d^2) + (b^2-c^2+d^2)^2
\end{array}
$$

11.

$$
\begin{array}{r|l}
 & x^2 - 4x + 2 + \frac{4}{x} + \frac{1}{x^2} \;\big(\; x - 2 - \frac{1}{x} \\
 & x^2 \\
2x - 2 & -4x + 2 + \frac{4}{x} + \frac{1}{x^2} \\
 & -4x + 4 \\
2x - 4 - \frac{1}{x} & -2 + \frac{4}{x} + \frac{1}{x^2} \\
 & -2 + \frac{4}{x} + \frac{1}{x^2}
\end{array}
$$

12.

$$
\begin{array}{r|l}
 & x^4 - x^3 + \frac{x^2}{4} + 4x - 2 + \frac{4}{x^2} \;\big(\; x^2 - \frac{x}{2} + \frac{2}{x} \\
 & x^4 \\
2x^2 - \frac{x}{2} & -x^3 + \frac{x^2}{4} + 4x - 2 + \frac{4}{x^2} \\
 & -x^3 + \frac{x^2}{4} \\
2x^2 - x + \frac{2}{x} & 4x - 2 + \frac{4}{x^2} \\
 & 4x - 2 + \frac{4}{x^2}
\end{array}
$$

13.

$$
\begin{array}{l}
\dfrac{a^4}{4}+\dfrac{a^3}{x}+\dfrac{a^2}{x^2}-ax-2+\dfrac{x^2}{a^2}\left(\dfrac{a^2}{2}+\dfrac{a}{x}-\dfrac{x}{a}\right. \\
\dfrac{a^4}{4} \\
\hline
a^2+\dfrac{a}{x}\Big)\ \dfrac{a^3}{x}+\dfrac{a^2}{x^2}-ax-2+\dfrac{x^2}{a^2} \\
\qquad\qquad \dfrac{a^3}{x}+\dfrac{a^2}{x^2} \\
\hline
a^2+\dfrac{2a}{x}-\dfrac{x}{a}\Big)\ -ax-2+\dfrac{x^2}{a^2} \\
\qquad\qquad\qquad -ax-2+\dfrac{x^2}{a^2} \\
\hline
\end{array}
$$

14.

$$
\begin{array}{l}
a^4+2(2b-c)a^3+(4b^2-4bc+3c^2)a^2+2c^2(2b-c)a+c^4\left(a^2+(2b-c)a+c^2\right. \\
a^4 \\
\hline
2a^2+(2b-c)a\Big)\ 2(2b-c)a^3+(4b^2-4bc+3c^2)a^2+2c^2(2b-c)a+c^4 \\
\qquad\qquad\qquad 2(2b-c)a^3+(2b-c)^2a^2 \\
\hline
2a^2+2(2b-c)a+c^2\Big)\ 2c^2a^2+2c^2(2b-c)a+c^4 \\
\qquad\qquad\qquad\qquad\quad 2c^2a^2+2c^2(2b-c)a+c^4 \\
\hline
\end{array}
$$

15. For shortness put A for $a^2+4ab-6a-8b^2+12b$, B for $4ab-6a$, and C for $4b^2-12b+9$.

$$
\begin{array}{l}
(a-2b)^2x^4-2a(a-2b)x^3+Ax^2-Bx+C\left((a-2b)x^2-ax+2b-3\right. \\
(a-2b)^2x^4 \\
\hline
2(a-2b)x^2-ax\Big)\ -2a(a-2b)x^3+Ax^2-Bx+C \\
\qquad\qquad\qquad -2a(a-2b)x^3+a^2x^2 \\
\hline
2(a-2b)x^2-2ax+2b-3\Big)\ (4ab-6a-8b^2+12b)x^2-Bx+C \\
\qquad\qquad\qquad\qquad\qquad (4ab-6a-8b^2+12b)x^2-Bx+C \\
\hline
\end{array}
$$

16. The sum of the squares is 1·2996.

17.

$$
\begin{array}{l}
\left.\begin{array}{r}3x^2-3x\\-6x\end{array}\right\} \\
\hline
3x^2-9x+2
\end{array}
\qquad\qquad
\begin{array}{l}
3x^4 \\
\qquad -3x(3x^2-3x) \\
\hline
3x^4-\ \ 9x^3+\ \ 9x^2 \\
\qquad\qquad\qquad 9x^2 \\
\hline
3x^4-18x^3+27x^2 \\
\qquad\qquad +2(3x^2-9x+2) \\
\hline
3x^4-18x^3+33x^2-18x+4
\end{array}
$$

$$
\begin{array}{l}
x^6-9x^5+33x^4-63x^3+66x^2-36x+8\left(x^2-3x+2\right. \\
x^6 \\
\hline
-9x^5+33x^4-63x^3+66x^2-36x+8 \\
-9x^5+27x^4-27x^3 \\
\hline
\qquad\qquad 6x^4-36x^3+66x^2-36x+8 \\
\qquad\qquad 6x^4-36x^3+66x^2-36x+8 \\
\hline
\end{array}
$$

18.

$$\begin{array}{r} 6x^2+4cx \\ 8cx \\ \hline 6x^2+12cx-3c^2 \end{array} \qquad \begin{array}{r} 12x^4 \\ 4cx\,(6x^2+4cx) \\ \hline 12x^4+24cx^3+16c^2x^2 \\ 16c^2x^2 \\ \hline 12x^4+48cx^3+48c^2x^2 \\ -18c^2x^2-36c^3x+9c^4 \\ \hline 12x^4+48cx^3+30c^2x^2-36c^3x+9c^4 \end{array}$$

$$\begin{array}{l} 8x^6+48cx^5+60c^2x^4-80c^3x^3-90c^4x^2+108c^5x-27c^6 \;\big(\; 2x^2+4cx-3c^2 \\ 8x^6 \\ \hline \quad 48cx^5+60c^2x^4-80c^3x^3-90c^4x^2+108c^5x-27c^6 \\ \quad 48cx^5+96c^2x^4+64c^3x^3 \\ \hline \qquad -36c^2x^4-144c^3x^3-90c^4x^2+108c^5x-27c^6 \\ \qquad -36c^2x^4-144c^3x^3-90c^4x^2+108c^5x-27c^6 \\ \hline \end{array}$$

19.

$$\begin{array}{r} 6x^2-3cx \\ -6cx \\ \hline 6x^2-9cx+4c^2 \end{array} \qquad \begin{array}{r} 12x^4 \\ -3cx\,(6x^2-3cx) \\ \hline 12x^4-18cx^3+9c^2x^2 \\ 9c^2x^2 \\ \hline 12x^4-36cx^3+27c^2x^2 \\ +24c^2x^2-36c^3x+16c^4 \\ \hline 12x^4-36cx^3+51c^2x^2-36c^3x+16c^4 \end{array}$$

$$\begin{array}{l} 8x^6-36cx^5+102c^2x^4-171c^3x^3+204c^4x^2-144c^5x+64c^6 \;\big(\; 2x^2-3cx+4c^2 \\ 8x^6 \\ \hline \quad -36cx^5+102c^2x^4-171c^3x^3+204c^4x^2-144c^5x+64c^6 \\ \quad -36cx^5+54c^2x^4-27c^3x^3 \\ \hline \qquad 48c^2x^4-144c^3x^3+204c^4x^2-144c^5x+64c^6 \\ \qquad 48c^2x^4-144c^3x^3+204c^4x^2-144c^5x+64c^6 \\ \hline \end{array}$$

20.

$$\begin{array}{r} 155 \\ 10 \\ \hline 1651 \end{array} \qquad \begin{array}{r} 75.. \\ 775 \\ \hline 8275 \\ 25 \\ \hline 9075.. \\ 1651 \\ \hline 909151 \end{array} \qquad \begin{array}{l} 16\dot{7}\cdot28\dot{4}15\dot{1} \;\big(\; 5\cdot51 \\ 125 \\ \hline \;42284 \\ \;41375 \\ \hline \quad 909151 \\ \quad 909151 \\ \hline \end{array}$$

21.

$$\begin{array}{r} 27009 \end{array} \qquad \begin{array}{r} 2430000.. \\ 243081 \\ \hline 243243081 \end{array} \qquad \begin{array}{l} 73\dot{1}18\dot{9}18\dot{7}72\dot{9} \;\big(\; 9009 \\ 729 \\ \hline \quad 2189187729 \\ \quad 2189187729 \\ \hline \end{array}$$

22.

$$\begin{array}{r} 62 \\ 4 \\ \hline 662 \\ 4 \\ \hline 6662 \end{array} \qquad \begin{array}{r} 12.. \\ 124 \\ \hline 1324 \\ 4 \\ \hline 1452.. \\ 1324 \\ \hline 146524 \\ 4 \\ \hline 147852.. \\ 13324 \\ \hline 14798524 \end{array} \qquad \begin{array}{l} 1\dot{0}97\dot{0}\cdot64\dot{5}04\dot{8} \;\big(\; 22\cdot22 \\ 8 \\ \hline 2970 \\ 2648 \\ \hline \;322645 \\ \;293048 \\ \hline \quad 29597048 \\ \quad 29597048 \\ \hline \end{array}$$

23.

```
      31 }
       2 }
     331 }
       2 }
    3331 }
       2 }
   33331 }
       2 }
  333331 }
       2 }
 3333331 }
       2 }
33333331
       2
333333331

1371742108367626890260631 ( 111111111
1
 371
 331
  40742
  36631
   4111108
   3699631
    411477367
    370329631
     41147736626
     37036629631
      4111106995890
      3703699629631
       407407366259260
       370370329629631
        37037036629629631
        37037036629629631
```

```
3..
 31 }
331 }
  1 }
363..
  331 }
36631 }
    1 }
36963..
    3331 }
3699631 }
      1 }
3702963..
     33331 }
370329631 }
        1 }
370362963..
       333331 }
37036629631 }
          1 }
37036962963..
        3333331 }
3703699629631 }
            1 }
3703702962963..
         33333331 }
370370329629631 }
              1 }
370370362962963..
        333333331
37037036629629631
```

24.

$$
\begin{array}{rl}
 & x^4 - 4x^2 + 6 - \dfrac{4}{x^2} + \dfrac{1}{x^4} \left(x^2 - 2 + \dfrac{1}{x^2} \right. \\
 & x^4 \\
\hline
2x^2 - 2 \,\big) & -4x^2 + 6 - \dfrac{4}{x^2} + \dfrac{1}{x^4} \\
 & -4x^2 + 4 \\
\hline
2x^2 - 4 + \dfrac{1}{x^2} \,\big) & 2 - \dfrac{4}{x^2} + \dfrac{1}{x^4} \\
 & 2 - \dfrac{4}{x^2} + \dfrac{1}{x^4} \\
\hline
\end{array}
$$

The square root of $x^2 - 2 + \dfrac{1}{x^2}$ is $x - \dfrac{1}{x}$.

25. First suppose n an *even* number, say $=2m$; then there are m digits in the square root: and $\frac{1}{4}\left\{2n+1-(-1)^n\right\}=\frac{1}{4}\left\{4m+1-1\right\}=m$.

Next suppose n an *odd* number, say $=2m+1$; then there are $m+1$ digits in the square root: and $\frac{1}{4}\left\{2n+1-(-1)^n\right\}=\frac{1}{4}\left\{4m+2+1+1\right\}=m+1$.

XVIII.

5. $$\frac{\left\{a^{\frac{m}{r}}\times a^{\frac{q}{n}}\right\}^{nr}}{\left\{b^{\frac{n}{q}}\times b^{\frac{r}{m}}\right\}^{mq}}\div\left(\frac{a}{b}\right)^{qr}=\frac{a^{mn}\times a^{qr}}{b^{mn}\times b^{qr}}\times\frac{b^{qr}}{a^{qr}}=\left(\frac{a}{b}\right)^{mn}.$$

16. The numerator $=a\left(a^{\frac{1}{2}}-x^{\frac{1}{2}}\right)+x\left(a^{\frac{1}{2}}-x^{\frac{1}{2}}\right)=(a+x)\left(a^{\frac{1}{2}}-x^{\frac{1}{2}}\right)$; the denominator $=a^2\left(a^{\frac{1}{2}}-x^{\frac{1}{2}}\right)+3ax\left(a^{\frac{1}{2}}-x^{\frac{1}{2}}\right)+x^2\left(a^{\frac{1}{2}}-x^{\frac{1}{2}}\right)=\left(a^2+3ax+x^2\right)\left(a^{\frac{1}{2}}-x^{\frac{1}{2}}\right)$: then remove the common factor $a^{\frac{1}{2}}-x^{\frac{1}{2}}$.

17. $$\begin{array}{r|l}
 & \dfrac{y^2}{x}+\dfrac{2y^{\frac{5}{4}}}{x^{\frac{1}{4}}}-\dfrac{x^{\frac{5}{4}}}{y^{\frac{1}{4}}}+\dfrac{x^2}{4y}\ \left(\dfrac{y}{x^{\frac{1}{2}}}+y^{\frac{1}{4}}x^{\frac{1}{4}}-\dfrac{x}{2y^{\frac{1}{2}}}\right. \\
 & \dfrac{y^2}{x} \\ \hline
\dfrac{2y}{x^{\frac{1}{2}}}+y^{\frac{1}{4}}x^{\frac{1}{4}} & \dfrac{2y^{\frac{5}{4}}}{x^{\frac{1}{4}}}-\dfrac{x^{\frac{5}{4}}}{y^{\frac{1}{4}}}+\dfrac{x^2}{4y} \\
 & \dfrac{2y^{\frac{5}{4}}}{x^{\frac{1}{4}}}+y^{\frac{1}{2}}x^{\frac{1}{2}} \\ \hline
\dfrac{2y}{x^{\frac{1}{2}}}+2y^{\frac{1}{4}}x^{\frac{1}{4}}-\dfrac{x}{2y^{\frac{1}{2}}} & -y^{\frac{1}{2}}x^{\frac{1}{2}}-\dfrac{x^{\frac{5}{4}}}{y^{\frac{1}{4}}}+\dfrac{x^2}{4y} \\
 & -y^{\frac{1}{2}}x^{\frac{1}{2}}-\dfrac{x^{\frac{5}{4}}}{y^{\frac{1}{4}}}+\dfrac{x^2}{4y} \\ \hline
\end{array}$$

18. $$\begin{array}{r|l}
 & 4a-4a^{\frac{1}{2}}\left(3b^{\frac{1}{3}}-4c^{\frac{1}{4}}\right)+9b^{\frac{2}{3}}-24b^{\frac{1}{3}}c^{\frac{1}{4}}+16c^{\frac{1}{2}}\ \left(2a^{\frac{1}{2}}-\left(3b^{\frac{1}{3}}-4c^{\frac{1}{4}}\right)\right. \\
 & 4a \\ \hline
4a^{\frac{1}{2}}-\left(3b^{\frac{1}{3}}-4c^{\frac{1}{4}}\right) & -4a^{\frac{1}{2}}\left(3b^{\frac{1}{3}}-4c^{\frac{1}{4}}\right)+9b^{\frac{2}{3}}-24b^{\frac{1}{3}}c^{\frac{1}{4}}+16c^{\frac{1}{2}} \\
 & -4a^{\frac{1}{2}}\left(3b^{\frac{1}{3}}-4c^{\frac{1}{4}}\right)+9b^{\frac{2}{3}}-24b^{\frac{1}{3}}c^{\frac{1}{4}}+16c^{\frac{1}{2}} \\ \hline
\end{array}$$

$$\begin{array}{rl}
19. & 256x^{\frac{4}{3}}-512x+640x^{\frac{2}{3}}-512x^{\frac{1}{3}}+304-128x^{-\frac{1}{3}}+40x^{-\frac{2}{3}}-8x^{-1}+x^{-\frac{4}{3}}\,\Big(\,16x^{\frac{2}{3}}-16x^{\frac{1}{3}}+12-4x^{-\frac{1}{3}}+x^{-\frac{2}{3}} \\
 & 256x^{\frac{4}{3}} \\ \hline
32x^{\frac{2}{3}}-16x^{\frac{1}{3}}\,\Big) & -512x+640x^{\frac{2}{3}}-512x^{\frac{1}{3}}+304-128x^{-\frac{1}{3}}+40x^{-\frac{2}{3}}-8x^{-1}+x^{-\frac{4}{3}} \\
 & -512x+256x^{\frac{2}{3}} \\ \hline
32x^{\frac{2}{3}}-32x^{\frac{1}{3}}+12\,\Big) & 384x^{\frac{2}{3}}-512x^{\frac{1}{3}}+304-128x^{-\frac{1}{3}}+40x^{-\frac{2}{3}}-8x^{-1}+x^{-\frac{4}{3}} \\
 & 384x^{\frac{2}{3}}-384x^{\frac{1}{3}}+144 \\ \hline
32x^{\frac{2}{3}}-32x^{\frac{1}{3}}+24-4x^{-\frac{1}{3}}\,\Big) & -128x^{\frac{1}{3}}+160-128x^{-\frac{1}{3}}+40x^{-\frac{2}{3}}-8x^{-1}+x^{-\frac{4}{3}} \\
 & -128x^{\frac{1}{3}}+128-\ 96x^{-\frac{1}{3}}+16x^{-\frac{2}{3}} \\ \hline
32x^{\frac{2}{3}}-32x^{\frac{1}{3}}+24-8x^{-\frac{1}{3}}+x^{-\frac{2}{3}}\,\Big) & 32-\ 32x^{-\frac{1}{3}}+24x^{-\frac{2}{3}}-8x^{-1}+x^{-\frac{4}{3}} \\
 & 32-\ 32x^{-\frac{1}{3}}+24x^{-\frac{2}{3}}-8x^{-1}+x^{-\frac{4}{3}} \\ \hline
\end{array}$$

20. $a^b=b^a$; extract the b^{th} root; thus $a=b^{\frac{a}{b}}$: and

$$\left(\frac{a}{b}\right)^{\frac{a}{b}}=a^{\frac{a}{b}}\div b^{\frac{a}{b}}=a^{\frac{a}{b}}\div a=a^{\frac{a}{b}-1}.$$

If $a=2b$ we have $(2b)^b=b^{2b}$; extract the b^{th} root; $2b=b^2$: divide by b; then $2=b$.

XIX.

4. $$\frac{1}{2+\sqrt{3}}=\frac{1}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}=\frac{2-\sqrt{3}}{4-3}=2-\sqrt{3}.$$

5. $$\frac{(3+\sqrt{3})(3+\sqrt{5})(\sqrt{5}-2)}{(5-\sqrt{5})(1+\sqrt{3})}=\frac{\sqrt{3}\,(3+\sqrt{5})(\sqrt{5}-2)}{5-\sqrt{5}}$$
$$=\frac{\sqrt{3}\,(\sqrt{5}-1)}{5-\sqrt{5}}=\frac{\sqrt{3}}{\sqrt{5}}=\frac{\sqrt{3}\times\sqrt{5}}{\sqrt{5}\times\sqrt{5}}=\frac{\sqrt{15}}{5}.$$

6. $\sqrt{10}=\sqrt{5}\times\sqrt{2}$; $\sqrt{20}=2\sqrt{5}$; $\sqrt{40}=2\sqrt{5}\times\sqrt{2}$; $\sqrt{80}=4\sqrt{5}$. Thus the fraction

$$=\frac{15}{\sqrt{5}\,\{\sqrt{2}+2+2\sqrt{2}-1-4\}}=\frac{3\sqrt{5}\times\sqrt{5}}{\sqrt{5}\,\{3\sqrt{2}-3\}}$$
$$=\frac{\sqrt{5}}{\sqrt{2}-1}=\frac{\sqrt{5}\,(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}=\sqrt{5}\,(\sqrt{2}+1).$$

8. Multiply out: then the expression becomes

$$a^2-4a^{\frac{3}{2}}b^{\frac{1}{2}}+2ab+4a^{\frac{1}{2}}b^{\frac{3}{2}}+b^2.$$

15. Assume $\sqrt{\{ab+c^2+\sqrt{(a^2-c^2)(b^2-c^2)}\}}=\sqrt{x}+\sqrt{y}$; then

$$x+y=ab+c^2, \quad \sqrt{(a^2-c^2)(b^2-c^2)}=2\sqrt{(xy)};$$

$$(x-y)^2=(ab+c^2)^2-(a^2-c^2)(b^2-c^2)=c^2(a^2+b^2+2ab);$$

therefore $$x-y=c\,(a+b), \text{ \&c.}$$

16. $\sqrt{27}+\sqrt{15}=\sqrt{3}\,(3+\sqrt{5})$; therefore $\sqrt{\{\sqrt{27}+\sqrt{15}\}}=\sqrt[4]{3}\sqrt{\{3+\sqrt{5}\}}$; &c.

17. $-9+6\sqrt{3}=\sqrt{3}(6-3\sqrt{3})$; therefore $\sqrt{\{-9+6\sqrt{3}\}}=\sqrt[4]{3}\sqrt{\{6-3\sqrt{3}\}}$; &c.

18. $1+(1-c^2)^{-\frac{1}{2}}=\dfrac{\sqrt{(1-c^2)}+1}{\sqrt{(1-c^2)}}$; therefore $\sqrt{\{1+(1-c^2)^{-\frac{1}{2}}\}}=\dfrac{\sqrt{\{\sqrt{(1-c^2)}+1\}}}{\sqrt[4]{(1-c^2)}}$; &c.

19. $$\sqrt{(1+x)}=\sqrt{\left(1+\frac{\sqrt{3}}{2}\right)}; \text{ we find this }=\frac{\sqrt{3}+1}{2}.$$

$$\sqrt{(1-x)}=\sqrt{\left(1-\frac{\sqrt{3}}{2}\right)}; \text{ we find this }=\frac{\sqrt{3}-1}{2}.$$

$$\frac{1+\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}+1}{2}}+\frac{1-\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}-1}{2}}=\frac{2+\sqrt{3}}{3+\sqrt{3}}+\frac{2-\sqrt{3}}{1+\sqrt{3}}=\frac{2+\sqrt{3}}{3+\sqrt{3}}+\frac{2\sqrt{3}-3}{\sqrt{3}+3}$$

$$=\frac{3\sqrt{3}-1}{3+\sqrt{3}}=\frac{(3\sqrt{3}-1)(3-\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})}=\frac{-12+10\sqrt{3}}{6}=\frac{5\sqrt{3}-6}{3}.$$

20. $$\frac{1+\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}+1}{2}}+\frac{1-\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}-1}{2}}=\frac{2+\sqrt{3}}{3+\sqrt{3}}+\frac{2-\sqrt{3}}{3-\sqrt{3}}$$

$$=\frac{(2+\sqrt{3})(3-\sqrt{3})+(2-\sqrt{3})(3+\sqrt{3})}{9-3}=\frac{6}{6}=1.$$

23. We observe that four of the radicals have the negative sign, and two the positive sign: this leads us to assume for the square root $\sqrt{x}+\sqrt{y}-\sqrt{z}-\sqrt{t}$. Square; thus, $x+y+z+t=15$; also we may put $2\sqrt{(xy)}=6\sqrt{2}$, $2\sqrt{(zt)}=2\sqrt{5}$; then we must try to adjust the remaining quantities. Thus $2\sqrt{(xz)}$ will be equal to one of the other four radical expressions, and $2\sqrt{(xt)}$ to another of them. After trial we take $2\sqrt{(xz)}=2\sqrt{30}$, and $2\sqrt{(xt)}=2\sqrt{6}$. Hence by multiplication $x\sqrt{(zt)}=\sqrt{30}\times\sqrt{6}=6\sqrt{5}$. But $\sqrt{(zt)}=\sqrt{5}$. Thus $x=6$. Proceeding in this way we obtain the required square root.

24. Proceed as in Art. 310: we have $a=7$, $b=50$; thus $c^3=-1$, $c=-1$; the cubic equation is $4x^3+3x=7$; and a root is $x=1$.

25. Here $a=16$, $b=320$; $c=-4$: a root of $4x^3+12x=16$ is $x=1$.

26. $9\sqrt{3}-11\sqrt{2}=3\sqrt{3}\left(3-\frac{11}{3}\sqrt{\frac{2}{3}}\right)$. Take $a=3$, $\sqrt{b}=\frac{11}{3}\sqrt{\frac{2}{3}}$; then $c=\frac{1}{3}$: a root of $4x^3-x=3$ is $x=1$.

27. $21\sqrt{6}-23\sqrt{5}=6\sqrt{6}\left(\frac{7}{2}-\frac{23}{6}\sqrt{\frac{5}{6}}\right)$. Take $a=\frac{7}{2}$, $\sqrt{b}=\frac{23}{6}\sqrt{\frac{5}{6}}$; then $c=\frac{1}{6}$: a root of $4x^3-\frac{x}{2}=\frac{7}{2}$ is $x=1$.

28. We first find $\sqrt[3]{(2+\sqrt{5})}$. Here $a=2$, $b=5$, $c=-1$: a root of $4x^3+3x=2$ is $x=\frac{1}{2}$. Thus $\sqrt[3]{(\sqrt{5}+2)}=\frac{\sqrt{5}+1}{2}$. Then $\sqrt[3]{(\sqrt{5}-2)}=\frac{\sqrt{5}-1}{2}$.

29. Transpose and square; $x+11=x+2\sqrt{x}+1$; $10=2\sqrt{x}$, &c.

30. Transpose and square; $3x+4=3x-5-18\sqrt{(3x-5)}+81$;

$$18\sqrt{(3x-5)}=72;\ \sqrt{(3x-5)}=4;\ \text{square, \&c.}$$

31. Square; $a^2(b-x)=b^2(a-x)$; $(a^2-b^2)x=ab(a-b)$, &c.

32. Transpose and square; $x+a=x+b-2\sqrt{c(x+b)}+c$;

$$2\sqrt{c(x+b)}=b+c-a;\quad 4c(x+b)=(b+c-a)^2, \text{ \&c.}$$

XX.

10. $4x^2-4x-3=0$, &c.

11. $6x^2-18=x-3$, &c.

15. $x^2-3x+2=6$, &c.

16. $3x^2-5x+2=14$, &c.

20. $2x^2-9x+10=12x-30$, &c.

21. $4x^2-12x+9=8x$, &c.

24. $100x^2+196x+73=0$, &c.

25. $x^2-\frac{5x}{6}+\frac{1}{6}+x^2-\frac{7x}{12}+\frac{1}{12}=x^2-\frac{9x}{20}+\frac{1}{20}$, &c.

26. Multiply by $6x$; $3x^2+12=2x^2+18$, &c.

27. $25x(x+1)-15(2x^2+x-1)=12(x+1)$, &c.

30. $x(x+5)+147=23(x+5)$, &c.

31. $147-x(5-x)=23(5-x)$, &c.

32. Multiply by $4(x^2-1)$; $2(x+1)+12=x^2-1$, &c.

33. Multiply by $8(x^2-1)$; $12+2x(x-1)=3(x^2-1)$, &c.

34. $19x(10-x)+40\times 95=9(100-x^2)$, &c.

35. $76x(10+x)+190(3x-50)=3(10+x)(12x+70)$, &c.

36. $x(x^2-5x)=x(x^2-9)+x+3$; therefore $-5x^2=-8x+3$, &c.

37. $6x(x+2)-3(x-1)(4-x)=14x(x-1)$, &c.

38. $2x^2=3x(x-1)+2(x-1)^2$, &c. Or we may proceed thus: put y for $\frac{x}{x-1}$, then the equation becomes $y=\frac{3}{2}+\frac{1}{y}$; therefore $y^2=\frac{3}{2}y+1$. By solving this quadratic equation in the usual way we obtain $y=2$ or $-\frac{1}{2}$. Taking the former value we have $\frac{x}{x-1}=2$, which gives $x=2$; taking the latter value we have $\frac{x}{x-1}=-\frac{1}{2}$, which gives $x=\frac{1}{3}$. This method may be applied to other examples, as for instance to 39, 40, 41, 42, 46, and 47: it will in some cases diminish the work.

43. $5(x+2)-10(x-2)=3(x^2-4)$, &c.

44. $4(x+2)(x+3)+5(x+1)(x+3)=12(x+1)(x+2)$, &c.

45. $5x(x+4)+3(x+2)(x+4)=14x(x+2)$, &c.

48. $(x+3)(x-2)(x-1)+(x-3)(x+2)(x-1)=(x+2)(x-2)(2x-3)$, &c. Or thus $\frac{x+3}{x+2}=\frac{x+2+1}{x+2}=1+\frac{1}{x+2}$; and treating the other fractions similarly we have $1+\frac{1}{x+2}+1-\frac{1}{x-2}=2-\frac{1}{x-1}$; therefore $\frac{1}{x+2}-\frac{1}{x-2}=-\frac{1}{x-1}$, &c.

50. $49x^2-28x-21=0$, &c.

51. $x^2+\frac{2-\sqrt{3}}{7-4\sqrt{3}}x=\frac{2}{7-4\sqrt{3}}$; that is, making the denominators rational,

$$x^2+\frac{(2-\sqrt{3})(7+4\sqrt{3})}{49-48}=\frac{2(7+4\sqrt{3})}{49-48}; \text{ that is } x^2+(2+\sqrt{3})x=14+8\sqrt{3}.$$

Complete the square by adding $\left(\frac{2+\sqrt{3}}{2}\right)^2$ to both sides; thus

$$\left(x+\frac{2+\sqrt{3}}{2}\right)^2=\frac{63+36\sqrt{3}}{4}; \text{ therefore } x+\frac{2+\sqrt{3}}{2}=\pm\frac{6+3\sqrt{3}}{2}, \text{ \&c.}$$

52. $x^2-2ax+a^2=b^2$, that is $(x-a)^2=b^2$, &c.

53. $x^2-2ax=-b^2$; therefore $(x-a)^2=a^2-b^2$, &c.

54. $2(a^2-b^2)x^2-4(a^2+b^2)x=2(b^2-a^2)$; therefore $x^2-\frac{2(a^2+b^2)}{a^2-b^2}=-1$;

$$\text{add } \left(\frac{a^2+b^2}{a^2-b^2}\right)^2; \text{ thus } \left(x-\frac{a^2+b^2}{a^2-b^2}\right)^2=\left(\frac{a^2+b^2}{a^2-b^2}\right)^2-1=\frac{4a^2b^2}{(a^2-b^2)^2};$$

$$\text{therefore } x-\frac{a^2+b^2}{a^2-b^2}=\frac{\pm 2ab}{a^2-b^2}; \text{ therefore } x=\frac{a^2\pm 2ab+b^2}{a^2-b^2}=\frac{(a\pm b)^2}{a^2-b^2}.$$

With the upper sign $x=\frac{a+b}{a-b}$; with the lower sign $x=\frac{a-b}{a+b}$.

55. $(x-b)(x-c)+(x-a)(x-c)+(x-a)(x-b)=0$; therefore

$$3x^2-2x(a+b+c)+bc+ca+ab=0,$$

$$x^2-\frac{2(a+b+c)x}{3}+\left(\frac{a+b+c}{3}\right)^2=\left(\frac{a+b+c}{3}\right)^2-\frac{bc+ca+ab}{3}, \text{ \&c.}$$

56. $(a+c)(a+b)+(x-b)(x-c)=(a+b)(x-b)+(a+c)(x-c)$,

$$x^2-2x(a+b+c)+a^2+b^2+c^2+2bc+2ca+2ab=0;$$

therefore $(x-a-b-c)^2=0$; therefore $x=a+b+c$.

57. $abx=bx(a+b+x)+ax(a+b+x)+ab(a+b+x)$;

$$(a+b)x^2+(a^2+2ab+b^2)x+ab(a+b)=0;$$

therefore $$x^2+(a+b)x=-ab;$$

therefore $$\left(x+\frac{a+b}{2}\right)^2=\left(\frac{a+b}{2}\right)^2-ab=\frac{(a-b)^2}{4}, \text{ \&c.}$$

58. $abx^2-(a^2+b^2)x=c^2-ab, \quad x^2-\dfrac{a^2+b^2}{ab}x=\dfrac{c^2-ab}{ab}$,

$$\left(x-\frac{a^2+b^2}{2ab}\right)^2=\frac{(a^2+b^2)^2}{4a^2b^2}+\frac{c^2-ab}{ab}=\frac{(a^2-b^2)^2+4abc^2}{4a^2b^2}; \text{ \&c.}$$

59. $a(x-b)(x-c)+b(x-a)(x-c)=2c(x-a)(x-b)$,

$(a+b-2c)x^2-\{2ab-c(a+b)\}x=0$; therefore $x=0$ or $\dfrac{2ab-ac-bc}{a+b-2c}$.

60. $abx^2+\dfrac{3a^2+b^2}{c}x=\dfrac{6a^2+ab-2b^2}{c^2}$; therefore $x^2+\dfrac{3a^2+b^2}{abc}x=\dfrac{6a^2+ab-2b^2}{abc^2}$;

therefore $$\left(x+\frac{3a^2+b^2}{2abc}\right)^2=\frac{6a^2+ab-2b^2}{abc^2}+\left(\frac{3a^2+b^2}{2abc}\right)^2$$

$$=\frac{9a^4+24a^3b+10a^2b^2-8ab^3+b^4}{4a^2b^2c^2};$$

therefore $$x+\frac{3a^2+b^2}{2abc}=\pm\frac{3a^2+4ab-b^2}{2abc}; \text{ \&c.}$$

61. Clear of fractions and simplify; then

$$x^2(a+b+c)-2x(bc+ca+ab)+3abc=0, \text{ \&c.}$$

The process may with advantage be conducted thus:

$\dfrac{x+a}{x-a}-1+\dfrac{x+b}{x-b}-1+\dfrac{x+c}{x-c}-1=0$; therefore $\dfrac{2a}{x-a}+\dfrac{2b}{x-b}+\dfrac{2c}{x-c}=0$

therefore $a(x-b)(x-c)+b(x-c)(x-a)+c(x-a)(x-b)=0$, &c.

62. Clear of fractions and reduce; thus we shall get

$$(2c^2+3c)x^2+ax(2c^2+2c-1)-a^2(1+c)=0;$$

$$\left\{x+\frac{a(2c^2+2c-1)}{4c^2+6c}\right\}^2=\frac{a^2(1+c)}{2c^2+3c}+\frac{a^2(2c^2+2c-1)^2}{4(2c^2+3c)^2}=\frac{a^2(4c^4+16c^3+20c^2+8c+1)}{4(2c^2+3c)^2};$$

therefore $$x+\frac{a(2c^2+2c-1)}{4c^2+6c}=\pm\frac{a(2c^2+4c+1)}{4c^2+6c}, \text{ \&c.}$$

XXI.

1. $x+\frac{2\sqrt{x}}{3}+\frac{1}{9}=\frac{1}{3}+\frac{1}{9}$, &c.

2. $x^{10}+31x^5+\left(\frac{31}{2}\right)^2=32+\left(\frac{31}{2}\right)^2=\frac{1089}{4}$, &c.

3. $x^3+14x^{\frac{3}{2}}+49=1107+49=1156$, &c.

4. $x^{\frac{1}{n}}-13x^{\frac{1}{2n}}+\left(\frac{13}{2}\right)^2=14+\frac{169}{4}=\frac{225}{4}$; therefore $x^{\frac{1}{2n}}-\frac{13}{2}=\pm\frac{15}{2}$, &c.

5. $x^6-35x^3+\left(\frac{35}{2}\right)^2=-216+\left(\frac{35}{2}\right)^2=\frac{361}{4}$, &c.

6. $x^{\frac{2}{n}}-x^{\frac{1}{n}}=2$; $x^{\frac{2}{n}}-x^{\frac{1}{n}}+\frac{1}{4}=2+\frac{1}{4}=\frac{9}{4}$; $x^{\frac{1}{n}}-\frac{1}{2}=\pm\frac{3}{2}$, &c.

7. $x+2\sqrt{a}\sqrt{x}=-c$, $x+2\sqrt{a}\sqrt{x}+a=a-c$; $\sqrt{x}+\sqrt{a}=\pm\sqrt{(a-c)}$, &c.

8. $x^4-\frac{7}{3}x^2+\frac{49}{36}=\frac{43076}{3}+\frac{49}{36}=\frac{516961}{36}$, &c.

9. $x^4-14x^2+49=-40+49$, &c.

10. $x^{\frac{2}{3}}-\frac{13}{4}x^{\frac{1}{3}}=-\frac{5}{2}$; $x^{\frac{2}{3}}-\frac{13}{4}x^{\frac{1}{3}}+\frac{169}{64}=-\frac{5}{2}+\frac{169}{64}=\frac{9}{64}$; $x^{\frac{1}{3}}-\frac{13}{8}=\pm\frac{3}{8}$, &c.

11. $7x-\sqrt{2}\sqrt{x}=52$, $x-\frac{\sqrt{2}}{7}\sqrt{x}+\left(\frac{\sqrt{2}}{14}\right)^2=\frac{52}{7}+\frac{1}{98}=\frac{729}{98}$,

$$\sqrt{x}-\frac{\sqrt{2}}{14}=\pm\frac{27}{7\sqrt{2}}=\pm\frac{27\sqrt{2}}{14};\quad \sqrt{x}=2\sqrt{2} \text{ or } -\frac{13}{7}\sqrt{2}.$$

12. $3x^{\frac{4n}{3}}+2x^{\frac{2n}{3}}=16$; $x^{\frac{4n}{3}}+\frac{2}{3}x^{\frac{2n}{3}}+\frac{1}{9}=\frac{16}{3}+\frac{1}{9}=\frac{49}{9}$; $x^{\frac{2n}{3}}+\frac{1}{3}=\pm\frac{7}{3}$, &c.

13. $x+5-\sqrt{(x+5)}+\frac{1}{4}=6+\frac{1}{4}=\frac{25}{4}$; $\sqrt{(x+5)}-\frac{1}{2}=\pm\frac{5}{2}$, &c.

14. $2x-5\sqrt{x}=-2$; $x-\frac{5}{2}\sqrt{x}=-1$; $x-\frac{5}{2}\sqrt{x}+\frac{25}{16}=-1+\frac{25}{16}=\frac{9}{16}$, &c.

15. $5x^{\frac{1}{2}}+x^{\frac{1}{4}}=22$; $x^{\frac{1}{2}}+\frac{1}{5}x^{\frac{1}{4}}=\frac{22}{5}$; $x^{\frac{1}{2}}+\frac{1}{5}x^{\frac{1}{4}}+\frac{1}{100}=\frac{22}{5}+\frac{1}{100}=\frac{441}{100}$, &c.

16. $x^{\frac{3}{2}}-\frac{4}{3}x^{\frac{3}{4}}=\frac{7}{3}$; $x^{\frac{3}{2}}-\frac{4}{3}x^{\frac{3}{4}}+\frac{4}{9}=\frac{7}{3}+\frac{4}{9}=\frac{25}{9}$, &c.

17. $4x+2\sqrt{(4x+8)}=7$; $4x+8+2\sqrt{(4x+8)}=15$,
$\{\sqrt{(4x+8)}+1\}^2=15+1=16$, &c.

18. $x^{\frac{2}{n}}+1=\frac{5}{2}x^{\frac{1}{n}}$, $x^{\frac{2}{n}}-\frac{5}{2}x^{\frac{1}{n}}+\frac{25}{16}=-1+\frac{25}{16}=\frac{9}{16}$, &c.

19. Square; $2x+7+3x-18+2\sqrt{\{(2x+7)(3x-18)\}}=7x+1$; therefore $\sqrt{\{(2x+7)(3x-18)\}}=x+6$; therefore $(2x+7)(3x-18)=(x+6)^2$, &c.

20. $\sqrt{(x^2-16)}+\sqrt{(x^2-9)}=7$; transpose and square; thus
$x^2-16=x^2-9-14\sqrt{(x^2-9)}+49$; simplify $\sqrt{(x^2-9)}=4$, &c.

21. Square; $a+x+2\sqrt{(a^2-x^2)}+a-x=b$; therefore $2\sqrt{(a^2-x^2)}=b-2a$; therefore $4(a^2-x^2)=b^2-4ab+4a^2$, &c.

22. Square; $x+9=4x-12\sqrt{x}+9$; $x=4\sqrt{x}$; $x^2=16x$; $x=0$ or 16.

23. Transpose and square; $5x+10=(8-x)^2$, &c.

24. $2^{x+1}+2^{2x}=80$; put y for 2^x, thus $2y+y^2=80$: solve this quadratic in y.

25. Performing the divisions $x(x+2)+x-1=39$, &c.

26. Clear of fractions $\sqrt{(a^2+ax)}-\sqrt{(a^2-x^2)}=\sqrt{(a^2-ax)}+\sqrt{(a^2-x^2)}$;
therefore $\sqrt{(a^2+ax)}-\sqrt{(a^2-ax)}=2\sqrt{(a^2-x^2)}$; square
$2a^2-2\sqrt{(a^4-a^2x^2)}=4(a^2-x^2)$; therefore $2x^2-a^2=\sqrt{(a^4-a^2x^2)}$; square, &c.

27. Clear of fractions $x^2(x+1)^2+x^2(x-1)^2=n(n-1)(x^2-1)^2$;
therefore $x^4\{n(n-1)-2\}-2x^2\{n(n-1)+1\}+n(n-1)=0$; therefore

$$x^4-2x^2\frac{n(n-1)+1}{n(n-1)-2}+\left\{\frac{n(n-1)+1}{n(n-1)-2}\right\}^2$$
$$=-\frac{n(n-1)}{n(n-1)-2}+\left\{\frac{n(n-1)+1}{n(n-1)-2}\right\}^2=\frac{4n^2-4n+1}{\{n(n-1)-2\}^2};$$

therefore $$x^2-\frac{n(n-1)+1}{n(n-1)-2}=\pm\frac{2n-1}{n(n-1)-2},\ \text{\&c.}$$

28. $(a+b)^2(a^2+b^2+x^2)+(a-b)^2(a^2+b^2-x^2)-2(a^2-b^2)\sqrt{\{(a^2+b^2)^2-x^4\}}$
$=(a^2+b^2)^2$; therefore $4abx^2+(a^2+b^2)^2=2(a^2-b^2)\sqrt{\{(a^2+b^2)^2-x^4\}}$;
therefore $16a^2b^2x^4+8ab(a^2+b^2)^2x^2+(a^2+b^2)^4=4(a^2-b^2)^2\{(a^2+b^2)^2-x^4\}$;
by simplifying, $4x^4+8abx^2=3a^4+3b^4-10a^2b^2$.
Divide by 4, and then add a^2b^2 to both sides, &c.

29. $\sqrt{(x+2)}+\sqrt{(x^2+2x)}=a-x-\sqrt{x}$; square

$$x+2+x^2+2x+2\sqrt{\{(x+2)(x^2+2x)\}}=a^2-2a(x+\sqrt{x})+(x+\sqrt{x})^2;$$

therefore $$3x+2+2(x+2)\sqrt{x}=a^2-2a(x+\sqrt{x})+2x\sqrt{x}+x;$$

therefore $2(a+1)x+2(a+2)\sqrt{x}=a^2-2$; therefore

$$x+\frac{a+2}{a+1}\sqrt{x}+\left(\frac{a+2}{2a+2}\right)^2=\frac{a^2-2}{2a+2}+\frac{(a+2)^2}{(2a+2)^2}=\frac{2a^3+3a^2}{(2a+2)^2}, \text{ \&c.}$$

30. $2+2x=\{c-(2+c)x\}^2$; simplifying $(c+2)^2x^2-2(c+1)^2x=2-c^2$: divide by $(c+2)^2$, and add $\left(\frac{c+1}{c+2}\right)^4$ to both sides; $\left\{x-\left(\frac{c+1}{c+2}\right)^2\right\}^2=\frac{(2c+3)^2}{(c+2)^4}$, &c.

31. Clear of fractions, $2a^{\frac{3}{2}}+a\{\sqrt{(a+x)}+\sqrt{(a-x)}\}+x\{\sqrt{(a-x)}-\sqrt{(a+x)}\}$

$$=a^{\frac{3}{2}}+a\{\sqrt{(a+x)}+\sqrt{(a-x)}\}+\sqrt{a}\sqrt{(a^2-x^2)}; \text{ therefore}$$

$$x\{\sqrt{(a-x)}-\sqrt{(a+x)}\}=\sqrt{a}\,\{\sqrt{(a^2-x^2)}-a\}.$$

Square; $x^2\{2a-2\sqrt{(a^2-x^2)}\}=a\{2a^2-x^2-2a\sqrt{(a^2-x^2)}\}$;

$$3ax^2-2a^3=-2(a^2-x^2)\sqrt{(a^2-x^2)};$$

square and simplify, $4x^6-3a^2x^4=0$; therefore $x=0$ or $\pm\frac{a\sqrt{3}}{2}$.

32. $\frac{\sqrt{(x+2a)}}{\sqrt{(x-2a)}}=\frac{2a+x}{2a-x}$; see *Algebra*, page 182: therefore

$$(x+2a)(2a-x)^2=(2a+x)^2(x-2a); \quad (2a+x)(2a-x)\{2a-x+2a+x\}=0;$$

therefore $$(2a+x)(2a-x)=0, \text{ \&c.}$$

33. Transpose and square, $x+8=x+x+3+2\sqrt{(x^2+3x)}$; transpose and square again; $(5-x)^2=4(x^2+3x)$, &c.

34. Transpose and square, $x+8=25x-10\sqrt{(x^2+3x)}+x+3$; transpose and simplify, $5x-1=2\sqrt{(x^2+3x)}$; square, &c.

35. Put y for $\frac{x^2-a^2}{x^2+a^2}$; thus $y+\frac{1}{y}=\frac{34}{15}$; hence we get $y=\frac{3}{5}$ or $\frac{5}{3}$, &c.

36. Transpose and square; $a+bx^n=a+2c\sqrt{(abx^n)}+bc^2x^n$; therefore $$b(1-c^2)x^n=2c\sqrt{(abx^n)}; \text{ square, \&c.}$$

37. Square; $x+4+x-2\sqrt{(x^2+4x)}=x+\frac{3}{2}$; therefore

$$x+\frac{5}{2}=2\sqrt{(x^2+4x)}; \text{ square again, \&c.}$$

38. $x^4 - x^2\left(a^2+\frac{1}{a^2}\right) = -1$; therefore

$$x^4 - x^2\left(a^2+\frac{1}{a^2}\right)+\frac{1}{4}\left(a^2+\frac{1}{a^2}\right)^2 = -1+\frac{1}{4}\left(a^2+\frac{1}{a^2}\right)^2=\frac{1}{4}\left(a^2-\frac{1}{a^2}\right)^2, \text{ \&c.}$$

Or thus: $x^2-a^2=\frac{1}{a^2}-\frac{1}{x^2}=\frac{x^2-a^2}{a^2x^2}$; $(x^2-a^2)\left(1-\frac{1}{a^2x^2}\right)=0$, &c. See Art. 332.

39. $\frac{850}{931}=\frac{x^2(x^2+a^2)}{x^4+x^2a^2+a^4}$; therefore $81x^4+81x^2a^2=850a^4$;

$$x^4+x^2a^2+\frac{a^4}{4}=\frac{850a^4}{81}+\frac{a^4}{4}=\frac{3481a^4}{81\times 4}, \text{ \&c.}$$

40. Multiply each term of the first fraction by its numerator, and each term of the second fraction by its numerator: thus

$$\frac{\{\sqrt{(x^2+1)}+\sqrt{(x^2-1)}\}^2}{2}+\frac{\{\sqrt{(x^2+1)}-\sqrt{(x^2-1)}\}^2}{2}=4\sqrt{(x^2-1)},$$

therefore $2x^2=4\sqrt{(x^2-1)}$; therefore $x^4=4(x^2-1)$, &c.

41. Raise both sides to the sixth power; $\left(a^{\frac{1}{2}}+x^{\frac{1}{2}}\right)^2=\left(a^{\frac{1}{3}}+x^{\frac{1}{3}}\right)^3$; therefore $a+2a^{\frac{1}{2}}x^{\frac{1}{2}}+x=a+3a^{\frac{2}{3}}x^{\frac{1}{3}}+3a^{\frac{1}{3}}x^{\frac{2}{3}}+x$; therefore $2a^{\frac{1}{2}}x^{\frac{1}{2}}=3a^{\frac{2}{3}}x^{\frac{1}{3}}+3a^{\frac{1}{3}}x^{\frac{2}{3}}$.

This is satisfied by $x=0$; dividing by $x^{\frac{1}{3}}$ we have $2a^{\frac{1}{2}}x^{\frac{1}{6}}=3a^{\frac{2}{3}}+3a^{\frac{1}{3}}x^{\frac{1}{3}}$;

$$x^{\frac{1}{3}}-\frac{2a^{\frac{1}{6}}x^{\frac{1}{6}}}{3}=-a^{\frac{1}{3}};\quad x^{\frac{1}{3}}-\frac{2a^{\frac{1}{6}}x^{\frac{1}{6}}}{3}+\left(\frac{a^{\frac{1}{6}}}{3}\right)^2=-a^{\frac{1}{3}}+\frac{a^{\frac{1}{3}}}{9}=-\frac{8a^{\frac{1}{3}}}{9}, \text{ \&c.}$$

42. $\frac{a^2+x^2}{a+x}+a+x=4a$; $a^2+x^2+(a+x)^2=4a(a+x)$; therefore $x^2-ax=a^2$, &c.

43. Square $2+2x^2-2\sqrt{\{(1+x^2)^2-x^2\}}=m^2$; therefore $(2+2x^2-m^2)^2=4(1+x^2+x^4)$; therefore $4x^2-4m^2(1+x^2)+m^4=0$, &c.

44. Proceed as in Example 40; $\{x+\sqrt{(x^2-1)}\}^2+\{x-\sqrt{(x^2-1)}\}^2=34$: thus $4x^2-2=34$, &c.

45. Square, $2(x^2+a^2)+2\sqrt{\{(a^2+x^2)^2-9a^2x^2\}}=2a^2+2b^2$; therefore $(a^2+x^2)^2-9a^2x^2=(b^2-x^2)^2$, &c.

46. $x^2\left(\frac{6}{x}-x\right)=\frac{(1+x^2)^2}{x}$; therefore $6x^2-x^4=(1+x^2)^2$, &c.

47. $x^{\frac{p+q}{2pq}} - \frac{1}{2c}\left(x^{\frac{1}{p}} + x^{\frac{1}{q}}\right) = 0$; this is satisfied by $x=0$; divide by $x^{\frac{1}{q}}$;

$$x^{\frac{q-p}{2pq}} = \frac{1}{2c}\left(x^{\frac{q-p}{pq}} + 1\right); \quad x^{\frac{q-p}{pq}} - 2cx^{\frac{q-p}{2pq}} = -1; \text{ add } c^2 \text{ to both sides, \&c.}$$

48. Transpose and square; $x - \sqrt{(1-x)} = (1-\sqrt{x})^2$; $-\sqrt{(1-x)} = 1 - 2\sqrt{x}$;
square, $1 - x = 1 - 4\sqrt{x} + 4x$; $5x = 4\sqrt{x}$; &c.

49. $2\{5x^4a + 10x^2a^3 + a^5\} = 242a^5$; therefore $x^4 + 2x^2a^2 = 24a^4$, &c.

50. $\frac{x^2-x+1}{x-1} - x = \sqrt{\frac{6}{x}}$; therefore $\frac{1}{x-1} = \sqrt{\frac{6}{x}}$; square, &c.

51. Transpose and square; $x^2+ax+b^2 = (a+b)^2 - 2(a+b)\sqrt{(x^2+bx+a^2)} + x^2+bx+a^2$,

$$(b-a)x + 2a(a+b) = 2(a+b)\sqrt{(x^2+bx+a^2)}; \text{ square}$$

$$(b-a)^2x^2 + 4a(b^2-a^2)x = 4(a+b)^2x^2 + 4(a+b)^2bx, \text{ \&c.}$$

52. $\frac{25x^2-16}{2(5x-4)} = \frac{3(x^2-4)x}{2(x-2)}$; $\frac{5x+4}{2} = \frac{3}{2}(x+2)x$. &c.

53. Square; $2x + 9 + 3x - 15 + 2\sqrt{\{(2x+9)(3x-15)\}} = 7x + 8$;
therefore $\sqrt{\{(2x+9)(3x-15)\}} = x + 7$; square, &c.

54. Transpose and square; $\frac{(b-c)(ac-bx)}{abc} = 1 - 2\sqrt{\frac{x}{a}} + \frac{x}{a}$,

therefore $\frac{b-c}{b} - \frac{b-c}{ac}x = 1 - 2\sqrt{\frac{x}{a}} + \frac{x}{a}$; therefore $\frac{bx}{ac} - 2\sqrt{\frac{x}{a}} + \frac{c}{b} = 0$;

that is $\left\{\sqrt{\frac{bx}{ac}} - \sqrt{\frac{c}{b}}\right\}^2 = 0$; therefore $\sqrt{x} = \frac{c\sqrt{a}}{b}$.

55. Transpose and square;

$$x^2 + 2x - 1 = 5 + 2\sqrt{6} + x^2 + x + 1 - 2(\sqrt{2}+\sqrt{3})\sqrt{(x^2+x+1)};$$

therefore $(x - 7 - 2\sqrt{6})^2 = 4(5+2\sqrt{6})(x^2+x+1)$;

therefore $(19+8\sqrt{6})x^2 + (34+12\sqrt{6})x = 53 + 20\sqrt{6}$;

therefore $x^2 + \frac{34+12\sqrt{6}}{19+8\sqrt{6}}x = \frac{53+20\sqrt{6}}{19+8\sqrt{6}}$;

therefore $x^2 + \frac{(34+12\sqrt{6})(8\sqrt{6}-19)x}{23} = \frac{(53+20\sqrt{6})(8\sqrt{6}-19)}{23}$;

that is $x^2 + \frac{44\sqrt{6}-70}{23}x = \frac{44\sqrt{6}-47}{23}$.

Completing the square we have

$$\left(x+\frac{44\sqrt{6}-70}{46}\right)^2=\frac{12192-2112\sqrt{6}}{(46)^2}=\left(\frac{44\sqrt{6}-24}{46}\right)^2;\ \&c.$$

56. Transpose and square;

$$x^2+ax-1=(\sqrt{a}+\sqrt{b})^2+x^2+bx-1-2(\sqrt{a}+\sqrt{b})\sqrt{(x^2+bx-1)};$$

therefore $$2\sqrt{(x^2+bx-1)}=\sqrt{a}+\sqrt{b}-(\sqrt{a}-\sqrt{b})x.$$

Square; $x^2\{4-(\sqrt{a}-\sqrt{b})^2\}+2(a+b)x=4+(\sqrt{a}+\sqrt{b})^2$.

Divide by the coefficient of x^2, and complete the square;

$$\left\{x+\frac{a+b}{4-(\sqrt{a}-\sqrt{b})^2}\right\}^2=\frac{4+(\sqrt{a}+\sqrt{b})^2}{4-(\sqrt{a}-\sqrt{b})^2}+\frac{(a+b)^2}{\{4-(\sqrt{a}-\sqrt{b})^2\}^2}=\left\{\frac{4+2\sqrt{(ab)}}{4-(\sqrt{a}-\sqrt{b})^2}\right\}^2;\ \&c.$$

57. $x^3+2x^2+x=0$; therefore $x=0$ or $x^2+2x+1=0$, &c.

58. $x^3+bx^2+ax=0$; therefore $x=0$ or $x^2+bx+a=0$, &c.

59. $$x^3-x^2(a+b+c)+x(bc+ca+ab)=0;$$

therefore $$x=0 \text{ or } x^2-x(a+b+c)=-(bc+ca+ab);$$

therefore $$\left(x-\frac{a+b+c}{2}\right)^2=-(bc+ca+ab)+\frac{(a+b+c)^2}{4}=\&c.$$

60. $\dfrac{2x}{1-x^2}=\dfrac{4x}{1+x^2}$; therefore $x(1+x^2)=2x(1-x^2)$; therefore $x=0$ or $1+x^2=2(1-x^2)$, &c.

61. $$\frac{1}{x+a+b}+\frac{1}{x-a-b}+\frac{1}{x-a+b}+\frac{1}{x+a-b}=0;\ \text{therefore}$$

$$\frac{2x}{x^2-(a+b)^2}+\frac{2x}{x^2-(a-b)^2}=0;\quad x\{x^2-(a-b)^2\}+x\{x^2-(a+b)^2\}=0;$$

therefore $$x=0 \text{ or } 2x^2=(a+b)^2+(a-b)^2.$$

62. $$(a-x)(x+m)(x-n)=(a+x)(x+n)(x-m);$$

therefore $$-x^3+x^2(a+n-m)+x(mn+ma-na)-amn$$
$$=x^3+x^2(a+n-m)+x(na-ma-mn)-amn;$$

therefore $$x=0 \text{ or } x^2=mn+(m-n)a.$$

63. $$\left(\frac{a+x}{a-x}\right)^2-1=\frac{cx}{ab};\ \text{therefore}\ \frac{4ax}{(a-x)^2}=\frac{cx}{ab};$$

therefore $4a^2bx=cx(a-x)^2$; therefore $x=0$ or $4a^2b=c(a-x)^2$; &c.

64. Transpose and square;

$$(2x+1)^2+2x(2x+1)\sqrt{(x^2+2)}+x^2(x^2+2)=(x+1)^2(x^2+2x+3);$$

therefore $2x(2x+1)\sqrt{(x^2+2)}=4x^3+2x^2+4x+2=2(x^2+1)(2x+1);$

therefore $(2x+1)\{x\sqrt{(x^2+2)}-(x^2+1)\}=0;$

therefore either $2x+1=0$ or $x\sqrt{(x^2+2)}-(x^2+1)=0.$

The former gives $x=-\frac{1}{2}$; from the latter we have $x^2(x^2+2)=(x^2+1)^2$,

that is, $x^4+2x^2=x^4+2x^2+1$, which is impossible.

65. $$x^2-2x+2-2\sqrt{(x^2-2x+2)}+1=0;$$

that is, $\{\sqrt{(x^2-2x+2)}-1\}^2=0$; therefore $\sqrt{x^2-2x+2}=1$; square, &c.

66. $$x^2+5x+28-5\sqrt{(x^2+5x+28)}=24;\text{ therefore}$$

$\left\{\sqrt{(x^2+5x+28)}-\frac{5}{2}\right\}^2=\frac{25}{4}+24=\frac{121}{4}$; therefore $\sqrt{(x^2+5x+28)}=\frac{5}{2}\pm\frac{11}{2}$, &c.

67. $x^2-2x+9-2\sqrt{(x^2-2x+9)}=8$; therefore $\{\sqrt{(x^2-2x+9)}-1\}^2=4$; &c.

68. $$3(x^2+5x+1)-2\sqrt{(x^2+5x+1)}=5,$$

$x^2+5x+1-\frac{2}{3}\sqrt{(x^2+5x+1)}+\frac{1}{9}=\frac{5}{3}+\frac{1}{9}=\frac{16}{9}$, $\sqrt{(x^2+5x+1)}-\frac{1}{3}=\pm\frac{4}{3}$, &c.

69. $x^2+3x+3\sqrt{(x^2+3x)}=10$; therefore $\left\{\sqrt{(x^2+3x)}+\frac{3}{2}\right\}^2=\frac{9}{4}+10$; &c.

70. $2x^2-3x+2-2\sqrt{(2x^2-3x+2)}+1=0$; that is, $\{\sqrt{(2x^2-3x+2)}-1\}^2=0$, &c.

71. $2x^2+6x+5+6\sqrt{(2x^2+6x+5)}=55$; therefore $\{\sqrt{(2x^2+6x+5)}+3\}^2=64$, &c.

72. $$3x^2-2ax+4-6\sqrt{(3x^2-2ax+4)}=a^2+2a-8;$$

therefore $\{\sqrt{(3x^2-2ax+4)}-3\}^2=(a+1)^2$, &c.

73. $2x^2-3x+2+6\sqrt{(2x^2-3x+2)}=16$; therefore $\{\sqrt{(2x^2-3x+2)}+3\}^2=25$, &c.

74. $9=5+4(x+x^2)-(x+x^2)^2$. Put y for $x+x^2$;

thus $y^2-4y+4=0$; therefore $y=2$; thus $x^2+x=2$, &c.

75. $$(x+a)(x+4a)(x+2a)(x+3a)=c^4;$$

that is, $(x^2+5ax+4a^2)(x^2+5ax+6a^2)=c^4$. Put y for x^2+5ax;

therefore $(y+4a^2)(y+6a^2)=c^4$; therefore $y^2+10a^2y+25a^4=c^4+a^4$, &c.

76. $16x(x+3)(x+1)(x+2)=9$; that is, $16(x^2+3x)(x^2+3x+2)=9$; put y for x^2+3x; therefore $16y(y+2)=9$, &c.

77. $\dfrac{a^2+x^2}{ax}=\dfrac{a^2+x^2}{a^2-x^2}$: see *Algebra*, page 182: $(a^2+x^2)(a^2-x^2-ax)=0$;

therefore $a^2+x^2=0$ or $a^2-x^2-ax=0$. The former gives impossible values to x.

$$x^2+ax=a^2;\quad\text{therefore}\quad x^2+ax+\frac{a^2}{4}=\frac{5a^2}{4};\ \&\text{c.}$$

78. $a=\left(x-\dfrac{1}{2}+\dfrac{1}{2}\right)^4+\left(x-\dfrac{1}{2}-\dfrac{1}{2}\right)^4$; put y for $x-\dfrac{1}{2}$;

therefore $\quad\left(y+\dfrac{1}{2}\right)^4+\left(y-\dfrac{1}{2}\right)^4=a$; therefore $2y^4+3y^2+\dfrac{1}{8}=a$;

therefore $\quad y^4+\dfrac{3y^2}{2}+\dfrac{9}{16}=\dfrac{a}{2}-\dfrac{1}{16}+\dfrac{9}{16}=\dfrac{a}{2}+\dfrac{1}{2}$, &c.

79. $x^4-2x^3+x^2-x^2+x=a$; that is, $(x^2-x)^2-(x^2-x)=a$; put y for x^2-x; therefore $y^2-y=a$; &c.

80. $(x^2-x)^2-(x^2-x)=132$; therefore $\left(x^2-x-\dfrac{1}{2}\right)^2=132+\dfrac{1}{4}=\dfrac{529}{4}$;

therefore $\quad x^2-x-\dfrac{1}{2}=\pm\dfrac{23}{2}$; &c.

81. $\quad 2x+7+2\sqrt{(x^2+7x)}+\sqrt{x}+\sqrt{(x+7)}=42$;

therefore $\quad \{\sqrt{x}+\sqrt{(x+7)}\}^2+\sqrt{x}+\sqrt{(x+7)}=42$;

therefore $\left\{\sqrt{x}+\sqrt{(x+7)}+\dfrac{1}{2}\right\}^2=42+\dfrac{1}{4}=\dfrac{169}{4}$; $\sqrt{x}+\sqrt{(x+7)}+\dfrac{1}{2}=\pm\dfrac{13}{2}$.

Take the upper sign $\sqrt{(x+7)}=6-\sqrt{x}$; square, &c. Then take the lower sign.

82. $(x-4\sqrt{x})^2+2(x-4\sqrt{x})+1=0$; therefore $(x-4\sqrt{x}+1)^2=0$;

therefore $\quad x-4\sqrt{x}=-1$, &c.

83. $\quad \{\sqrt{x}+\sqrt{(a+x)}\}^2+\sqrt{x}+\sqrt{(a+x)}=b+a$;

therefore $\quad \left\{\sqrt{x}+\sqrt{(a+x)}+\dfrac{1}{2}\right\}^2=b+a+\dfrac{1}{4}=c^2$ say:

therefore $\quad \sqrt{x}+\sqrt{(a+x)}+\dfrac{1}{2}=\pm c$;

therefore $\quad \sqrt{(a+x)}=\pm c-\dfrac{1}{2}-\sqrt{x}$; square, &c.

84. $(x^2+x)^2+4(x^2+x)+4=16x^2$; therefore $x^2+x+2=\pm 4x$, &c.

85. $(x^2+a^2)^2=2a^2(x-a)^2$; therefore $x^2+a^2=\pm a\sqrt{2}\,(x-a)$. The upper sign gives impossible values; take the lower sign $x^2+ax\sqrt{2}=-a^2+a^2\sqrt{2}$;

therefore $\left(x+\dfrac{a}{\sqrt{2}}\right)^2=-a^2+a^2\sqrt{2}+\dfrac{a^2}{2}=a^2\sqrt{2}-\dfrac{a^2}{2}$; $x=-\dfrac{a}{\sqrt{2}}\pm\dfrac{a\sqrt{(2\sqrt{2}-1)}}{\sqrt{2}}$.

86. Divide by x^2; $\left(x+\dfrac{c}{ax}\right)^2+a\left(x+\dfrac{c}{ax}\right)+b=\dfrac{2c}{a}$;

therefore $\left(x+\dfrac{c}{ax}+\dfrac{a}{2}\right)^2=\dfrac{2c}{a}-b+\dfrac{a^2}{4}$; &c. From this we could find $x+\dfrac{c}{ax}$; suppose that $x+\dfrac{c}{ax}=m$; clear of fractions and we have a quadratic in x.

87. Square $1+2\sqrt{\left(1-\dfrac{a}{x}\right)}+1-\dfrac{a}{x}=1+\dfrac{x}{a}$; $2\sqrt{\left(1-\dfrac{a}{x}\right)}=\left(\dfrac{x}{a}+\dfrac{a}{x}-1\right)$;

square $\quad 4-\dfrac{4a}{x}=\left(\dfrac{x}{a}+\dfrac{a}{x}\right)^2-2\left(\dfrac{x}{a}+\dfrac{a}{x}\right)+1$; therefore

$\left(\dfrac{x}{a}-\dfrac{a}{x}\right)^2-2\left(\dfrac{x}{a}-\dfrac{a}{x}\right)+1=0$; therefore $\dfrac{x}{a}-\dfrac{a}{x}=1$; therefore $x^2-ax=a^2$; &c.

88. $$\left(x+\frac{1}{x}\right)^2+2\left(x+\frac{1}{x}\right)=\frac{142}{9}+2;$$

therefore $\left(x+\dfrac{1}{x}+1\right)^2=\dfrac{142}{9}+3=\dfrac{169}{9}$; therefore $x+\dfrac{1}{x}+1=\pm\dfrac{13}{3}$; &c.

89. $\dfrac{\sqrt{(x^2-1)}}{\sqrt{x}}-\dfrac{\sqrt{(x-1)}}{\sqrt{x}}=\dfrac{x-1}{x}$; therefore $\sqrt{(x-1)}\left\{\sqrt{(x+1)}-1-\dfrac{\sqrt{(x-1)}}{\sqrt{x}}\right\}=0$;

therefore either $\quad\sqrt{(x-1)}=0$ or $\sqrt{(x+1)}-1=\dfrac{\sqrt{(x-1)}}{\sqrt{x}}$; square

$x+1+1-2\sqrt{(x+1)}=\dfrac{x-1}{x}$; therefore $x+1+\dfrac{1}{x}=2\sqrt{(x+1)}$; square

$\left(x+\dfrac{1}{x}\right)^2+2\left(x+\dfrac{1}{x}\right)+1=4x+4$; therefore $\left(x-\dfrac{1}{x}\right)^2-2\left(x-\dfrac{1}{x}\right)+1=0$.

90. $2(x^4+1)=(x+1)^4$; therefore $x^4-4x^3-6x^2-4x+1=0$:

divide by x^2; $\quad x^2+\dfrac{1}{x^2}-4x-\dfrac{4}{x}-6=0$; therefore

$\left(x+\dfrac{1}{x}\right)^2-4\left(x+\dfrac{1}{x}\right)=8$; therefore $\left(x+\dfrac{1}{x}-2\right)^2=12$; therefore

$x+\dfrac{1}{x}=2\pm\sqrt{12}=2\pm2\sqrt{3}$. Take the upper sign; $x^2-2x(1+\sqrt{3})=-1$;

therefore $\quad\{x-(1+\sqrt{3})\}^2=-1+(1+\sqrt{3})^2=3+2\sqrt{3}$; &c.

91. $(x+1)(x^2-x+1)=0$; therefore $x+1=0$ or $x^2-x+1=0$, &c.

92. $n(x^3+1)+x+1=0$; therefore $(x+1)\{n(x^2-x+1)+1\}=0$, &c.

93. It is obvious that $x=5$ is a solution; multiplying out we get $x^3-9x^2+26x-30=0$; and we know that $5^3-9\times5^2+26\times5-30=0$; subtract; $x^3-5^3-9(x^2-5^2)+26(x-5)=0$; divide by $x-5$;

$$x^2+5x+25-9(x+5)+26=0; \text{ this gives impossible values for } x.$$

94. It is obvious that $x=6$ is a solution; multiplying out we get $x^3-6x^2+11x-66=0$; that is $x^2(x-6)+11(x-6)=0$. Divide by $x-6$; $x^2+11=0$; this gives impossible values for x.

95. It is obvious that $x=5$ is a solution; multiplying out we get $x^3-6x^2+11x-30=0$; and we know that $5^3-6\times5^2+11\times5-30=0$;

subtract; $x^3-5^3-6(x^2-5^2)+11(x-5)=0$; divide by $x-5$;

$$x^2+5x+25-6(x+5)+11=0; \text{ this gives impossible values for } x.$$

96. $x(6x^2-5x+1)=0$; either $x=0$ or $6x^2-5x+1=0$, &c.

97. $x^2(x+1)-4(x+1)=0$; $(x^2-4)(x+1)=0$; either $x+1=0$ or $x^2-4=0$, &c.

98. It is obvious that $x=a$ is a solution; $\frac{x}{a}-1+\frac{b}{x}-\frac{b}{a}+\frac{b^2}{x^2}-\frac{b^2}{a^2}=0$;

$(x-a)\left\{\frac{1}{a}-\frac{b}{ax}-\frac{b^2(x+a)}{a^2x^2}\right\}=0$; divide by $x-a$; $\frac{1}{a}-\frac{b}{ax}-\frac{b^2(x+a)}{a^2x^2}=0$;

therefore $ax^2-abx-b^2x-b^2a=0$; therefore

$$x^2-\frac{b(a+b)}{a}x+\left\{\frac{b(a+b)}{2a}\right\}^2=b^2+\left\{\frac{b(a+b)}{2a}\right\}^2=\frac{b^2(b^2+2ab+5a^2)}{4a^2}, \text{ \&c.}$$

99. $8x^3-1+8(2x-1)=0$; therefore $(2x-1)\{4x^2+2x+9\}=0$; either $2x-1=0$ or $4x^2+2x+9=0$; the latter gives impossible values for x.

100. $x^2-\frac{4}{9}=\frac{1}{x}\left(x+\frac{2}{3}\right)$; $\left(x+\frac{2}{3}\right)\left\{x-\frac{2}{3}-\frac{1}{x}\right\}=0$; therefore $x+\frac{2}{3}=0$ or $x-\frac{2}{3}-\frac{1}{x}=0$; the latter gives $x^2-\frac{2x}{3}=1$; therefore $\left(x-\frac{1}{3}\right)^2=1+\frac{1}{9}$; &c.

101. $3(x^6-1)+8x^2(x^2-1)=0$; therefore $(x^2-1)\{3(x^4+x^2+1)+8x^2\}=0$; therefore either $x^2-1=0$ or $3x^4+11x^2+3=0$; &c.

102. It is obvious that $x=-m$ is a solution;

$$x^3-mx^2-2(m^2+1)x-2m=0;$$

and we know that $-m^3-m^3+2(m^2+1)m-2m=0$;

subtract; $x^3+m^3-m(x^2-m^2)-2(m^2+1)(x+m)=0$;

divide by $x+m$; $x^2-mx+m^2-m(x-m)-2(m^2+1)=0$;

therefore $x^2-2mx=2$; therefore $(x-m)^2=m^2+2$; &c.

103. It is obvious that $x=a$ is a solution; multiply out, and divide by $b-a$; thus $x^3-(a^2+ab+b^2)x+ab(a+b)=0$; and we know that $a^3-(a^2+ab+b^2)a+ab(a+b)=0$; subtract; $x^3-a^3-(a^2+ab+b^2)(x-a)=0$; divide by $x-a$; $x^2+ax+a^2-(a^2+ab+b^2)=0$; &c.

104. $x(x^2+px+p-1)+\frac{x+p-1}{p-1}=0$; $x\{x^2-1+p(x+1)\}+\frac{x+p-1}{p-1}=0$;

$$x(x+1)(x-1+p)+\frac{x+p-1}{p-1}=0; \quad (x+p-1)\left\{x(x+1)+\frac{1}{p-1}\right\}=0;$$

therefore either $x+p-1=0$ or $x^2+x+\frac{1}{p-1}=0$, &c.

105. $$x\left\{(p-1)^2x^2+px+\frac{1}{p-1}\right\}+(p-1)x+1=0;$$

therefore $$x\left\{(p-1)^2x^2-1+px+1+\frac{1}{p-1}\right\}+(p-1)x+1=0;$$

therefore $$x\left\{(p-1)^2x^2-1+\frac{p(p-1)x+p}{p-1}\right\}+(p-1)x+1=0;$$

therefore $$\left\{(p-1)x+1\right\}\left\{x[(p-1)x-1]+\frac{px}{p-1}+1\right\}=0;$$

therefore either $(p-1)x+1=0$ or $x[(p-1)x-1]+\frac{px}{p-1}+1=0$, &c.

XXII.

10. For equal roots $8^2=4\times 2\times m$; therefore $m=8$.

11. $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{p^2-2q}{q}$;

$\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=p^3-3pq$.

12. $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=-\frac{b}{a}\div\frac{c}{a}=-\frac{b}{c}$; $\frac{1}{\alpha}\times\frac{1}{\beta}=\frac{1}{\alpha\beta}=\frac{a}{c}$; thus the required equation is $x^2+\frac{b}{c}x+\frac{a}{c}=0$.

13. The roots are $\frac{-p\pm\sqrt{(p^2-4q)}}{2}$; and $p^2-4q=\left(k+\frac{q}{k}\right)^2-4q$

$$=k^2+2q+\frac{q^2}{k^2}-4q=\left(k-\frac{q}{k}\right)^2, \text{ which is a perfect square.}$$

14. Suppose that x is a value which satisfies both $ax^2+bx+c=0$, and $a'x^2+b'x+c'=0$; multiply the first by c' and the second by c, and subtract;

$$(ac'-a'c)x^2+(bc'-b'c)x=0; \text{ therefore } (ac'-a'c)x=b'c-bc' \text{.........(1)}.$$

Again, multiply the first by a' and the second by a, and subtract;

$$(a'b-ab')x+a'c-ac'=0; \text{ therefore } (a'b-ab')x=ac'-a'c \text{...........(2)}.$$

Multiply (1) and (2) crosswise; thus $(ac'-a'c)^2=(a'b-ab')(b'c-bc')$.

15. Put $\dfrac{2x-7}{2x^2-2x-5}=y$; then we get $x=\dfrac{1+y\pm\sqrt{\{(y-1)(11y-1)\}}}{2y}$, as in Art. 345. This shews that y must not lie between 1 and $\dfrac{1}{11}$.

16. Put $\dfrac{x^2-2x+p^2}{x^2+2x+p^2}=y$; then $x=\dfrac{-(y+1)\pm\sqrt{\{(y+1)^2-p^2(y-1)^2\}}}{y-1}$.

Here the expression under the radical sign will be found to be $(1-p^2)\left(y-\dfrac{p-1}{p+1}\right)\left(y-\dfrac{p+1}{p-1}\right)$; and as $1-p^2$ is negative, *one* of the other factors must be negative to make x real: thus y must lie between $\dfrac{p-1}{p+1}$ and $\dfrac{p+1}{p-1}$.

XXIII.

1. Multiply the second equation by 7, and add to the first, &c.

2. From the first equation $y=100-x$; substitute in the second, &c.

3. The second equation gives $x+y=xy$; therefore $4=xy$; substitute $4-x$ for y, &c.

4. From the first equation $y=7-x$; substitute in the second, &c.

5. From the first equation $y=x-12$; substitute in the second, &c.

6. $x+y=8$, $yx-y=2+2x$; from the first equation $y=8-x$; substitute in the second, &c.

7. $x^2+2xy+y^2=65+56$; that is $(x+y)^2=121$; also
$x^2-2xy+y^2=65-56$; that is $(x-y)^2=9$, &c.

8. From the second equation $x=\dfrac{2+5y}{3}$; substitute in the first, &c.

9. $x+y=2xy$; therefore $1=xy$; substitute $2-x$ for y, &c.

10. Multiply the first equation by 2, and subtract the second, &c.

11. $2x+3y=37$, $45(x+y)=14xy$; from the first equation $x=\dfrac{37-3y}{2}$; substitute in the second, &c.

12. Put $y=vx$; thus $\frac{4v^2+v}{1+3v}=\frac{115}{54}$; hence $v=\frac{5}{3}$ or $-\frac{23}{72}$, &c.

Or, add the equations, extract the square root, &c.

13. Put $y=vx$; thus $\frac{v-v^2}{1+v}=\frac{2}{15}$; hence $v=\frac{2}{3}$ or $\frac{1}{5}$, &c.

14. Put $y=vx$; thus $\frac{4v^2+v+1}{8v^2+3}=\frac{6}{14}$; hence $v=\frac{1}{4}$ or -2, &c.

15. Put $y=vx$; thus $\frac{v-2v^2}{1+v}=\frac{1}{12}$; hence $v=\frac{1}{3}$ or $\frac{1}{8}$, &c.

16. Put $y=vx$; thus $\frac{v^2-v+1}{v^2-2v}=-\frac{21}{15}$; hence $v=\frac{5}{4}$ or $\frac{1}{3}$, &c.

17. Put $y=vx$; thus $\frac{1-4v^2}{v+2v^2}=\frac{9}{3}$, that is $\frac{(1+2v)(1-2v)}{v(1+2v)}=3$; therefore $\frac{1-2v}{v}=3$, &c.

18. From the second equation $y=\frac{7-5x}{2}$; substitute in the first, &c.

19. From the second equation $y=2-x$; substitute in the first, &c.

Or, add the first equation to the square of the second, &c.

20. $(x+y)^2+(x-y)^2=\frac{10}{3}(x^2-y^2)$; therefore $6(x^2+y^2)=10(x^2-y^2)$; therefore $4x^2=16y^2$; therefore $x=\pm 2y$, &c.

21. $(x+y)^2+(x-y)^2=\frac{5}{2}(x^2-y^2)$; therefore $4(x^2+y^2)=5(x^2-y^2)$; therefore $x^2=9y^2$; therefore $x=\pm 3y$, &c.

22. $\frac{y}{10}+\frac{x}{8}=y-x$, $y-\frac{x}{2}=\frac{3xy}{4}-3x$: from the first equation we get $y=\frac{5x}{4}$; substitute in the second, &c.

23. $\frac{3x}{10}+\frac{y}{8}=3x-y$, $3x-\frac{y}{2}=\frac{9xy}{4}+3y$: from the first equation we get $y=\frac{12x}{5}$; substitute in the second, &c.

24. Multiply the first equation by 4, and add it to the second; thus $9(y^2-6xy+9x^2)=0$: therefore $(y-3x)^2=0$; therefore $y=3x$, &c.

25. Subtract the first equation from the second; $2(y-x)=y^2-x^2$; therefore $(y-x)(y+x-2)=0$; therefore either $y-x=0$ or $y+x-2=0$; thus $y=x$ or $y=2-x$: substitute in either of the given equations, &c.

26. Square the second equation; $x^2-2xy+y^2=\frac{x^2y^2}{16}$; substitute from the first $\frac{5}{2}xy-2xy=\frac{x^2y^2}{16}$; therefore $x^2y^2=8xy$: therefore $xy=0$ or 8. The former gives $x^2+y^2=0$, $x-y=0$, so that $x=0$, $y=0$; the latter gives $x^2+y^2=20$, $x-y=2$; &c.

27. $16-x-2y=\frac{3x}{y}$, $23-y-3x=\frac{3x}{y}$; therefore $16-x-2y=23-y-3x$; therefore $y=2x-7$: substitute in either of the given equations, &c.

28. $xy=\frac{4}{3}(x+y)$; therefore $2xy-\frac{8}{3}(x+y)=0$, add to the second equation: thus $(x+y)^2-\frac{5}{3}(x+y)=26$; therefore $\left(x+y-\frac{5}{6}\right)^2=26+\frac{25}{36}=\frac{961}{36}$; therefore $x+y-\frac{5}{6}=\pm\frac{31}{6}$; therefore $x+y=6$ or $-\frac{13}{3}$: substitute in the first equation, &c.

29. $\frac{x^3-y^3}{x-y}=\frac{8}{2}$; that is $x^2+xy+y^2=4$; substitute $x-2$ for y, &c.

30. $\frac{x^3+y^3}{x+y}=\frac{65}{5}$; that is $x^2-xy+y^2=13$; substitute $5-x$ for y, &c.

31. $\frac{x^3+y^3}{x+y}=\frac{1001}{11}$; that is $x^2-xy+y^2=91$; substitute $11-x$ for y.

32. $\frac{x^3+y^3}{xy(x+y)}=\frac{35}{30}$; that is $\frac{x^2-xy+y^2}{xy}=\frac{7}{6}$. Put $y=vx$; thus $\frac{1-v+v^2}{v}=\frac{7}{6}$; therefore $v=\frac{3}{2}$ or $\frac{2}{3}$, &c. Or, add three times the first equation to the second, extract the cube root; thus $x+y=5$; then from the first equation $xy=6$; &c.

33. $x^3+y^3=18xy$, $x+y=12$; therefore, by division, $x^2-xy+y^2=\frac{3xy}{2}$. Put $y=vx$; thus $1-v+v^2=\frac{3v}{2}$; therefore $v=2$ or $\frac{1}{2}$, &c.

34. $\frac{x^3+y^3}{x+y}=\frac{4914}{18}$; that is $x^2-xy+y^2=273$; substitute $18-x$ for y.

35. $x^3+y^3=9xy$, $x+y=\frac{3}{4}xy$; by division $x^2-xy+y^2=12$; and by squaring the second equation we get $\frac{9x^2y^2}{16}-3xy=12$; this gives $xy=8$ or $-\frac{8}{3}$, &c.

36. $x^2(x+y)=x^2(2x-3y)$; therefore $x+y=2x-3y$; therefore $x=4y$; substitute for x in either of the given equations.

37. $xy(x+y)=20$, $\frac{5}{4}xy=x+y$; multiply $\frac{5}{4}x^2y^2(x+y)=20(x+y)$; therefore $x^2y^2=16$; therefore $xy=\pm 4$; substitute in the first equation, &c.

38. $\frac{x^3+y^3}{x^2-xy+y^2}=\frac{6xy-1}{7}$; that is $x+y=\frac{6xy-1}{7}$; square; thus

$$x^2+y^2+2xy=\frac{(6xy-1)^2}{49};\ 7+3xy=\frac{(6xy-1)^2}{49};\ 12x^2y^2-53xy-114=0;$$

from this quadratic we get $xy=6$ or $-\frac{19}{12}$: substitute in $x+y=\frac{6xy-1}{7}$, &c.

39. $x^2+y^2=\frac{x^2y^2}{2}$; therefore $8=\frac{x^2y^2}{2}$; $16=x^2y^2$; substitute $8-x^2$ for y^2, &c.

40. $x+y=4$; square; thus $x^2+y^2=16-2xy$; square; thus $x^4+y^4+2x^2y^2=256-64xy+4x^2y^2$; therefore $82+2x^2y^2=256-64xy+4x^2y^2$; $x^2y^2-32xy=-87$; therefore $xy=3$ or 29. Use these with the first given equation.

41. $\frac{x^5-y^5}{x-y}=\frac{3093}{3}$, that is $x^4+x^3y+x^2y^2+xy^3+y^4=1031$, that is $x^4+y^4+xy(x^2+y^2)+x^2y^2=1031$. Now $x-y=3$; therefore $x^2+y^2=9+2xy$; therefore $x^4+y^4=(9+2xy)^2-2x^2y^2=81+36xy+2x^2y^2$. Substitute: thus $81+36xy+2x^2y^2+xy(9+2xy)+x^2y^2=1031$; that is $5x^2y^2+45xy=950$; therefore $xy=10$ or -19. Use these with the second given equation.

42. $\left(\frac{3x-3y}{x+y}\right)^2+\left(\frac{3x+3y}{x-y}\right)^2=82$. Put u for $\frac{x-y}{x+y}$: thus $9u^2+\frac{9}{u^2}=82$; therefore $9u^4-82u^2+9=0$; from this we get $u^2=9$ or $\frac{1}{9}$; therefore $\frac{x-y}{x+y}=\pm\frac{1}{3}$ or ± 3. Take $\frac{x-y}{x+y}=\frac{1}{3}$; thus $3(x-y)=x+y$; therefore $x=2y$: substitute in the second given equation, &c.

43. $x+y=4+xy$; therefore $x^2+y^2=(4+xy)^2-2xy=16+6xy+x^2y^2$: substitute in the first equation; $16+6xy=19$; therefore $xy=\frac{1}{2}$. Use this with the second given equation.

44. $\frac{x^4+x^2y^2+y^4}{x^2-xy+y^2}=\frac{133}{7}$; this will give $x^2+xy+y^2=19$. From this and the first given equation, by addition and subtraction $x^2+y^2=13$, $xy=6$, &c.

45. $\frac{x^4+x^2y^2+y^4}{x^2+xy+y^2}=\frac{931}{49}$; this will give $x^2-xy+y^2=19$. From this and the first given equation, by addition and subtraction $x^2+y^2=34$, $xy=15$.

46. Add twice the second equation to the first; thus $(x^2+y^2)^2+x^2+y^2=182$; from this quadratic $x^2+y^2=13$ or -14. Take the former and substitute in the second equation; thus we get $xy=\pm 6$, &c.

47. The first equation gives $xy(y+x-3)=3(4x+y-xy)$. Multiply this and the second together: $x^2y^2(y+4x-xy)(y+x-3)=36(x+y-3)(4x+y-xy)$. Therefore either $(y+4x-xy)(y+x-3)=0$ or $x^2y^2=36$. Take $xy=+6$, and substitute in the first given equation; next take $xy=-6$. Next take $y+x-3=0$, this will lead to $y+4x-xy=0$; from these find x and y.

48. Divide the second equation by the first: thus $x+y-\sqrt{(xy)}=6$; therefore by addition and subtraction $x+y=10$, $\sqrt{(xy)}=4$; therefore $xy=16$; substitute $10-x$ for y, &c.

49. Divide the second equation by the first: thus $x+y+\sqrt{(xy)}=19$; therefore by addition and subtraction $x+y=13$, $\sqrt{(xy)}=6$, &c.

50. $x+y=72$; $x^{\frac{1}{3}}+y^{\frac{1}{3}}=6$; divide the first equation by the second: thus $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}=12$; substitute $6-x^{\frac{1}{3}}$ for $y^{\frac{1}{3}}$: thus we get $x^{\frac{2}{3}}-6x^{\frac{1}{3}}+12=4$; therefore $(x^{\frac{1}{3}}-3)^2=1$, &c.

51. $x^2-y^2=(8-x)^2$; substitute $x-1$ for y, &c.

52. $x+y=7+\sqrt{(xy)}$, $(x+y)\sqrt{(xy)}=78$; therefore $\{7+\sqrt{(xy)}\}\sqrt{(xy)}=78$; from this quadratic we get $\sqrt{(xy)}=6$ or -13; &c.

53. $x+y=10$, $x+y=\frac{5}{2}\sqrt{(xy)}$; therefore $10=\frac{5}{2}\sqrt{(xy)}$; $\sqrt{(xy)}=4$, &c.

54. Square the first equation; $x+y-2\sqrt{(xy)}=4xy$; therefore

$$20-2\sqrt{(xy)}=4xy; \text{ from this quadratic we get } \sqrt{(xy)}=2 \text{ or } -\frac{5}{2}, \text{ \&c.}$$

55. Put $y=vx$ in the second equation; $\frac{v^2+1}{v}=\frac{34}{15}$; hence $v=\frac{3}{5}$ or $\frac{5}{3}$; the first equation gives $\sqrt{(x^2-y^2)}-2y=-2$; put $x=\frac{5y}{3}$, &c.

56. $3+x^2=(8-2y)^2$; $5y^2+4x^4=(9-2x^2)^2$; from the second equation we get $x^2=\frac{81-5y^2}{36}$; substitute in the first equation, &c.

57. $ay+bx=4xy$; substitute $b-\frac{bx}{a}$ for y, &c.

58. $y=\frac{b^2}{x}$; substitute in the first equation; $x^2-\frac{b^4}{x^2}=a^2$; therefore $x^4-a^2x^2=b^4$; therefore $\left(x^2-\frac{a^2}{2}\right)^2=b^4+\frac{a^4}{4}$, &c.

59. $a^4=(x+y)^4=x^4+y^4+4xy(x^2+y^2)+6x^2y^2=b^4+4xy(a^2-2xy)+6x^2y^2$; see *Algebra*, page 199; thus $2x^2y^2-4a^2xy=b^4-a^4$; find xy from this quadratic, &c.

60. $x^4+y^4+2x^2y^2=16x^2y^2$; therefore $x^2+y^2=\pm 4xy$; substitute $a-x$ for y, &c.

61. Clear the first equation of fractions; thus $ab=xy$; substitute $a+b-x$ for y; thus $ab=(a+b-x)x$; therefore $x^2-(a+b)x=-ab$, &c.

62. Clear the first equation of fractions; and substitute $2b-\frac{bx}{a}$ for y; thus we get $(a+b)x^2-4abx+a^2(3b-a)=0$, &c.

63. Divide the second equation by the first; $x^4+y^4+xy(x^2+y^2)+x^2y^2=\frac{b^5}{a}$; Now $x^2+y^2=a^2+2xy$; $x^4+y^4=a^4+4a^2xy+2x^2y^2$; thus we get $5x^2y^2+5a^2xy+a^4=\frac{b^5}{a}$; find xy from this quadratic, &c.

64. Square the first equation; $x^2+\sqrt{(x^4-y^4)}=2y^2$; thus $x^2\pm a^2=2y^2$; therefore $x^2=2y^2\mp a^2$; substitute in the second equation, &c.

65. The second equation gives $y\{2(a+b)-x\}=-x(2ab-y)$; and the first gives $y(2ab-y)=x\{2(a+b)-x\}ab$; multiply the two together; thus $y^2(2ab-y)\{2(a+b)-x\}=-x^2(2ab-y)\{2(a+b)-x\}ab$; therefore either $(2ab-y)\{2(a+b)-x\}=0$, or $y^2=-x^2ab$. Take $2ab-y=0$; then from the second equation we must also have $2(a+b)-x=0$, &c.

66. $4(x^2-y^2)=(1-xy)^2$, $x^2-y^2=axy$; therefore $4axy=(1-xy)^2$; find xy from this quadratic, &c.

67. Divide the first equation by the second; also multiply the two together; thus $\frac{x+y}{x-y}=\frac{ay}{c}$; $x^2-y^2=cax$; from the former $x=\frac{(ay+c)y}{ay-c}$; substitute in the latter; thus $4acy^3=cay(a^2y^2-c^2)$, &c.

68. Square the first equation; $2x+2\sqrt{(x^2-y^2)}=a$; transpose and square; therefore $4(x^2-y^2)=(a-2x)^2$; therefore $4y^2=4ax-a^2$. Square the second equation; $2x^2+2\sqrt{(x^4-y^4)}=b^2$; transpose and square; therefore $4(x^4-y^4)=(b^2-2x^2)^2$; therefore $4y^4=4b^2x^2-b^4$; substitute $\frac{4ax-a^2}{4}$ for y^2, &c.

69. Substitute $\frac{ab}{x}$ for y in the first equation; thus $2\left(\frac{b^2}{x^2}+\frac{x^2}{b^2}\right)^{\frac{1}{2}}=4$; therefore $\frac{b^2}{x^2}+\frac{x^2}{b^2}=4$; therefore $x^4-4b^2x^2+b^4=0$, &c.

70. Add: thus $x^2(x-1)^2+y^2(y-1)^2=a+b$; and $x(x-1)+y(y-1)=a$; put u for $x(x-1)$ and v for $y(y-1)$; thus we have $u^2+v^2=a+b$, $u+v=a$; find u and v from these, and then x and y can be found.

71. From the second and the third equations $\frac{y}{b}=\frac{z}{c}$; therefore $y=\frac{bz}{c}$: substitute in the first, &c.

72. $2y+3x=13xy$; substitute $\frac{5-8x}{3}$ for y; thus find x and y, and then z from the first given equation.

73. Clear of fractions; then we find $yz=\frac{1}{2}$, $zx=\frac{1}{2}$, $xy=\frac{1}{2}$; therefore, by multiplication, $xyz^2=\frac{1}{4}$; and by division, $z^2=\frac{1}{2}$, &c. Thus $x=y=z=\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$.

74. $\frac{1}{a^2}=\frac{1}{xy}+\frac{1}{xz}$, $\frac{1}{b^2}=\frac{1}{yz}+\frac{1}{xy}$, $\frac{1}{c^2}=\frac{1}{xz}+\frac{1}{yz}$; thus $\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}=\frac{2}{xy}$; this gives xy; similarly we find yz and xz. Then as in Example 73 we can find x, y, and z.

75. $x^2+yz=c$, $y^2+xz=c$, $z^2+xy=a$. Subtract the second equation from the first; $(x-y)(x+y-z)=0$; therefore either $x-y=0$ or $x+y=z$. First take $y=x$; thus we have $x^2+xz=c$, $z^2+x^2=a$: therefore $z=\frac{c-x^2}{x}$, and by substitution $\frac{(c-x^2)^2}{x^2}+x^2=a$, &c. Next take $y=z-x$: thus we have $(z-x)^2+xz=c$, $z^2+x(z-x)=a$; add these two equations, &c.

76. $x+\frac{y}{z}=\frac{29}{6}$, $y+\frac{x}{z}=\frac{34}{6}$; by addition $(x+y)\left(1+\frac{1}{z}\right)=\frac{21}{2}$; therefore $x+y=\frac{21z}{2(1+z)}$: thus $\frac{21z}{2+2z}+z=15$, &c.

77. $z=\frac{1}{xy}$; substitute in the other two equations; $x+y+\frac{1}{xy}=\frac{7}{2}$, $\frac{1}{x}+\frac{1}{y}+xy=\frac{7}{2}$; by subtraction $(x+y)\left(1-\frac{1}{xy}\right)+\frac{1-x^2y^2}{xy}=0$; therefore either $xy-1=0$ or $x+y=\frac{1+xy}{xy}$, &c.

78. $(x+y+z)^3=1$, $(x+y+z)(x^2+y^2+z^2)=1$, $x^3+y^3+z^3=1$; therefore $(x+y+z)^3+2(x^3+y^3+z^3)-3(x+y+z)(x^2+y^2+z^2)=0$; that is $6xyz=0$, &c.

79. Add the three equations together; thus $(x+y+z)^2=a^2+b^2+c^2$; therefore $x+y+z=\pm\sqrt{(a^2+b^2+c^2)}$, &c.

80.
$$xy+yz+zx=26 \quad\ldots\ldots(1),$$
$$(x+y+z)(xy+yz+zx)-3xyz=162 \quad\ldots\ldots(2),$$
$$(x^2+y^2+z^2)(xy+yz+zx)-xyz(x+y+z)=538 \quad\ldots\ldots(3).$$
Substitute from (1) in (2); thus $26(x+y+z)-3xyz=162$ ....(4).

Substitute from (1) and (4) in (3) thus

$$26(x^2+y^2+z^2)-\frac{26(x+y+z)-162}{3}(x+y+z)=538;$$

to this add (1) multiplied by 52; thus

$$26(x+y+z)^2-\frac{26}{3}(x+y+z)^2+54(x+y+z)=1890;$$

that is $$\frac{52}{3}(x+y+z)^2+54(x+y+z)=1890.$$

Solving this quadratic we get 9 as one value of $x+y+z$; and thus $xyz=24$. We have now $x+y+z=9$, $xy+yz+zx=26$, $xyz=24$; therefore $x+y+\frac{24}{xy}=9$, $xy+\frac{24(x+y)}{xy}=26$; therefore $xy+\left(9-\frac{24}{xy}\right)\frac{24}{xy}=26$; therefore $(xy)^3-26(xy)^2+216xy-576=0$. By trial we find $xy=6$ or 8 or 12; &c.

XXIV.

1. Let x denote one number and y the other. Then

$$x+y=39, \quad x^3+y^3=17199.$$

2. Let $x(x+1)(x+2)$ denote the number. Thus

$$(x+1)(x+2)+x(x+2)+x(x+1)=47.$$

3. Let the length be x yards, and the breadth $x-1$ yards. Then

$$x(x-1)=3\times 4840.$$

4. Suppose the crew could row at the rate of x miles per hour in still water. Then *with* the current $3\frac{1}{2}$ miles are passed over in $\frac{3\frac{1}{2}}{x+2}$ hours, and *against* the current in $\frac{3\frac{1}{2}}{x-2}$ hours. Therefore $\frac{3\frac{1}{2}}{x+2}+\frac{3\frac{1}{2}}{x-2}=1\frac{2}{3}$.

5. Suppose x hurdles are placed in each of two opposite sides of the rectangle; and y hurdles in each of the other two opposite sides; then $2x+2y=176$. And as each hurdle is two yards long the area of the rectangle is $4xy$ square yards. Therefore $4xy=4840+968$.

6. Let x denote the number of acres he rents. Then he pays $\frac{84}{x}$ pounds for each acre; and he lets $x-4$ acres at $\frac{84}{x}+\frac{1}{2}$ pounds an acre. Therefore $(x-4)\left(\frac{84}{x}+\frac{1}{2}\right)=84.$

7. Let x denote the number of sheep he purchased. Then he pays $\frac{35}{x}$ pounds for each sheep; and he sells $x-2$ sheep at $\frac{35}{x}+\frac{1}{2}$ pounds each. Therefore $(x-2)\left(\frac{35}{x}+\frac{1}{2}\right)=36$.

8. Let a denote the length of the given line, and x the length of the produced part. Then $\frac{a}{2}\left(\frac{a}{2}+x\right)=x^2$.

9. Let x denote the dividend, and y the divisor. Then $\frac{x}{y}=3\frac{1}{3}$, $xy=750$.

10. Let x denote the number the gentleman got. Then the market price of each is $\frac{1}{x+2}$ of a shilling, and the gentleman paid for each $\frac{1}{x}$ of a shilling. Therefore $\frac{15}{x}-\frac{15}{x+2}=\frac{1}{12}$.

11. Suppose eggs cost x pence per dozen. Then for a shilling we get $\frac{12}{x}$ dozen, that is $\frac{144}{x}$ eggs; and if the price were lowered one penny per dozen we should get $\frac{144}{x-1}$ for a shilling. Therefore $\frac{144}{x-1}-\frac{144}{x}=2$.

12. Suppose that x Austrian kreuzers are worth a shilling. Then $x+6$ Bavarian kreuzers are worth a shilling. The worth of 15 Austrian kreuzers is $\frac{15}{x}$ of a shilling; and the worth of 15 Bavarian kreuzers is $\frac{15}{x+6}$ of a shilling. Therefore $\frac{15}{x}-\frac{15}{x+6}=\frac{1}{12}$.

13. Let x denote the greater number, and y the less. Then

$$x+y=9(x-y), \quad xy=\frac{12x}{y}+x.$$

14. Let x denote the number of days the first worked; and therefore $x-6$ the number of days the second worked. Then the first received $\frac{96}{x}$ shillings a day, and the second received $\frac{54}{x-6}$ shillings a day. Therefore $(x-6)\frac{96}{x}=x\times\frac{54}{x-6}$.

15. Let x denote the number of persons in the party, and y the number of shillings each spent. Therefore

$$(x+5)(y+1)=120, \quad (x-3)\left(y-\frac{2}{3}\right)=52.$$

16. Let x denote the number of shares he bought, and y the rate per cent. discount. Then he paid $\frac{20(100-y)}{100}$ pounds per share and received

$\frac{20\,(100+y)}{100}$ pounds per share. Therefore

$$\frac{20\,(100-y)\,x}{100}=1500, \quad \frac{20\,(100+y)\,(x-60)}{100}=1000.$$

17. Let x denote the number. Then $x^2+x^3=9\,(x+1)$. Divide by $x+1$.

18. Suppose that he lends x pounds at the rate of y per cent., and $1300-x$ pounds at the rate of z per cent. Then

$$\frac{xy}{100}=\frac{(1300-x)\,z}{100}, \quad \frac{xz}{100}=36, \quad \frac{(1300-x)\,y}{100}=49.$$

Substitute in the first equation the values of z and y found from the second and third equations; &c.

19. Let x denote the number of miles in the rest of his journey; and suppose that the coach goes y miles per hour and the train z miles per hour. Then $\frac{56+x}{4z}=\frac{5}{y}$, $\frac{56}{z}+\frac{x}{y}=\frac{56+35+x}{z}$. From the first equation $x=\frac{20z}{y}-56$; and from the second equation $x=\frac{35y}{z-y}$. Therefore $\frac{20z}{y}-56=\frac{35y}{z-y}$. Put u for $\frac{z}{y}$: thus $20u-56=\frac{35}{u-1}$; therefore $20u^2-76u+21=0$. The only admissible root of this quadratic is $\frac{7}{2}$. Thus $x=14$.

20. Let c denote the number of miles from London to York; and suppose that A travels x miles per hour, and that B travels y miles per hour. Suppose that they meet at the distance of z miles from London. Then as they have travelled for the same time when they meet $\frac{z}{x}=\frac{c-z}{y}$; therefore $z=\frac{cx}{x+y}$. Thus when they meet B has still $\frac{cx}{x+y}$ miles to travel, and A has $c-\frac{cx}{x+y}$ miles to travel, that is $\frac{cy}{x+y}$. Therefore

$$\frac{cy}{x\,(x+y)}=16\ldots\ldots(1), \quad \frac{cx}{y\,(x+y)}=36\ldots\ldots(2).$$

Divide (2) by (1); thus $\frac{x^2}{y^2}=\frac{36}{16}$; therefore $\frac{x}{y}=\frac{3}{2}$. And from (1) we have $\frac{c}{x}=16\left(\frac{x}{y}+1\right)=16\times\frac{5}{2}$; and $\frac{c}{x}$ is the time in which A performs the journey, &c.

21. Let x denote the number of miles in the distance. Then the first courier goes $\frac{x}{14}$ miles an hour, and the second goes $\frac{x+10}{14}$ miles an hour. Therefore $\frac{20\times 14}{x}-\frac{20\times 14}{x+10}=\frac{1}{2}$.

22. Let x denote the number of miles in the distance; and suppose that A travels y miles per day, and that B travels z miles per day. Then as in

the solution of Example 20 we shall find that A and B meet at the distance of $\frac{xy}{y+z}$ miles from P and $\frac{xz}{y+z}$ miles from Q. Therefore

$$\frac{xy}{y+z} - \frac{xz}{y+z} = 30, \quad \frac{xz}{y(y+z)} = 4, \quad \frac{xy}{z(y+z)} = 9.$$

Divide the third equation by the second; thus we get $y = \frac{3z}{2}$; substitute in the first, &c.

23. Let x denote the number of hours in which the one pipe alone would fill the vessel, and $x-2$ the number of hours in which the second pipe alone would fill the vessel. Then $\frac{1}{x} + \frac{1}{x-2} = \frac{1}{1\frac{7}{8}} = \frac{8}{15}$.

24. Let x denote the number of hours in which the first pipe alone would fill the vessel, and y the number of hours in which the second pipe alone would fill the vessel. The first pipe is kept open for $\frac{3y}{5}$ hours, and therefore fills $\frac{3y}{5x}$ of the vessel; there remains $1 - \frac{3y}{5x}$ of the vessel to fill: this is filled by the second pipe in $y\left(1 - \frac{3y}{5x}\right)$ hours. Thus the whole time in hours is $\frac{3y}{5} + y\left(1 - \frac{3y}{5x}\right)$. If the two pipes had been kept open together the time would have been $\frac{xy}{x+y}$ hours. Thus $\frac{xy}{x+y} = \frac{3y}{5} + y\left(1 - \frac{3y}{5x}\right) - 6$. Also $\frac{1}{x} \times \frac{xy}{x+y} = \frac{2}{3}\left(1 - \frac{3y}{5x}\right)$. Put u for $\frac{y}{x}$ in the second equation; thus $\frac{u}{1+u} = \frac{2}{3}\left(1 - \frac{3u}{5}\right)$; therefore $6u^2 + 11u - 10 = 0$. The only admissible root of this quadratic is $\frac{2}{3}$. Then put $\frac{2x}{3}$ for y in the first equation.

25. Let x denote the number of workmen, and y the number of pounds each carried at a time; and suppose that z journeys are made in an hour. Then $8xyz$ is the total number of pounds moved. Thus

$$8xyz = 7(x+8)(y-5)z, \quad 8xyz = 9(x-8)(y+11)z.$$

Therefore $\quad 8xy = 7(x+8)(y-5), \quad 8xy = 9(x-8)(y+11).$

XXV.

1. The expression $= \frac{a^2(c-b) + b^2(a-c) + c^2(b-a)}{(a-b)(b-c)(c-a)}$; it will be found that the numerator is equal to the denominator.

2. All three statements reduce to $a - b + c - d - bcd + acd - abd + abc = 0$.

5. Let a and b denote the quantities; and suppose that $\sqrt{(ab)} = k$: then $ab = k^2$; therefore $\frac{a}{b} = \frac{k^2}{b^2}$; therefore $\sqrt{\frac{a}{b}} = \frac{k}{b}$.

6.
$$
\begin{array}{r|l}
 & 1+2x^{\frac{1}{2}}+x+2x^{\frac{3}{2}}-2x^{\frac{5}{2}}+x^3-2x^{\frac{7}{2}}+x^4 \ \big(\ 1+x^{\frac{1}{2}}+x^{\frac{3}{2}}-x^2 \\
 & 1 \\ \hline
2+x^{\frac{1}{2}} & 2x^{\frac{1}{2}}+x+2x^{\frac{3}{2}}-2x^{\frac{5}{2}}+x^3-2x^{\frac{7}{2}}+x^4 \\
 & 2x^{\frac{1}{2}}+x \\ \hline
2+2x^{\frac{1}{2}}+x^{\frac{3}{2}} & 2x^{\frac{3}{2}} \qquad -2x^{\frac{5}{2}}+x^3-2x^{\frac{7}{2}}+x^4 \\
 & 2x^{\frac{3}{2}}+2x^2 \qquad +x^3 \\ \hline
2+2x^{\frac{1}{2}}+2x^{\frac{3}{2}}-x^2 & -2x^2-2x^{\frac{5}{2}} \qquad -2x^{\frac{7}{2}}+x^4 \\
 & -2x^2-2x^{\frac{5}{2}} \qquad -2x^{\frac{7}{2}}+x^4 \\ \hline
\end{array}
$$

7. The roots are $\pm\sqrt{\{a\pm\sqrt{(a^2-b^2)}\}}$. Assume $\sqrt{\{a+\sqrt{(a^2-b^2)}\}}=\sqrt{x}+\sqrt{y}$: then as in Art. 301, $x+y=a$; $2\sqrt{xy}=\sqrt{(a^2-b^2)}$; therefore $(x-y)^2=b^2$; &c.

8. The roots are $\pm a\sqrt{\dfrac{1+n^2\pm\sqrt{(1+n^2+n^4)}}{2}}$. Assume
$$\sqrt{\frac{1+n^2+\sqrt{(1+n^2+n^4)}}{2}}=\sqrt{x}+\sqrt{y}\text{; &c.}$$

9.
$$
\begin{array}{r|l}
 & x^4+6x^3+11x^2+3x+31 \ \big(\ x^2+3x+1 \\
 & x^4 \\ \hline
2x^2+3x & 6x^3+11x^2+3x+31 \\
 & 6x^3+\ 9x^2 \\ \hline
2x^2+6x+1 & 2x^2+3x+31 \\
 & 2x^2+6x+\ 1 \\ \hline
 & -3x+30
\end{array}
$$

Thus if $-3x+30=0$ the process terminates; that is if $x=10$.

10. If $x^4+ax^3+bx^2+cx+d$ be a perfect square the square root must be of the form x^2+px+q, where p and q do not contain x. Thus we must have *identically* $x^4+ax^3+bx^2+cx+d=(x^2+px+q)^2=x^4+2px^3+(p^2+2q)x^2+2pqx+q^2$.
Thus we see that $\quad a=2p,\ \ b=p^2+2q,\ \ c=2pq,\ \ d=q^2$;
therefore $\quad a(4b-a^2)=2p\times 8q=16pq=8c,\ \ (4b-a^2)^2=(8q)^2=64q^2=64d.$

11. $(1+xx'+yy')^2=(1+x^2+y^2)(1+x'^2+y'^2)$; working it out we have
$$x^2+y^2+x'^2+y'^2+x^2y'^2+x'^2y^2-2xx'-2yy'-2xx'yy'=0,$$
that is
$$(x-x')^2+(y-y')^2+(xy'-x'y)^2=0.$$
Each of the squares then must vanish; therefore $x=x'$ and $y=y'$.

13. Suppose that the cost was x pounds; then the loss was $x-24\frac{3}{8}$; therefore $x-24\frac{3}{8}=\dfrac{18x}{100}$, &c.

14. Let x, y, z denote the parts in descending order of magnitude. Then $x+y+z=16$, $y-z=\sqrt{x}$, $x-y=z^2$. From the second $x=y^2+z^2-2yz$; substitute in the third and divide by y; thus $y=2z+1$. Then from the second $x=(z+1)^2$. Substitute for x and y in the first; $z^2+5z=14$, &c.

15. Put p for $\dfrac{-1+\sqrt{-3}}{2}$ and q for $\dfrac{-1-\sqrt{-3}}{2}$. First suppose n a multiple of 3, say $n=3m$. By Art. 360 we have $p^3=1$; therefore $p^{3m}=1$; $q^3=1$; therefore $q^{3m}=1$: thus the sum $=2$. Next suppose that when n is divided by 3 there is a remainder 1; so that n is of the form $3m+1$. $p^{3m+1}=p^{3m}\times p=p$; $q^{3m+1}=q^{3m}\times q=q$; the sum$=p+q=-1$. Last suppose that when n is divided by 3 there is a remainder 2; so that n is of the form $3m+2$. $p^{3m+2}=p^{3m}\times p^2=p^2$, $q^{3m+2}=q^{3m}\times q^2=q^2$; the sum$=p^2+q^2$ which$=-1$.

16. $$\frac{(x+1)(x-2)+(x-1)(x+2)}{(x-1)(x-2)}=\frac{2(x+3)}{x-3};\text{ therefore}$$
$(x^2-2)(x-3)=(x-1)(x-2)(x+3)$; therefore $x^3-3x^2-2x+6=x^3-7x+6$, &c.

17. $\dfrac{4}{x^2-2x}-\dfrac{2}{x^2-x}=x^2-x$; therefore $\dfrac{4(x-1)-2(x-2)}{x(x-2)(x-1)}=x^2-x$;

therefore $$2=x(x-2)(x-1)^2=(x^2-2x)(x^2-2x+1).$$
Put y for x^2-2x; thus $2=y(y+1)$; therefore $y=1$ or -2, &c.

18. $(x^2-1)(x^2-4)(x^2-9)-x^3(x-1)(x-2)(x-3)=0$; thus either
$(x-1)(x-2)(x-3)=0$ or $(x+1)(x+2)(x+3)-x^3=0$, &c.

19. $(x^2-4x)^2-4(x^2-4x)=16$; $x^2-4x=2\pm2\sqrt{5}$, &c.

20. $\sqrt{(2x-1)}-\sqrt{(5x-4)}=\sqrt{(4x-3)}-\sqrt{(3x-2)}$; square
$$7x-5-2\sqrt{\{(2x-1)(5x-4)\}}=7x-5-2\sqrt{\{(4x-3)(3x-2)\}};$$
therefore $\sqrt{(2x-1)}\sqrt{(5x-4)}=\sqrt{\{(4x-3)(3x-2)\}}$; square, &c.

21. Multiply by 2, and arrange thus;
$$x-a+4c\sqrt{(x-a)}+4c^2=x+a-4b\sqrt{(x+a)}+4b^2;$$
extract the square root; $\sqrt{(x-a)}+2c=\pm\{\sqrt{(x+a)}-2b\}$.
Thus $\sqrt{(x-a)}\mp\sqrt{(x+a)}=-2(c\pm b)$; square $x\mp\sqrt{(x^2-a^2)}=2(c\pm b)^2$;
transpose and square $\{x-2(c\pm b)^2\}^2=x^2-a^2$, &c.

22. $\{\sqrt{(a+x)}-\sqrt{a}\}\{\sqrt{(a-x)}+\sqrt{a}\}=n\{\sqrt{(a+x)}+\sqrt{a}\}\{\sqrt{(a+x)}-\sqrt{a}\}$. Either $\sqrt{(a+x)}-\sqrt{a}=0$ or $\sqrt{(a-x)}+\sqrt{a}=n\{\sqrt{(a+x)}+\sqrt{a}\}$. The former gives $x=0$; take the latter; transpose $\sqrt{(a-x)}-n\sqrt{(a+x)}=(n-1)\sqrt{a}$; square $a(1+n^2)+x(n^2-1)-2n\sqrt{(a^2-x^2)}=(n-1)^2a$, $2na+x(n^2-1)=2n\sqrt{(a^2-x^2)}$. Square again, &c.

23. $ay+bx=2xy$; put $a+b-x$ for y; thus $2x^2-(3a+b)x+a^2+ab=0$.

24. $\dfrac{ab(x+y)+(a+b)xy}{(a+x)(b+y)}=\dfrac{(a+b)c}{a+b+c}$; therefore, using the second equation,
$$(a+b+c)\{abc+(a+b)xy\}=(a+b)c(a+x)(b+y);$$
therefore $(a+b)^2xy+abc^2=(a+b)c(ay+bx)$. Substitute $c-x$ for y; thus we get $(a+b)^2x^2-2acx(a+b)+a^2c^2=0$, that is $\{(a+b)x-ac\}^2=0$.

25. $6(x^2-y^2)=5xy$, $6(x+y)=5xy$; by division $x-y=1$, &c.

26. From the first equation $ay+bx=\frac{xy(x+y)}{c}$; substitute in the second; $\frac{x^2y^2}{c}(x+y-c)=abc(x+y-c)$. Thus either $x+y-c=0$, or $x^2y^2=abc^2$ Take the former then we have also from the second equation $ay+bx-xy=0$. Substitute $c-x$ for y, &c. Next take $xy=c\sqrt{ab}$; substitute in the first equation; thus $ay+bx=\sqrt{(ab)}(x+y)$; therefore $y\sqrt{a}(\sqrt{a}-\sqrt{b})=x\sqrt{b}(\sqrt{a}-\sqrt{b})$, &c.

27. From the second equation $x+\frac{1}{z}=9y$; substitute in the first equation; thus $y(x+z)=1$. Hence the second and third equations become $(x+z)\left(x+\frac{1}{z}\right)=9$, $(x+z)\left(1+\frac{1}{xz}\right)=\frac{9}{2}$; divide the former by the latter; $x+\frac{1}{z}=2\left(1+\frac{1}{xz}\right)$; therefore $x-2=\frac{1}{z}\left(\frac{2}{x}-1\right)$; therefore either $x=2$ or $xz=-1$; only the former will be found admissible. Hence $z+\frac{1}{z}=2$; &c.

28. Add the four equations; thus we get $(v+x+y+z)^2=4(a+b+c)$. By subtracting from this four times the first, second, and third equations in succession we get $(v+x-y-z)^2=8a$, $(v-x+y-z)^2=8b$, $(v-x-y+z)^2=8c$. Extract the square roots; thus we have four simple equations.

XXVI.

1. $\left(\frac{2}{3}\right)^2=\frac{4}{9}$; $\sqrt{\frac{100}{144}}=\frac{10}{12}$.

2. $\frac{3}{5}\times\frac{7}{9}=\frac{7}{15}$.

3. Let $2x$ and $3x$ denote the numbers: then $\frac{2x+9}{3x+9}=\frac{3}{4}$, &c.

4. $\left(\frac{a+c}{b+c}\right)^2=\frac{a^2+2ac+c^2}{b^2+2bc+c^2}=\frac{a^2+2ac+ab}{b^2+2bc+ab}=\frac{a(a+2c+b)}{b(b+2c+a)}=\frac{a}{b}$.

5. Let x be the number of miles in the distance from A to B by the shorter road; then $x+14$ is the number of miles by the longer road. The distance from B to C by the shorter road is $2x$ miles, and by the longer road $2x+8$ miles. Therefore $\frac{x+14}{2x+8}=\frac{2}{3}$.

6. $\frac{ax+by}{cz}=\frac{cz+ax}{by}=\frac{by+cz}{ax}=\frac{2(ax+by+cz)}{ax+by+cz}$. If $ax+by+cz$ is not $=0$ we thus get $ax+by=2cz$, $cz+ax=2by$, $by+cz=2ax$. Subtracting the second of these from the first, $by=cz$; similarly $by=ax$; and $x+y+z=2$. If $ax+by+cz=0$ then $\frac{ax+by}{cz}=-1$; thus $x+y+z=-1$.

7. By Art. 384 each fraction $=$

$$\frac{a_1+a_2x+a_2+a_3x+a_3+a_1x}{a_2+a_3y+a_3+a_1y+a_1+a_2y}=\frac{(a_1+a_2+a_3)(1+x)}{(a_1+a_2+a_3)(1+y)}=\frac{1+x}{1+y}.$$

8. By Art. 384 each fraction $=$

$$\frac{a-b+b-c+c-a+a+b+c}{ay+bx+bz+cx+cy+az+ax+by+cz}=\frac{a+b+c}{(a+b+c)(x+y+z)};$$

and this $=\dfrac{1}{x+y+z}$ if $a+b+c$ is not zero.

9. $$\frac{ay-bx}{c}=\frac{cx-az}{b}=\frac{bz-cy}{a}=\frac{c(ay-bx)+b(cx-az)+a(bz-cy)}{c^2+b^2+a^2}=0.$$

Thus $ay-bx=0$, $cx-az=0$, $bz-cy=0$.

10. $$\frac{a-a'}{a'-a''}=\frac{b-b'}{b'-b''}=\frac{b'(a-a')-a'(b-b')}{b'(a'-a'')-a'(b'-b'')}=\frac{b'a-ba'}{b''a'-b'a''}.$$

Similarly the other cases are established.

11. From the first two equations $\frac{x}{3}=\frac{y}{4}=\frac{z}{5}=k$ say; thus $x=3k$, $y=4k$, $z=5k$: substitute in the last equation, and we have $k^3=8$; therefore $k=2$.

12. From the first two equations $\dfrac{x}{a(b^2-c^2)}=\dfrac{y}{b(c^2-a^2)}=\dfrac{z}{c(a^2-b^2)}=k$ say: substitute in the last equation: thus we find $k=\pm 1$ or 0.

XXVII.

4. Let z be the required number of days; and let t be the number of days in which x men could copy the manuscript working 8 hours a day. Then $t : 32 :: 2 : x$, and $z : t :: 8 : y$; therefore $t=\dfrac{2\times 32}{x}$, $z=\dfrac{8t}{y}=\dfrac{2\times 8\times 32}{xy}$.

5. $\dfrac{x}{y}=\dfrac{(x+z)^2}{(y+z)^2}=\dfrac{x^2+2xz+z^2}{y^2+2yz+z^2}$; by Art. 394, $\dfrac{x-y}{y}=\dfrac{x^2-y^2+2(x-y)z}{y^2+2yz+z^2}$; divide by $x-y$; thus $\dfrac{1}{y}=\dfrac{x+y+2z}{y^2+2yz+z^2}$; clear of fractions; $z^2=xy$.

6. Suppose $\frac{a}{b}=r$: then $\frac{p}{q}=r$; thus $a=rb$, $p=rq$;

$$(a^2+b^2)\div\frac{a^3}{a+b}=b^2(1+r^2)\div\frac{b^3r^3}{b(1+r)}=\frac{(1+r)(1+r^2)}{r^3};$$

$$(p^2+q^2)\div\frac{p^3}{p+q}=q^2(1+r^2)\div\frac{q^3r^3}{q(1+r)}=\frac{(1+r)(1+r^2)}{r^3};$$

therefore the required result follows.

7. Suppose that $a : b :: c : d$; and let $b^2=cd$; then $ad=bc$; therefore $\frac{ab^2}{c}=bc$; therefore $ab=c^2$.

8. $(a+d)^2-(b+c)^2=(a-d)^2-(b-c)^2$; therefore $2ad-2bc=-2ad+2bc$; therefore $ad=bc$.

10. Suppose that in the first vessel the fraction x of the whole is wine, and $1-x$ of the whole is water; and that in the second vessel the fraction y of the whole is wine, and $1-y$ of the whole is water. If we take a measure from each the mixture contains as much wine as water; therefore $x+y=1-x+1-y$. If we take four measures from the first and one measure from the second, we obtain $4x+y$ measures of wine and $4(1-x)+1-y$ measures of water; therefore $4x+y : 4(1-x)+1-y :: 2 : 3$.

11. Suppose that A has x pounds, and that B has y pounds; and suppose that A stakes mx pounds, then B stakes my pounds. Then

$$x+y=168, \quad x+my=2(y-my), \quad y+mx=3(x-mx);$$

therefore $x=(2-3m)y$, and $y=(3-4m)x$; therefore by multiplication $1=(2-3m)(3-4m)$. The only admissible root of this quadratic is $\frac{5}{12}$.

12. Suppose x the original number of male criminals, and y the original number of female criminals; then $x\left(1-\frac{4\cdot6}{100}\right)$ and $y\left(1+\frac{9\cdot8}{100}\right)$ are the new numbers of male and female criminals respectively; therefore

$$x\left(1-\frac{4\cdot6}{100}\right)+y\left(1+\frac{9\cdot8}{100}\right)=(x+y)\left(1+\frac{1\cdot8}{100}\right).$$ Hence we get $y=\frac{4x}{5}$.

XXVIII.

1. Put $y=mx$; then $3=m\times1$; therefore $m=3$: thus $y=3x$, and $y=9$ when $x=3$.

2. Put $a=mb$; then $15=m\times3$; therefore $m=5$: thus $a=5b$.

3. Put $z=mxy$; then $1=m\times1$; therefore $m=1$: thus $z=xy$, and $z=4$ when $x=2$ and $y=2$.

4. Put $z=m(px+y)$; then $3=m(p+2)$, and $5=m(2p+3)$; therefore $\frac{5}{3}=\frac{2p+3}{p+2}$; therefore $p=1$.

5. x varies as y when $\frac{1}{z}$ is constant, and varies as $\frac{1}{z}$ when y is constant: therefore by Art. 425 when both y and $\frac{1}{z}$ vary x varies as their product.

6. Put $x=\frac{my}{z}$; then $3=2m$; therefore $m=\frac{3}{2}$; thus $x=\frac{3y}{2z}$, and $x=\frac{3}{4}$ when $y=2$ and $z=4$.

7. The number of weeks varies as the sum of money directly, and the number of men inversely. Let x, y, z be corresponding values of the number of weeks, the number of pounds, and the number of men. Then $x=\frac{my}{z}$, $6=\frac{m14\frac{1}{4}}{5}$; therefore $m=\frac{40}{19}$; thus $x=\frac{40y}{19z}$. Now put 19 for y, and 4 for z; then $x=10$.

8. Put $x^2=my^3$; then $4=27m$; thus $m=\frac{4}{27}$, and $x^2=\frac{4y^3}{27}$.

9. Denote one quantity by ax and the other by $\frac{b}{x}$; then y varies as $ax+\frac{b}{x}$. Put $y=m\left(ax+\frac{b}{x}\right)$; then $4=ma+mb$, $5=2ma+\frac{mb}{2}$; therefore $ma=2$, and $mb=2$: thus $y=2x+\frac{2}{x}$.

10. Let x denote the former quantity, and y the latter. Put $x=my$; then $\frac{3}{4}=\frac{4m}{3}$; therefore $m=\frac{9}{16}$; thus $x=\frac{9y}{16}$; therefore $y=16$ when $x=9$.

11. Suppose that z varies as $x+y$ when $x-y$ is constant, and varies as $x-y$ when $x+y$ is constant. Put u for $x+y$, and v for $x-y$. Then z varies as u when v is constant, and varies as v when u is constant. Therefore by Art. 425 when both u and v vary z varies as their product, that is as x^2-y^2.

12. Let x denote the radius of a sphere, y the volume of the sphere; then y varies as x^3. Put $y=mx^3$. Thus the sum of the volumes of spheres, whose radii are 3, 4, and 5 inches is $m(3^3+4^3+5^3)$, that is $m\times 216$, that is $m6^3$, that is the volume of a sphere whose radius is 6 inches.

13. Let x denote the radius of a circle, y the area of the circle; then y varies as x^2. Put $y=mx^2$. Then the sum of the areas of circles whose diameters are 6 inches and 8 inches respectively is $m(6^2+8^2)$ that is $m(36+64)$ that is $m10^2$; that is the area of a circle whose diameter is 10 inches.

14. Since the volume of a globe varies as the cube of its radius we may denote the volumes of the two globes by mr^3 and mr'^3 respectively. Let R be the radius of the single globe which is formed; then the volume is mR^3; therefore $mR^3=m(r^3+r'^3)$.

16. The *speed* in the nth mile varies inversely as $n-1$; and therefore the *time* of describing the nth mile varies directly as $n-1$. Denote this time in hours by $m(n-1)$. Since the second mile is described in two hours we have $2=m(2-1)$; therefore $m=2$.

17. Denote the first quantity by p, the second by qx, and the third by rx^2; then y varies as $p+qx+rx^2$. Put $y=m(p+qx+rx^2)$; then

$$0=m(p+qa+ra^2),\quad a=m(p+2qa+4ra^2),\quad 4a=m(p+3qa+9ra^2);$$

therefore $mp=a$, $mq=-2$, $mr=\frac{1}{a}$; thus $y=a\left(1-\frac{2x}{a}+\frac{x^2}{a^2}\right)=a\left(\frac{x}{a}-1\right)^2$.

18. Let x denote the quantity of work done in y hours by z men; then x varies as $z^{\frac{1}{3}}$ when z alone varies, and varies as $y^{\frac{1}{2}}$ when y alone varies; therefore when both y and z vary x varies as $z^{\frac{1}{3}}y^{\frac{1}{2}}$. Put $x=mz^{\frac{1}{3}}y^{\frac{1}{2}}$. Put $z=24$, $y=25$, and $x=1$; thus $m=\dfrac{1}{5\,(24)^{\frac{1}{3}}}$; so that $x=\left(\dfrac{z}{24}\right)^{\frac{1}{3}}.\dfrac{y^{\frac{1}{2}}}{5}$. Next put $z=3$, $x=\dfrac{1}{5}$; and find y: thus $y^{\frac{1}{2}}=8^{\frac{1}{3}}=2$; therefore $y=4$.

XXIX.

1.
7 | 123456
7 | 17636...4
7 | 2519...3
7 | 359...6
7 | 51...2
7 | 7...2
1...0

2.
5 | 1357531
5 | 271506...1
5 | 54301...1
5 | 10860...1
5 | 2172...0
5 | 434...2
5 | 86...4
5 | 17...1
3...2

3.
7 | 357234
7 | 51033...3
7 | 7290...3
7 | 1041...3
7 | 148...5
7 | 21...1
3...0

4.
eleven | 333310
eleven | 30300...t
eleven | 2754...6
eleven | 250...4
eleven | 22...8
2...0

5.
ten | 545
ten | 32...9
2...0

6.
ten | 4444
ten | 222...4
ten | 11...2
...6

7.
seven | 3413
seven | 310...3
seven | 24...2
2...2

8.
twelve | 40234
twelve | 1324...1
twelve | 82...t
1...5

9.
eleven | 64520
eleven | 4151...t
eleven | 246...5
eleven | 15...0
1...1

10.
ten | 15951
ten | 1760...1
ten | 194...4
ten | 20...4
2...2

11.
8 | 15
1...7

$\dfrac{75}{100}\times 8=\dfrac{3}{4}\times 8=6.$

12.
8 | 31462
8 | 3932...6
8 | 491...4
8 | 61...3
7...5

$\dfrac{125}{1000}\times 8=1.$

13.

$$\begin{array}{r|l} 5 & 221 \\ 5 & 44\ldots1 \\ 5 & 8\ldots4 \\ & 1\ldots3 \end{array}$$

$$\frac{248}{1000}\times 5=\frac{248}{200}=1\tfrac{6}{25}.$$

$$\frac{6}{25}\times 5=\frac{6}{5}=1\tfrac{1}{5},\quad \frac{1}{5}\times 5=1.$$

14.

$$\begin{array}{r|l} \text{ten} & 444 \\ \text{ten} & 22\ldots4 \\ & 1\ldots2 \end{array}$$

$\cdot44$ here stands for $\frac{4}{5^2}+\frac{4}{5}$, that is $\frac{24}{25}$ in the common scale.

$$\frac{24}{25}\times 10=\frac{48}{5}=9\tfrac{3}{5};\quad \frac{3}{5}\times 10=6.$$

15.

$$\begin{array}{r|l} 12 & 1845 \\ 12 & 153\ldots9 \\ 12 & 12\ldots9 \\ & 1\ldots0 \end{array}$$

$$\begin{array}{r} \cdot3125 \\ 12 \\ \hline 3\cdot7500 \\ 12 \\ \hline 9\cdot0000 \end{array}$$

16.

$$\begin{array}{r|l} \text{ten} & 3065 \\ \text{ten} & 236\ldots9 \\ \text{ten} & 17\ldots8 \\ & 1\ldots5 \end{array}$$

$\cdot263$ here stands for $\frac{2}{8}+\frac{6}{8^2}+\frac{3}{8^3}$, that is $\frac{179}{512}$ in the common scale; and this $=\cdot349609375$.

17.

$$\begin{array}{r|l} 7 & 231 \\ 7 & 33 \\ & 4\ldots5 \end{array}\qquad \begin{array}{r|l} 7 & 452 \\ 7 & 64\ldots4 \\ 7 & 9\ldots1 \\ & 1\ldots2 \end{array}$$

$$\begin{array}{r} 1214 \\ 450 \\ \hline 64060 \\ 5162 \\ \hline \end{array}$$

$$\begin{array}{r|l} \text{ten} & 613260 \\ \text{ten} & 42304\ldots2 \\ \text{ten} & 3021\ldots1 \\ \text{ten} & 206\ldots4 \\ \text{ten} & 13\ldots4 \\ & 1\ldots0 \end{array}$$

The product is 104412 in the scale of ten.

18.

$$\begin{array}{r} 4685\,)\,17832126\,(\,3483 \\ 15276 \\ \hline 25451 \\ 21072 \\ \hline 43682 \\ 42154 \\ \hline 15276 \\ 15276 \\ \hline \end{array}$$

19.

$$\begin{array}{r} 33224\,(\,152 \\ 1 \\ \hline 25\,)\,232 \\ 221 \\ \hline 342\,)\,1124 \\ 1124 \\ \hline \end{array}$$

20.

$$\begin{array}{r|l}
 & 12345654321 \;(\,11111 \\
 & 1 \\
21 & 23 \\
 & 21 \\
221 & 245 \\
 & 221 \\
2221 & 2443 \\
 & 2221 \\
22221 & 22221 \\
 & 22221
\end{array}$$

21.

$$\begin{array}{r|l}
 & 3445{\cdot}44 \;(\,44{\cdot}4 \\
 & 24 \\
124 & 1045 \\
 & 544 \\
1324 & 10144 \\
 & 10144
\end{array}$$

$$\begin{array}{r|l}
3 & 44 \\
3 & 13 \ldots 1 \\
3 & 3 \ldots 0 \\
 & 1 \ldots 0
\end{array}$$

$\cdot 4$ means $\frac{4}{6}=\frac{2}{3}$.

22.

$$\begin{array}{r}
103050301 \\
20404020 \\
\hline
62444261
\end{array}$$

$$\begin{array}{r|l}
 & 62444261 \;(\,7071 \\
 & 61 \\
1607 & 14442 \\
 & 14261 \\
16161 & 16161 \\
 & 16161
\end{array}$$

23.

$$\begin{array}{r|l}
 & 110000000100001 \;(\,1101111 \\
 & 1 \\
101 & 1000 \\
 & 101 \\
11001 & 110000 \\
 & 11001 \\
110101 & 1011110 \\
 & 110101 \\
1101101 & 10100100 \\
 & 1101101 \\
11011101 & 11011101 \\
 & 11011101
\end{array}$$

24.

$$\begin{array}{r|l}
 & 6755 6t21 \;(\,8te7 \\
 & 54 \\
14t & 1355 \\
 & 1204 \\
158e & 1516t \\
 & 14321 \\
159t7 & t4921 \\
 & t4921
\end{array}$$

25.

$$\frac{117}{192}\times 12=\frac{117}{16}=7\tfrac{5}{16},$$

$$\frac{5}{16}\times 12=\frac{15}{4}=3\tfrac{3}{4},$$

$$\frac{3}{4}\times 12=9.$$

26. Let x denote the radix; then $95=x^2+3x+7$.

27. Let x denote the radix; then $2704=2x^4+3x^2+4$.

28. Let x denote the radix; then $1331=x^3$; therefore $x=11$.

29. Let x denote the radix; then $16000 = x^6 + 3x^3$;

therefore $16002\frac{1}{4} = \left(x^3 + \frac{3}{2}\right)^2$; that is $\frac{64009}{4} = \left(x^3 + \frac{3}{2}\right)^2$;

therefore $\frac{253}{2} = x^3 + \frac{3}{2}$; &c.

30. Let x denote the radix; then $35\frac{5}{6} = 5x + 5 + \frac{5}{x}$.

31. Let x denote the radix; then $\frac{1664}{10000} = \frac{4}{x^3} + \frac{4}{x^4}$.

32. Perform the division; the radix being supposed greater than six, it will be found that we have never to consider what the radix is; the quotient is 1002001.

33. The square root is found to be 12.

34. The cube root is found to be 11.

35.

$$\begin{array}{r|l}
2 & 1719 \\
2 & 859 \ldots 1 \\
2 & 429 \ldots 1 \\
2 & 214 \ldots 1 \\
2 & 107 \ldots 0 \\
2 & 53 \ldots 1 \\
2 & 26 \ldots 1 \\
2 & 13 \ldots 0 \\
2 & 6 \ldots 1 \\
2 & 3 \ldots 0 \\
 & 1 \ldots 1
\end{array}$$

36. $1027 = 3 \times 342 + 1$; $342 = 3 \times 114$; $114 = 3 \times 38$; $38 = 3 \times 13 - 1$; $13 = 3 \times 4 + 1$, $4 = 3 + 1$.

Hence reversing $4 = 3 + 1$; $13 = 3^2 + 3 + 1$; $38 = 3^3 + 3^2 + 3 - 1$;
$114 = 3^4 + 3^3 + 3^2 - 3$; $342 = 3^5 + 3^4 + 3^3 - 3^2$; $1027 = 3^6 + 3^5 + 3^4 - 3^3 + 1$.

37. $716 = 3 \times 239 - 1$; $239 = 3 \times 80 - 1$; $80 = 3 \times 27 - 1$. Hence reversing $80 = 3^4 - 1$; $239 = 3^5 - 3 - 1$; $716 = 3^6 - 3^2 - 3 - 1$.

38. $475 = 3 \times 158 + 1$; $158 = 3 \times 53 - 1$; $53 = 3 \times 18 - 1$; $18 = 3 \times 6$; $6 = 3 \times 2$; $2 = 3 - 1$. Hence reversing $2 = 3 - 1$; $6 = 3^2 - 3$; $18 = 3^3 - 3^2$; $53 = 3^4 - 3^3 - 1$; $158 = 3^5 - 3^4 - 3 - 1$; $475 = 3^6 - 3^5 - 3^2 - 3 + 1$.

39. 235 cubic inches $= \frac{235}{1728}$ of a cubic foot; $\frac{235}{1728} = \frac{1}{12} + \frac{7}{144} + \frac{7}{1728}$. Thus the volume of the parallelepiped *expressed in the scale of twelve* is $7t{\cdot}177$ cubic feet. The area of the base *expressed in the scale of twelve* is $20{\cdot}05$ square feet. By the rules of mensuration we find the height by dividing $7t{\cdot}177$ by $20{\cdot}05$,

$$\begin{array}{r|l|l}
20{\cdot}05 & 7t{\cdot}177 & 3{\cdot}e \\
 & 6013 & \\ \hline
 & 1t047 & \\
 & 1t047 & \\ \hline
\end{array}$$

40. $10\frac{1}{4}$ inches $=\frac{10\frac{1}{4}}{12}$ of a foot; $\frac{10\frac{1}{4}}{12}=\frac{10}{12}+\frac{3}{144}$. Thus the length *expressed in the scale of twelve* is $2{\cdot}t3$ feet. $79\frac{1}{6}$ square inches $=\frac{79\frac{1}{6}}{144}$ of a square foot. $\frac{79\frac{1}{6}}{144}=\frac{6}{12}+\frac{7}{144}+\frac{2}{1728}$. Thus the area *expressed in the scale of twelve* is $5{\cdot}672$ square feet.

$$
\begin{array}{r|l}
2{\cdot}t3 & 5{\cdot}672 \;(\,1{\cdot}e4 \\
 & \underline{2t3} \\
 & 2842 \\
 & \underline{2749} \\
 & e50 \\
 & \underline{e50}.
\end{array}
$$

41. The number consists of $100p_2+10p_1+p_0+$ some *multiple of a thousand.* Now a thousand is divisible by eight; and therefore the number is divisible by eight if $100p_2+10p_1+p_0$ is, that is if $8(12p_2+p_1)+4p_2+2p_1+p_0$ is, that is if $4p_2+2p_1+p_0$ is.

42. Let s denote the sum of the digits in either number: then each number is equal to $\frac{s}{9}$ increased by some multiple of nine; see Art. 446; and therefore the difference of the numbers is a multiple of nine.

43. Let n be the number of digits, and r the radix of the scale: the greatest number has the digit $r-1$ in every place, and is therefore equal to r^n-1: the least number has unity in the extreme left-hand place, and zero in every other, and is therefore equal to r^{n-1}. For example let $n=3$, and $r=10$: the greatest number is 999, and the least is 100.

44. Let a and b denote the numbers; we will suppose a the greater. It is given that $a+b$ is a multiple of the radix. Now $a^2-b^2=(a-b)(a+b)$ which is therefore a multiple of the radix. This demonstrates the first part of the proposition. Again, $ab+a^2=a(a+b)$, which is a multiple of the radix: this shews that if we divide a^2 and ab by the radix the sum of the two remainders is equal to the radix.

45. It is given that the number is divisible by 2^3, by 3 and 5. The number then has, besides 2, the following twelve divisors: 2^2, 2^3, 2.3, $2^2.3$, $2^3.3$, 2.5, $2^2.5$, $2^3.5$, 3.5, $2.3.5$, $2^2.3.5$, $2^3.3.5$. These twelve numbers are the twelve excepted scales. And as the number has no other divisor it must be $2^3.3.5$, that is 120.

XXX.

1. $\frac{20}{2}(4+19\times4)=800.$ 2. $\frac{32}{2}\left(8-\frac{31}{4}\right)=\frac{32}{2}\times\frac{1}{4}=4.$

3. $\frac{24}{2}\left(1-\frac{23\times5}{4}\right)=12-23\times15=-333.$

4. $\frac{20}{2}\left(10-\frac{19\times2}{3}\right)=10\left(-\frac{8}{3}\right)=-26\frac{2}{3}.$

5. $\frac{10}{2}\left(\frac{16}{5}-\frac{9\times 2}{5}\right)=\frac{10}{2}\left(-\frac{2}{5}\right)=-2.$

6. $\frac{12}{2}\left(2+\frac{11\times 3}{4}\right)=6\times\frac{41}{4}=61\frac{1}{2}.$

7. $\frac{21}{2}\left(\frac{10}{7}-\frac{20}{21}\right)=\frac{21}{2}\times\frac{10}{21}=5.$

8. $\frac{50}{2}\left(\frac{2}{3}+\frac{49}{3}\right)=25\times 17=425.$ 9. $\frac{30}{2}(232-29\times 8)=0.$

10. $\frac{n}{2}\left\{18+2(n-1)\right\}=\frac{n}{2}(18+2n-2)=\frac{n}{2}(16+2n)=n(8+n).$

11. $\frac{n}{2}\left\{2-\frac{n-1}{6}\right\}=\frac{n}{2}\times\frac{12-n+1}{6}=\frac{n(13-n)}{12}.$

12. Let b denote the common difference; the sum of the first five terms is $\frac{5}{2}(2+4b)$; the sum of the following five terms is $\frac{5}{2}\{2(1+5b)+4b\}$, for the first of these terms is $1+5b$. Therefore

$$\frac{5}{2}(2+4b)=\frac{1}{4}\times\frac{5}{2}\left\{2+14b\right\};\text{ therefore } 4(2+4b)=2+14b;\ \&c.$$

13. Here $a=2$; $7=2+4b$; therefore $b=\frac{5}{4}$;

then $$63=\frac{n}{2}\left\{4+\frac{5}{4}(n-1)\right\}=\frac{n}{8}(5n+11);\ \&c.$$

14. $88=\frac{n}{2}\left\{32+4(n-1)\right\}=n(2n+14)$; &c.

15. Here $\frac{m}{2}\left\{2+(m-1)b\right\}:\frac{n}{2}\left\{2+(n-1)b\right\}::m^2:n^2$; therefore $mn^2\{2+(m-1)b\}=nm^2\{2+(n-1)b\}$; therefore $n\{2+(m-1)b\}=m\{2+(n-1)b\}$; therefore $2(n-m)=b(n-m)$; therefore $b=2$.

The nth term $=a+(n-1)b=1+2n-2=2n-1$.

16. $120=\frac{n}{2}\left\{42-2(n-1)\right\}=n\{22-n\}$; hence we find $n=10$ or 12. If $n=10$ the last term $=21-9\times 2=3$. If $n=12$ the last term $=21-11\times 2=-1$.

17. $204=\frac{1}{2}\times n(l+a)=\frac{1}{2}\times n(50+1)$; therefore $n=8$; $50=1+7b$; therefore $b=7$.

18. $29=1+7d$; therefore $d=4$.

19. The sum of the $n+1$ terms 1, 5, 9, ... is $\frac{n+1}{2}\left\{2+4n\right\}$; the sum of the n terms 3, 7, 11, ... is $\frac{n}{2}\left\{6+4(n-1)\right\}$; that is $\frac{n}{2}\left\{2+4n\right\}$.

20. $a=1, b=4$; $s=\frac{n}{2}\left\{2+4(n-1)\right\}=n(2n-1)$.

21. $1234321=\frac{n}{2}\left\{2+2(n-1)\right\}=n^2$, &c.

22. $1840=\frac{n}{2}\left\{32+8(n-1)\right\}=n\{4n+12\}$, &c.

23. There are $n-1$ journeys; the first is of 2 yards, the second of $2(1+3)$ yards, the third of $2(1+3+5)$ yards; and so on. Thus the first journey is of 2×1 yards, the second of 2×4 yards, the third of 2×9 yards; and so on. The sum is found by Art. 460; we change n in the formula there given to $n-1$, and double tne result.

24. $66=a+13b$, $666=a+133b$, $6666=a+(n-1)b$. From the first two equations we find $a=1$, $b=5$; and from the third $n-1=1333$.

25. Let n denote the number of means; then $21=1+(n+1)b$; therefore $b=\frac{20}{n+1}$. The first mean is $1+b$; the last is $21-b$; therefore the sum $=\frac{n}{2}(1+b+21-b)=11n$. The two greatest means are the two last, namely $21-b$, and $21-2b$. Therefore $11n : 42-3b :: 11 : 4$; therefore $4n=42-3b=42-\frac{60}{n+1}$; therefore $4n^2-38n+18=0$. The only admissible root of this quadratic is 9.

26. $28\frac{1}{2}=\frac{n}{2}\left\{-24+\frac{3}{2}(n-1)\right\}=\frac{n}{4}\left\{3n-51\right\}$; &c.

27. $25=\frac{n}{2}\left\{6+n-1\right\}=\frac{n}{2}(n+5)$; &c.

28. $14=\frac{n}{2}\left\{10-(n-1)\right\}=\frac{n}{2}(11-n)$; &c.

29. $s=\frac{n}{2}\left\{2a+(n-1)b\right\}$; this vanishes if $2a+(n-1)b$ vanishes, that is if $n-1=\frac{-2a}{b}$: hence that n may be a positive integer b must divide $2a$, and the sign of b must be contrary to that of a.

30. $n=a+(m-1)b$, $m=a+(n-1)b$; by subtraction $n-m=(m-n)b$; therefore $b=-1$. Hence $a=m+n-1$. Let x denote the required number of terms; then $\frac{1}{2}(m+n)(m+n-1)=\frac{x}{2}\left\{2(m+n-1)-(x-1)\right\}$;

therefore $$x^2 - x(2m+2n-1) + (m+n)(m+n-1) = 0;$$

therefore $$\left(x - \frac{2m+2n-1}{2}\right)^2 = \left(\frac{2m+2n-1}{2}\right)^2 - (m+n)(m+n-1) = \frac{1}{4};$$

therefore $$x - \frac{2m+2n-1}{2} = \pm\frac{1}{2}, \text{ \&c.}$$

31. $72 = \frac{n}{2}\left\{48 - 4(n-1)\right\} = n(26-2n)$; &c.

32. $pn + qn^2 = \frac{n}{2}\left\{2a + (n-1)b\right\} = \frac{n}{2}\left\{2a - b + nb\right\} = \frac{n}{2}(2a-b) + \frac{n^2 b}{2}$.

Since this is true for *all* values of n we may put for n in succession 1, 2, 3, ..., we shall thus find that $\frac{2a-b}{2} = p$, $\frac{b}{2} = q$; therefore $b = 2q$, and $2a = 2p + b = 2p + 2q$. The mth term $= a + (m-1)b = p + q + 2(m-1)q$.

33. $$S_a = \frac{n}{2}\left\{2a + n - 1\right\},$$

$$S_{3a+n-1} = \frac{n}{2}\left\{2(3a+n-1) + n - 1\right\} = \frac{n}{2}\left\{6a + 3n - 3\right\}.$$

34. We have to shew that $(x^2 - 2x - 1)^2 + (x^2 + 2x - 1)^2 = 2(x^2+1)^2$.

35. $l^2 - a^2 = \{a + (n-1)b\}^2 - a^2 = (n-1)b\{2a + (n-1)b\}$. Twice the sum of all the terms $= n\{2a + (n-1)b\} = 2na + n(n-1)b$; from this take away the sum of the first and last term, that is take away $2a + (n-1)b$; the remainder is $(n-1)\{2a + (n-1)b\}$. Divide $l^2 - a^2$ by this, and the quotient is b.

36. $n = \frac{m}{2}\left\{2a + (m-1)b\right\}$...(1), $m = \frac{n}{2}\left\{2a + (n-1)b\right\}$... (2); from these equations we may deduce $b = -\frac{2(m+n)}{mn}$, $a = \frac{m^2+n^2+mn-m-n}{mn}$, and then the sum of any assigned number of terms can be found. Or we may proceed thus: the sum of $m+n$ terms $= \frac{m+n}{2}\left\{2a + (m+n-1)b\right\}$. Now from (1) and (2) by subtraction $2(n-m) = 2a(m-n) + (m-n)(m+n-1)b$; divide by $m-n$; thus $(m+n-1)b + 2a = -2$. Hence the sum of $m+n$ terms $= -(m+n)$. Again the sum of $m-n$ terms $= \frac{m-n}{2}\left\{2a + (m-n-1)b\right\}$. Now we have just seen that $-1 = a + \frac{(m+n-1)b}{2}$, and from (1) $\frac{2n}{m} = 2a + (m-1)b$; therefore by subtraction we obtain $\frac{2n}{m} + 1 = a + \frac{(m-1-n)b}{2}$. Hence the sum of $m-n$ terms $= (m-n)\left(1 + \frac{2n}{m}\right)$.

37. Suppose n the number of means. The second mean is $1+2b$; the last is $19-b$. Therefore $1+2b : 19-b :: 1 : 6$. Thus $b=1$. And $19=1+(n+1)b$; therefore $n=17$.

38. $5350=\frac{n}{2}\left\{8+n-1\right\}=\frac{n}{2}\left\{n+7\right\}$; &c.

39. Let $2n+1$ denote the whole number of terms. Then 44 is the sum of $n+1$ terms of the series $a+(a+2b)+(a+4b)\ldots$; therefore $44=\frac{n+1}{2}\left\{2a+2bn\right\}$. And 33 is the sum of n terms of the series $a+b+(a+3b)+\ldots$; therefore $33=\frac{n}{2}\left\{2a+2b+2b(n-1)\right\}=\frac{n}{2}\left\{2a+2bn\right\}$. Hence, by division, $\frac{44}{33}=\frac{n+1}{n}$; therefore $n=3$, and $2n+1=7$. The middle term $=a+nb$; and from above this $=\frac{44}{n+1}=11$.

40. $\frac{1}{b+c}$, $\frac{1}{c+a}$, and $\frac{1}{a+b}$ are in A.P. if $\frac{1}{b+c}+\frac{1}{a+b}=\frac{2}{c+a}$, that is if $(c+a)(a+2b+c)=2(b+c)(a+b)$, that is if $a^2+c^2=2b^2$; and this is true by hypothesis.

41. The first term is 1, and the common difference 2,

$$s=\frac{n}{2}\left\{2+2(n-1)\right\}=n^2.$$

42. Here we might first sum the series $1+5+9+\ldots$, and then the series $3+7+11+\ldots$; and subtract the second result from the first. Or we may proceed thus: the first and second terms together make -2, the third and fourth terms together make -2, and so on. Thus if n be even the sum is $-2\times\frac{n}{2}$, that is $-n$; and if n be odd the sum is $-(n-1)+$the last term, that is $-(n-1)+2n-1$, that is n. Thus the sum is $-n$ if n be even, and $+n$ if n be odd; that is the sum is $-n(-1)^n$.

43. Proceed as in the solution of Example 42. If n be even the sum is $-\frac{n}{2}$. If n be odd the sum is $-\frac{n-1}{2}+n$, that is $\frac{n+1}{2}$. It will be found on trial that this may be expressed thus in both cases: $\frac{1}{4}\left\{1-(2n+1)(-1)^n\right\}$.

44. $P=a+(p-1)b$, $Q=a+(q-1)b$. Therefore

$$b=\frac{P-Q}{p-q},\quad a=\frac{(p-1)Q-(q-1)P}{p-q}.$$

Hence the required expression can be obtained.

45. $x=a+(p-1)b$, $y=a+(q-1)b$, $z=a+(r-1)b$. From the first of these equations $p-1=-\frac{a}{b}+\frac{x}{b}$; therefore $p=1+\frac{1-a}{b}+(x-1)\frac{1}{b}$. Simi-

larly $q=1+\frac{1-a}{b}+(y-1)\frac{1}{b}$, $r=1+\frac{1-a}{b}+(z-1)\frac{1}{b}$. These shew that p, q, r are respectively the x^{th}, y^{th}, z^{th} terms of an A. P., of which $1+\frac{1-a}{b}$ is the first term and $\frac{1}{b}$ the common difference.

46. Suppose n the number of sides; then the number of degrees in all the angles $=\frac{n}{2}\left\{240+5(n-1)\right\}$; and this number, by Euclid I. 32, Cor. $=(2n-4)\,90$. Thus $\frac{n}{2}\left\{5n+235\right\}=(2n-4)\,90$; therefore $n^2-25n+144=0$. The roots of this quadratic are 9 and 16; the latter root is inadmissible for it would make some of the angles of the figure greater than two right-angles.

47. The first term $=1+1$; the second $=2^2+2$; the third $=3^2+3$; and so on. Thus we require the sum of the two series $1+2+3+\ldots+n$, and $1^2+2^2+3^2+\ldots+n^2$. The sum of the former series is $\frac{n(n+1)}{2}$; and the sum of the latter series is $\frac{n(n+1)(2n+1)}{6}$. By adding the two together we obtain $\frac{n(n+1)}{2}\left\{1+\frac{2n+1}{3}\right\}$, that is $\frac{n(n+1)(n+2)}{3}$.

48. We have $(a+b)^2=a(a+3b)$; therefore $b^2=ab$; therefore $b=a$; therefore $(a+5b)^2=(a+3b)(a+8b)$.

49. We have $\phi(n)=\frac{n}{2}\left\{2a+(n-1)b\right\}$. Let h denote the second term of the series; then $h=a+b$; therefore $b=h-a$, and $\phi(n)=\frac{n^2(h-a)}{2}+\frac{n(3a-h)}{2}$. Hence $\phi(n+3)-3\phi(n+2)+3\phi(n+1)-\phi(n)$

$$=\frac{h-a}{2}\left\{(n+3)^2-3(n+2)^2+3(n+1)^2-n^2\right\}$$
$$+\frac{3a-h}{2}\left\{n+3-3(n+2)+3(n+1)-n\right\}.$$

It is easily seen that each of these two expressions vanishes.

50. The first term is $5-\frac{1}{2}$, and the common difference is $-\frac{1}{2}$. Thus $s=\frac{n}{2}\left\{9-\frac{n-1}{2}\right\}=\frac{n}{4}\left\{19-n\right\}$.

51. Let the four parts be denoted by $x-3y$, $x-y$, $x+y$, $x+3y$ these are in A.P., the common difference being $2y$. Thus as their sum is unity $4x=1$. Also $(x-3y)^3+(x-y)^3+(x+y)^3+(x+3y)^3=\frac{1}{10}$; this reduces to

$4x^3+60xy^2=\frac{1}{10}$. Substitute the value of x, and we obtain $y^2=\frac{1}{400}$; therefore $y=\pm\frac{1}{20}$. It will be seen that owing to the peculiar form we have adopted for the unknown quantities the work is much simplified. The artifice should be noticed, as it is often useful. If we are engaged for example on a problem respecting three unknown quantities in A.P. instead of denoting them by x, $x+y$, $x+2y$ it may be more advantageous to denote them by $x-y$, x, $x+y$.

52. Let n denote the number of months, a the number of shillings in the wages for the first month. Then, as the wages are raised a shilling every month, $a+n-1=60$. And the total amount of the wages is $48n$ shillings; therefore $48n=\frac{n}{2}\left\{2a+n-1\right\}$; thus $48=\frac{1}{2}\left\{a+60\right\}$; therefore $a=36$, and $n=25$.

53. Suppose x the number of hours B travels; then A travels $x+4\frac{1}{2}$ hours. Therefore $5(x+4\frac{1}{2})=\frac{x}{2}\left\{6+\frac{x-1}{2}\right\}=\frac{x^2}{4}+\frac{11x}{4}$. Thus $x^2-9x=90$; therefore $x=15$ or -6.

54. Suppose that the last person worked for x hours, then the last but one worked for $2x$ hours, and so on; and the first person worked for rx hours. Then the total number of hours of work is $\frac{xr(r+1)}{2}$. And by supposition this is equal to mr; therefore $x=\frac{2m}{r+1}$, and $rx=\frac{2mr}{r+1}$.

55. Let x denote the digit in the hundreds' place, y the digit in the tens' place, z the digit in the units' place. Then the number is $100x+10y+z$. Therefore $100x+10y+z=26(x+y+z)$, $100x+10y+z+396=100z+10y+x$, $2y=x+z$.

56. Let a denote the first integer; then the sum of the $2n+1$ integers $=\frac{2n+1}{2}\left\{2a+2n\right\}=(2n+1)(a+n)$; this is divisible by $2n+1$.

XXXI.

1. $\frac{8}{5}\cdot\frac{\left(\frac{5}{3}\right)^6-1}{\frac{5}{3}-1}=\frac{12}{5}\left\{\left(\frac{5}{3}\right)^6-1\right\}$.

2. $2\frac{(-2)^{10}-1}{-2-1}=-\frac{2}{3}\left\{2^{10}-1\right\}$.

3. $3\frac{\left(\frac{2}{3}\right)^n-1}{\frac{2}{3}-1}=9\left\{1-\left(\frac{2}{3}\right)^n\right\}$.

4. $\frac{2}{3}\frac{\left(\frac{3}{4}\right)^n-1}{\frac{3}{4}-1}=\frac{8}{3}\left\{1-\left(\frac{3}{4}\right)^n\right\}$.

5. $\dfrac{\frac{2}{3}}{1-\frac{2}{3}}=2.$ 6. $\dfrac{\frac{4}{3}}{1-\frac{3}{4}}=\dfrac{16}{3}.$ 7. $\dfrac{\frac{1}{2}}{1-\frac{1}{2}}=1.$ 8. $\dfrac{3}{1-\frac{2}{3}}=9.$

9. $\dfrac{4}{1-\frac{3}{5}}=10.$ 10. $\dfrac{1}{1-\frac{1}{4}}=\dfrac{4}{3}.$ 11. $\dfrac{5}{1-\left(-\frac{1}{10}\right)}=\dfrac{5}{1+\frac{1}{10}}=\dfrac{50}{11}.$

12. $\dfrac{1}{1-\left(-\frac{1}{2}\right)}=\dfrac{1}{1+\frac{1}{2}}=\dfrac{2}{3}.$ 13. $\dfrac{\frac{3}{2}}{1-\left(-\frac{4}{9}\right)}=\dfrac{\frac{3}{2}}{1+\frac{4}{9}}=\dfrac{27}{26}.$

14. $\dfrac{\frac{1}{5}}{1-\left(-\frac{1}{5}\right)}=\dfrac{\frac{1}{5}}{1+\frac{1}{5}}=\dfrac{1}{6}.$ 15. $\dfrac{\frac{1}{2}}{1+\frac{1}{2}}=\dfrac{1}{3}.$

16. To find r divide the second term by the first; thus we get

$$\frac{1}{2-\sqrt{2}}\times\frac{\sqrt{2}-1}{\sqrt{2}+1}=\frac{\sqrt{2}-1}{\sqrt{2}\,(\sqrt{2}-1)\,(\sqrt{2}+1)}=\frac{\sqrt{2}-1}{\sqrt{2}}.$$

Thus $$1-r=1-\frac{\sqrt{2}-1}{\sqrt{2}}=\frac{1}{\sqrt{2}},$$

$$\frac{a}{1-r}=\frac{\sqrt{2}+1}{\sqrt{2}-1}\times\sqrt{2}=\frac{\sqrt{2}\,(\sqrt{2}+1)^2}{2-1}=4+3\sqrt{2}$$

17. Here we have *two* geometrical progressions,

$$\frac{2}{5}+\frac{2}{5^3}+\frac{2}{5^5}+\ldots,\ \text{and}\ \frac{3}{5^2}+\frac{3}{5^4}+\ldots.$$

$$\text{The sum}=\frac{\frac{2}{5}}{1-\frac{1}{25}}+\frac{\frac{3}{5^2}}{1-\frac{1}{25}}=\frac{25}{24}\left(\frac{2}{5}+\frac{3}{5^2}\right).$$

18. Put $a=r$ and $b=r$ in Art. 473; thus we get

$$\frac{r-nr^{n+1}}{1-r}+\frac{r^2(1-r^{n-1})}{(1-r)^2}.$$

19. Put $a=1$, $b=1$, $r=\frac{1}{2}$ in Art. 473; thus we get

$$\frac{1-\frac{n}{2^n}}{1-\frac{1}{2}}+\frac{\frac{1}{2}\left(1-\frac{1}{2^{n-1}}\right)}{\left(1-\frac{1}{2}\right)^2}=2-\frac{n}{2^{n-1}}+2\left(1-\frac{1}{2^{n-1}}\right)=4-\frac{n+2}{2^{n-1}}.$$

20. Put $a=1$, $b=2$, $r=\frac{1}{2}$ in Art. 473; thus we get

$$\frac{1-\frac{2n-1}{2^n}}{1-\frac{1}{2}}+\frac{1-\frac{1}{2^{n-1}}}{\left(1-\frac{1}{2}\right)^2}=2-\frac{2n-1}{2^{n-1}}+4-\frac{4}{2^{n-1}}=6-\frac{2n+3}{2^{n-1}}.$$

21. Put $a=1$, $b=2$, $r=-\frac{1}{2}$ in Art. 473; thus we get

$$\frac{1-\frac{2n-1}{(-2)^n}}{1+\frac{1}{2}}-\frac{1-\frac{1}{(-2)^{n-1}}}{\left(1+\frac{1}{2}\right)^2}=\frac{2}{3}+\frac{2n-1}{3(-2)^{n-1}}-\frac{4}{9}+\frac{4}{9(-2)^{n-1}}=\frac{2}{9}+\frac{6n+1}{9(-2)^{n-1}}.$$

22. Let a denote the first term, and c the third term; then $c=ar^2$; therefore $r=\pm\sqrt{\frac{c}{a}}$. We can now substitute this value of r in the known expression for the sum.

23. $(-3)^4=81$.

24. Suppose the least share to be a pounds, the next ar pounds, the third ar^2 pounds, and the greatest ar^3 pounds. Then

$$a+ar+ar^2+ar^3=700; \quad ar^3-a=\frac{37}{12}(ar^2-ar).$$

The second equation gives $r^3-1=\frac{37r(r-1)}{12}$; therefore $12(r^2+r+1)=37r$; &c.

25. The first term is $-a^4$, and the common ratio is $-a^4$. The sum

$$=-a^4\frac{(-a^4)^n-1}{-a^4-1}=\frac{a^4}{a^4+1}\left\{(-a^4)^n-1\right\}.$$

26. $$P=\frac{1}{1-r^p}, \quad Q=\frac{1}{1-r^q}, \quad P-1=\frac{r^p}{1-r^p}, \quad Q-1=\frac{r^q}{1-r^q},$$

$$P^q(Q-1)^p=\frac{r^{pq}}{(1-r^p)^q(1-r^q)^p}=Q^p(P-1)^q.$$

27. By Art. 468 we have $\cdot444\ldots=\frac{4}{9}$, $\cdot666\ldots=\frac{6}{9}=\frac{2}{3}$.

28. Suppose that in the first year he saved a pounds; then in the second year he saved $\frac{3a}{2}$ pounds, in the third year $\frac{3}{2}\times\frac{3a}{2}$ pounds; and so on.

Therefore $\dfrac{a\left\{\left(\frac{3}{2}\right)^7-1\right\}}{\frac{3}{2}-1}=102\tfrac{19}{20}$; that is $2a\left(\frac{2187}{128}-1\right)=\frac{2059}{20}$; thus $a=\frac{16}{5}$.

29. Take as the given term the n^{th}, that is ar^{n-1}; the $(n+p)^{\text{th}}$ term is ar^{n+p-1}; the $(n-p)^{\text{th}}$ term is ar^{n-p-1}: the product of these is a^2r^{2n-2}, which is the same whatever p may be.

30. Let the G. P. be a, ar, ar^2, ar^3 .... Subtract each term from the succeeding; thus we have $a(r-1)$, $ar(r-1)$, $ar^2(r-1)$, ... which is also a G. P., the common ratio being r.

31. Let a and b be the two quantities; the square of their arithmetical mean is $\left(\frac{a+b}{2}\right)^2$; the arithmetical mean of a^2 and b^2 is $\frac{a^2+b^2}{2}$, and the geometrical mean is $\sqrt{(a^2b^2)}$, that is ab. The arithmetical mean of $\frac{a^2+b^2}{2}$ and ab is $\frac{1}{2}\left\{\frac{a^2+b^2}{2}+ab\right\}$, that is $\left(\frac{a+b}{2}\right)^2$.

32. By Art. 466, $r=\frac{1}{1+10}=\frac{1}{11}$.

33. $S_1=\frac{a(r-1)}{r-1}$, $S_2=\frac{a(r^2-1)}{r-1}$, $S_3=\frac{a(r^3-1)}{r-1}$ ....

The sum of n of these terms $=\frac{a}{r-1}\left\{r+r^2+\ldots+r^n-n\right\}$.

$$\frac{a}{r-1}\left\{\frac{r(r^n-1)}{r-1}-n\right\}=\frac{ra(r^n-1)}{(r-1)^2}-\frac{na}{r-1}.$$

34. By Art. 469 the product $=a^nr^s$ where s stands for $1+2+\ldots+n$, that is $\frac{n(n+1)}{2}$; and $ar^{n+1}=c$. Thus $r^s=\left(\frac{c}{a}\right)^{\frac{n}{2}}$, and $a^nr^s=(ac)^{\frac{n}{2}}$.

35. $s=\frac{a(r^n-1)}{r-1}$, $s'=\frac{a\left(\frac{1}{r^n}-1\right)}{\frac{1}{r}-1}=\frac{a(r^n-1)}{r^{n-1}(r-1)}$; therefore

$$ls'=\frac{a^2r^{n-1}(r^n-1)}{r^{n-1}(r-1)}=\frac{a^2(r^n-1)}{r-1}=as.$$

36. $(a^2+b^2+c^2)(b^2+c^2+d^2)=a^2(1+r^2+r^4)\,a^2r^2(1+r^2+r^4)$ $=a^4r^2(1+r^2+r^4)^2$; and $(ab+bc+cd)^2=(a^2r+a^2r^3+a^2r^5)^2=a^4r^2(1+r^2+r^4)^2$.

37.
$$(a-d)^2=a^2(1-r^3)^2=a^2(1-r)^2(1+r+r^2)^2,$$
$$(b-c)^2+(c-a)^2+(d-b)^2=a^2r^2(1-r)^2+a^2(1-r^2)^2+a^2(r-r^3)^2$$
$$=a^2(1-r)^2\{r^2+(1+r)^2+r^2(1+r)^2\};$$

and it will be found that $r^2+(1+r)^2+r^2(1+r)^2=(1+r+r^2)^2$.

38.
$$a+ar+ar^2=21,\quad a+ar+ar^2+ar^3=45;$$

by division $\frac{1+r+r^2+r^3}{1+r+r^2}=\frac{45}{21}$; therefore $\frac{r^3}{1+r+r^2}=\frac{24}{21}=\frac{8}{7}$, therefore $7r^3=8(1+r+r^2)$. By trial we find $r=2$; &c.

39. Squaring out we have

$$r^2+r^4+r^6+\ldots+r^{2n}+\frac{1}{r^2}+\frac{1}{r^4}+\frac{1}{r^6}+\ldots+\frac{1}{r^{2n}}-2n, \text{ that is}$$

$$\frac{r^2(r^{2n}-1)}{r^2-1}+\frac{\frac{1}{r^2}\left(\frac{1}{r^{2n}}-1\right)}{\frac{1}{r^2}-1}-2n, \text{ that is } \frac{r^{2n}-1}{r^2-1}\left(r^2+\frac{1}{r^{2n}}\right)-2n.$$

40. $$5=\frac{5\,(10-1)}{10-1},\quad 55=\frac{5\,(10^2-1)}{10-1},\quad 555=\frac{5\,(10^3-1)}{10-1}, \text{ \&c.}$$

Thus the series $=\dfrac{5}{9}\left\{10+10^2+\ldots+10^n-n\right\}$

$$=\frac{5}{9}\left\{\frac{10\,(10^n-1)}{10-1}-n\right\}=\frac{50}{81}(10^n-1)-\frac{5n}{9}.$$

41. Let x denote one quantity, and y the other. Then $A=\dfrac{x+y}{2}$, $G=\sqrt{(xy)}$; therefore $A+G=\dfrac{(\sqrt{x}+\sqrt{y})^2}{2}$, $A-G=\dfrac{(\sqrt{x}-\sqrt{y})^2}{2}$; therefore $A^2-G^2=\dfrac{(x-y)^2}{4}$; therefore $\dfrac{x-y}{2}=\sqrt{(A^2-G^2)}$. Thus $x=A+\sqrt{(A^2-G^2)}$, $y=A-\sqrt{(A^2-G^2)}$.

42. Let a denote the first number, ar the second, ar^2 the third, b the fourth. Then $2ar^2=ar+b$, since the last three are in A.P.: also $a+b=14$, $ar+ar^2=12$. Put $14-a$ for b in the first equation: thus $14=2ar^2-ar+a$; therefore by division $\dfrac{2r^2-r+1}{r+r^2}=\dfrac{14}{12}=\dfrac{7}{6}$; therefore $5r^2-13r+6=0$; therefore $r=2$ or $\dfrac{3}{5}$, &c.

43. Let $x-y$, x, $x+y$ denote the numbers: then $x-y+x+x+y=15$; therefore $x=5$. And $x-y+1$, $x+4$, $x+y+19$ are in G.P.; thus

$$(x-y+1)(x+y+19)=(x+4)^2; \text{ that is } (6-y)(24+y)=81;$$

thus $y^2+18y-63=0$: the only admissible root of this quadratic is 3.

44. If a, b, c be in A.P., $\dfrac{2}{9}(a+b+c)^3=\dfrac{2}{9}\left(a+\dfrac{a+c}{2}+c\right)^3=\dfrac{3\,(a+c)^3}{4}$.

And $\quad a^2(b+c)+b^2(c+a)+c^2(a+b)=b^2(c+a)+b\,(a^2+c^2)+ac\,(a+c)$

$$=\frac{(a+c)^3}{4}+\frac{(a+c)(a^2+c^2)}{2}+ac\,(a+c)=\frac{(a+c)^3}{4}+\frac{(a+c)^3}{2}=\frac{3\,(a+c)^3}{4}.$$

If a, b, c be in G.P., $a^3b^3c^3\left(\dfrac{1}{a^3}+\dfrac{1}{b^3}+\dfrac{1}{c^3}\right)=a^6r^6\left(\dfrac{1}{a^3}+\dfrac{1}{a^3r^3}+\dfrac{1}{a^3r^6}\right)$

$$=a^3(r^6+r^3+1)=c^3+b^3+a^3.$$

45. $ar = \dfrac{ar(1-b)}{1-b}$, $(a+ab)r^2 = \dfrac{ar^2(1-b^2)}{1-b}$, $(a+ab+ab^2)r^3 = \dfrac{ar^3(1-b^3)}{1-b}$, &c.

Thus we have $\quad \dfrac{ar}{1-b}\left\{1+r+r^2+\ldots\right\} - \dfrac{arb}{1-b}\left\{1+br+b^2r^2+\ldots\right\}$,

that is $\quad \dfrac{ar}{(1-b)(1-r)} - \dfrac{arb}{(1-b)(1-rb)}$, that is $\dfrac{ar}{(1-r)(1-br)}$.

XXXII.

1. First continue $\frac{1}{3} + \frac{5}{6} + \frac{4}{3}$ which is in A.P. for two terms; the next two terms are $\frac{4}{3} + \frac{1}{2}$, and $\frac{4}{3} + 1$, that is $\frac{11}{6}$ and $\frac{7}{3}$. Hence the next two terms of the H.P. are $\frac{6}{11}$ and $\frac{3}{7}$.

2. We must insert 18 *arithmetical* means between 1 and 20; thus $20 = 1 + 19b$; therefore $b = 1$. The *arithmetical* means are 2, 3, 4, ...; and therefore the harmonical means are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$.

3. In the corresponding A.P. the first term is $\frac{1}{a}$; the second term is $\frac{1}{b}$; and therefore the n^{th} term is $\frac{1}{a} + (n-1)\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{b+(n-1)(a-b)}{ab}$. Invert and we have the required term of the H.P.

4. Denote the required term by x: let a be the first term and b the common difference of the corresponding A. P. Then $\frac{1}{P} = a + (p-1)b$, $\frac{1}{Q} = a + (q-1)b$, $\frac{1}{x} = a + (p+q-1)b$, find a and b from the first and second equations, and substitute in the third. Or thus: multiply the first by p and the second by q and subtract; hence $\frac{p}{P} - \frac{q}{Q} = (p-q)\{a + (p+q-1)b\} = (p-q) \times \frac{1}{x}$.

5. Let a, b, c be the three quantities; and suppose x subtracted from each: then $a-x$, $b-x$, $c-x$ are in H.P. Therefore $b - x = \dfrac{2(a-x)(c-x)}{a+c-2x}$. This gives $x = \dfrac{2ac - b(a+c)}{a+c-2b}$.

6. Let a, b, c be the three quantities in H.P.; then $b = \frac{2ac}{a+c}$; thus $a - \frac{b}{2} = \frac{a^2}{a+c}$; $\frac{b}{2} = \frac{ac}{a+c}$; $c - \frac{b}{2} = \frac{c^2}{a+c}$. These three remainders are obviously in G.P.

7. If a, b, c are in A. P. then $b=\frac{a+c}{2}$; thus $b^2-ac=\left(\frac{a+c}{2}\right)^2-ac=\frac{(a-c)^2}{4}$ which is *positive*. If a, b, c are in G. P. then $b^2-ac=0$. If a, b, c are in H. P. then $b^2-ac=\frac{4a^2c^2}{(a+c)^2}-ac=\frac{ac}{(a+c)^2}\left\{4ac-(a+c)^2\right\}=-\frac{ac(a-c)^2}{(a+c)^2}$ which is negative.

8. Let x and y denote the numbers. Then

$$\frac{1}{2}(x+y)=3;\quad \frac{2xy}{x+y}=\frac{8}{3};\ \&c.$$

9. Let a and b denote the numbers. The geometrical mean

$$=\sqrt{(ab)}=\sqrt{\left(\frac{a+b}{2}\times\frac{2ab}{a+b}\right)}.\quad \text{And } \frac{a+b}{2}-\frac{2ab}{a+b}=\frac{(a-b)^2}{2(a+b)}.$$

10. $$z=\frac{2ab}{a+b};\quad z-a=\frac{ab-a^2}{a+b};\quad z-b=\frac{ab-b^2}{a+b};$$

$$\frac{1}{z-a}+\frac{1}{z-b}=\frac{a+b}{a(b-a)}+\frac{a+b}{b(a-b)}=\frac{a+b}{b-a}\left(\frac{1}{a}-\frac{1}{b}\right)=\frac{a+b}{ab}=\frac{1}{a}+\frac{1}{b}.$$

11. Let x denote the least number, y the next, and z the greatest. Then

$$y=\frac{2xz}{x+z};\quad z=xy;\quad z+1=x+y+2.$$

Substitute from the second in the first: thus $y=2x-1$, &c.

12. Let x and y denote the two terms. Then $x+y=\frac{29}{104}$, $xy=\frac{1}{52}$. Hence these terms are $\frac{2}{13}$ and $\frac{1}{8}$; then the series can be found.

13. Let a and b denote the two numbers. Then $A_1=a+\frac{b-a}{3}$, $A_2=a+\frac{2(b-a)}{3}$, $\frac{1}{H_1}=\frac{1}{a}+\frac{1}{3}\left(\frac{1}{b}-\frac{1}{a}\right)$, $\frac{1}{H_2}=\frac{1}{a}+\frac{2}{3}\left(\frac{1}{b}-\frac{1}{a}\right)$. Thus $A_1=\frac{2a+b}{3}$, $A_2=\frac{2b+a}{3}$; the harmonical mean between these is $\frac{2(2a+b)(2b+a)}{3(2a+b+2b+a)}$, that is $\frac{2(2a+b)(2b+a)}{9(a+b)}$. Also $H_1=\frac{3ab}{a+2b}$, $H_2=\frac{3ab}{2a+b}$; the arithmetical mean between these is $\frac{9ab(a+b)}{2(2a+b)(2b+a)}$. Then $\frac{2(2a+b)(2b+a)}{9(a+b)}\times\frac{9ab(a+b)}{2(2a+b)(2b+a)}=ab$.

14. $\frac{x+y}{2}=A$, $\sqrt{(xy)}=G$, $\frac{2xy}{x+y}=H$. Thus $G^2=AH$; therefore we have $(A-a)^2=A(A-b)$: hence $A=\frac{a^2}{2a-b}$, $H=A-b=\frac{(b-a)^2}{2a-b}$, $G=\frac{a(b-a)}{2a-b}$. Then x and y can be found in terms of a and b from the first two equations; since A and G are known in terms of a and b.

15. $b=\frac{a+c}{2}$, $\beta=\frac{2a\gamma}{a+\gamma}$, $b^2\beta^2=aca\gamma$. Substitute from the first and second equations in the third: thus $\frac{a^2\gamma^2}{(a+\gamma)^2}=\frac{aca\gamma}{(a+c)^2}$; therefore $\frac{2a\gamma}{(a+\gamma)^2}=\frac{2ac}{(a+c)^2}$; therefore, by Art. 395, $\frac{2a\gamma}{a^2+\gamma^2}=\frac{2ac}{a^2+c^2}$; therefore $\frac{a^2+\gamma^2}{a\gamma}=\frac{a^2+c^2}{ac}$.

16. $$b=\frac{2ac}{a+c};\quad a-b=\frac{a^2-ac}{a+c};\quad b-c=\frac{ac-c^2}{a+c};$$

$$\frac{1}{a-b}+\frac{1}{b-c}+\frac{4}{c-a}=\frac{1}{c-a}\left(4-\frac{a+c}{a}-\frac{a+c}{c}\right)$$
$$=\frac{1}{c-a}\left(2-\frac{c}{a}-\frac{a}{c}\right)=-\frac{(c-a)^2}{ac\,(c-a)}=\frac{a-c}{ac}.$$

17. We have given that $b=\frac{2ac}{a+c}$, and we have to shew that

$$\frac{\frac{2ac}{(b+c)(a+b)}}{\frac{a}{b+c}+\frac{c}{a+b}}=\frac{b}{c+a}.$$ The left-hand expression

$$=\frac{2ac}{a(a+b)+c(b+c)}=\frac{2ac}{a^2+c^2+(a+c)\,b}=\frac{2ac}{a^2+c^2+2ac}=\frac{2ac}{(a+c)^2}=\frac{b}{c+a}.$$

18. Let c denote the r^{th} arithmetical mean, and γ the r^{th} harmonical mean. Then $c=a+\frac{r}{n+1}(b-a)$, $\frac{1}{\gamma}=\frac{1}{a}+\frac{r}{n+1}\left(\frac{1}{b}-\frac{1}{a}\right)$;

therefore $$\frac{c}{\gamma}=1+\frac{r}{n+1}\left(\frac{b-a}{a}+\frac{a}{b}-\frac{a}{a}\right)+\frac{r^2}{(n+1)^2}(b-a)\left(\frac{1}{b}-\frac{1}{a}\right)$$
$$=1+\frac{r(b-a)^2}{ab\,(n+1)}-\frac{r^2(b-a)^2}{(n+1)^2ab}.$$

We have then to find the sum obtained by giving to r in this expression all integral values from 1 to n inclusive. Thus we obtain

$$n+\frac{(b-a)^2}{ab\,(n+1)}\times\frac{n(n+1)}{2}-\frac{(b-a)^2}{(n+1)^2ab}\times\frac{n(n+1)(2n+1)}{6},$$

that is $n\left\{1+\frac{(b-a)^2}{2ab}-\frac{(b-a)^2}{6ab\,(n+1)}(2n+1)\right\}$; that is $n\left\{1+\frac{n+2}{n+1}\,\frac{(a-b)^2}{6ab}\right\}$.

19. Suppose x, y, z the three numbers: then $x+y+z=3a^2-b^2$, $x^2+y^2+z^2=3a^4+b^4$, $y=\frac{2xz}{x+z}$. Square the first equation, and subtract the second from the result: thus $2xy+2xz+2yz=6a^4-6a^2b^2$. Substitute for y: thus $6xz=6a^4-6a^2b^2$; therefore $xz=a^4-a^2b^2$. Eliminate y between the first and second: thus $x^2+z^2+(3a^2-b^2-x-z)^2=3a^4+b^4$; therefore

$$2(x+z)^2-2(3a^2-b^2)(x+z)+(3a^2-b^2)^2=3a^4+b^4+2(a^4-a^2b^2);$$

therefore $$(x+z)^2-(3a^2-b^2)(x+z)=-2a^4+2a^2b^2;$$

therefore $$\left(x+z-\frac{3a^2-b^2}{2}\right)^2=\left(\frac{3a^2-b^2}{2}\right)^2-2a^4+2a^2b^2=\left(\frac{a^2+b^2}{2}\right)^2.$$

Hence $x+z=2a^2$ or a^2-b^2. It will be found that the former is applicable; and that it gives $y=a^2-b^2$, $x=a^2\pm ab$, $z=a^2\mp ab$.

20. We will first shew that if the sum of the products of every $n-1$ terms be divided by the product of all the terms the quotient *is the sum of the reciprocals of the terms*. Suppose, for example, there are four terms, p, q, r, s. Then the sum of the products of every three

$$=qrs+prs+pqs+pqr=\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}+\frac{1}{s}\right)pqrs:$$

if then we divide the sum of the products of every three terms by the product of all four we obtain $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}+\frac{1}{s}$. Hence it is given in the present example that the *sum of the reciprocals of the harmonical terms*$=2n$; so that the sum of the terms of the corresponding A. P.$=2n$. Let b be the common difference in this A. P.; thus since the first term is unity we have the sum $=\frac{n}{2}\left\{2+(n-1)b\right\}$; this then$=2n$. Hence $b=\frac{2}{n-1}$.

XXXIII.

1.
$$\begin{array}{r|l} 5 & 221 \\ \hline 5 & 44\ldots1 \\ \hline 5 & 8\ldots4 \\ \hline & 1\ldots3 \end{array}
\qquad
\begin{aligned}
\frac{342}{1000}\times5&=\frac{342}{200}=1\tfrac{71}{100},\\
\frac{71}{100}\times5&=\frac{71}{20}=3\tfrac{11}{20},\\
\frac{11}{20}\times5&=\frac{11}{4}=2\tfrac{3}{4},\\
\frac{3}{4}\times5&=\frac{15}{4}=3\tfrac{3}{4},\ \&c.
\end{aligned}$$

2. Put r for $4m+2$. First let the last digit of the number be $2m+1$; then the number is of the form $pr+2m+1$, and its square is of the form $qr+(2m+1)^2$. Now $(2m+1)^2=4m^2+4m+1=mr+2m+1$; thus the last digit is $2m+1$. Next let the last digit of the number be $2m+2$; then the number is of the form $pr+2m+2$, and its square is of the form $qr+(2m+2)^2$. Now $(2m+2)^2=4m^2+8m+4=(m+1)r+2m+2$; thus the last digit is $2m+2$.

3. The first number$=a=\frac{a(r-1)}{r-1}$; the second number$=a(r+1)=\frac{a(r^2-1)}{r-1}$; the third number$=a(r^2+r+1)=\frac{a(r^3-1)}{r-1}$; and so on. Thus the sum

$$=\frac{a}{r-1}\left\{r+r^2+\ldots+r^n-n\right\}=\frac{a}{r-1}\left\{\frac{r(r^n-1)}{r-1}-n\right\}=\frac{rb}{r-1}-\frac{na}{r-1}.$$

4. $$g=\sqrt{(mn)},\quad a_1=\frac{m+g}{2}=\frac{\sqrt{m}\,(\sqrt{m}+\sqrt{n})}{2},\quad h_1=\frac{2gm}{g+m}=\frac{2m\sqrt{n}}{\sqrt{m}+\sqrt{n}},$$

$$a_2=\frac{n+g}{2}=\frac{\sqrt{n}\,(\sqrt{m}+\sqrt{n})}{2},\quad h_2=\frac{2gn}{g+n}=\frac{2n\sqrt{m}}{\sqrt{m}+\sqrt{n}};$$

therefore $$a_1h_2=mn=g^2=a_2h_1.$$

5. Let c denote the r^{th} arithmetical mean, and γ the r^{th} harmonical mean. Then $$c=b+\frac{r}{n+1}(a-b),\quad \frac{1}{\gamma}=\frac{1}{a}+\frac{r}{n+1}\left(\frac{1}{b}-\frac{1}{a}\right);$$

therefore $\gamma=\dfrac{(n+1)\,ab}{(n+1-r)\,b+ra}$; therefore $\gamma c=ab$. Hence the sum of the products of the corresponding means $=nab$; and from the product of the two first terms we get ab, and also from the product of the two last terms. Thus on the whole we have $(n+2)\,ab$.

6. Let x denote the first harmonical mean and y the last. Then $\dfrac{1}{x}=\dfrac{1}{a}+\dfrac{1}{n+1}\left(\dfrac{1}{b}-\dfrac{1}{a}\right)$, and $\dfrac{1}{y}=\dfrac{1}{a}+\dfrac{n}{n+1}\left(\dfrac{1}{b}-\dfrac{1}{a}\right)$. Thus $x=\dfrac{(n+1)\,ab}{nb+a}$, $y=\dfrac{(n+1)\,ab}{b+na}$. Suppose a less than b; then x is less than y. And $y-x=\dfrac{(n+1)\,ab\,(n-1)\,(b-a)}{(nb+a)\,(na+b)}$. We have to shew that $\dfrac{(n+1)\,ab\,(n-1)}{(nb+a)\,(na+b)}$ is less than $\dfrac{n-1}{n+1}$; that is $(n+1)^2ab$ less than $(nb+a)\,(na+b)$; that is $(n^2+2n+1)\,ab$ less than $(n^2+1)\,ab+n\,(a^2+b^2)$; that is $2ab$ less than a^2+b^2: this is obviously the case.

7. Suppose that A travels x days before B overtakes him; then A travels $\dfrac{x}{2}\left\{2+(x-1)\right\}$ miles, and B travels $12\,(x-5)$ miles; therefore $\dfrac{x}{2}(x+1)=12\,(x-5)$, that is $x^2-23x+120=0$. The roots of this quadratic are 8 and 15. It will be found that B overtakes A in 8 days, and if both continue travelling after the same laws, in 7 more days A overtakes B.

8. Suppose x gallons are taken away each time; after the first operation $256-x$ gallons of wine remain, that is $256\,\dfrac{256-x}{256}$ gallons; the second operation removes $\dfrac{x}{256}$ of the wine which remains after the first operation, and therefore leaves $\dfrac{256-x}{256}$ of it, that is $256\left(\dfrac{256-x}{256}\right)^2$. Proceeding thus we find that after the fourth operation $256\left(\dfrac{256-x}{256}\right)^4$ gallons remain. Therefore $256\left(\dfrac{256-x}{256}\right)^4=81$; thus $\dfrac{256-x}{256}=\left(\dfrac{81}{256}\right)^{\frac{1}{4}}=\dfrac{3}{4}$; therefore $x=64$.

9. Suppose that A has x pounds, and that B has y pounds; and suppose that A stakes $\frac{m}{x}$ pounds, then B stakes $\frac{m}{y}$ pounds. Then

$$\frac{m}{x}+\frac{m}{y}=90, \quad x+\frac{m}{y}=5\left(y-\frac{m}{y}\right), \quad y+\frac{m}{x}=2\left(x-\frac{m}{x}\right).$$

From the second equation we get $5y-x=\frac{6m}{y}$, from the third equation we get $2x-y=\frac{3m}{x}$; by addition $5y-x+4x-2y=\frac{6m}{x}+\frac{6m}{y}=540$; thus $x+y=180$. Combine this with the first equation, and we get $m=\frac{xy}{2}$. Substitute this value of m in the second equation; &c.

10. $\frac{a+b+c}{c+d+a}=\frac{c+d+b}{a+b+d}$; therefore by Art. 394 $\frac{b-d}{c+d+a}=\frac{c-a}{a+b+d}$; therefore $\frac{b-d}{c-a}=\frac{c+d+a}{a+b+d}$; therefore $\frac{a+b-c-d}{c-a}=\frac{c-b}{a+b+d}$; therefore $a+b+d=\frac{(a-c)(b-c)}{a+b-c-d}$. Similarly we can shew that $a+b+c=\frac{(a-d)(b-d)}{a+b-c-d}$;

therefore $$(a+b+c)(a+b+d)=\frac{(a-c)(b-c)(a-d)(b-d)}{(a+b-c-d)^2}.$$

11. In order that the roots of $ax^2+2bx+c=0$ be possible and different b^2-ac must be positive. The second equation is

$$(a^2-ac+2b^2)x^2+2b(a+c)x+c^2-ac+2b^2=0;$$

in order that the roots of this equation may be possible and different $b^2(a+c)^2-(a^2-ac+2b^2)(c^2-ac+2b^2)$ must be positive: the latter expression will be found to reduce to $-(b^2-ac)\{4b^2+(a-c)^2\}$, so that its sign is *opposite* to that of b^2-ac. Hence if the roots of either equation are possible and different the roots of the other are impossible.

12. $a+b=-c$, $x+y=-(z+w)$; therefore $(a+b)(x+y)=cz+cw$; therefore $ax+by-cz=cw-ay-bx$.....(1). Now take $\sqrt{(ax)}+\sqrt{(by)}=-\sqrt{cz}$; square; thus $ax+by+2\sqrt{(abxy)}=cz$......(2). Now by means of (1) we may put (2) in the form $cw-ay-bx+2\sqrt{(abxy)}=0$; therefore $\sqrt{(cw)}=\sqrt{ay}-\sqrt{(bx)}$; that is $\sqrt{(bx)}-\sqrt{(ay)}+\sqrt{(cw)}=0$. Thus we have deduced the second equation from the first. Similarly we might deduce the first from the second.

XXXIV.

1. $\frac{\underline{|8}}{\underline{|3}\,\underline{|3}}=1120.$

2. $\frac{\underline{|10}}{\underline{|2}\,\underline{|2}\,\underline{|2}}=453600.$

3. $\frac{\underline{|14}}{\underline{|2}\,\underline{|2}\,\underline{|2}\,\underline{|2}\,\underline{|2}\,\underline{|3}}=454053600.$

4. $\frac{\underline{|11}}{\underline{|4}\,\underline{|4}\,\underline{|2}}=34650.$

5. $n(n-1)(n-2)(n-3)=12n(n-1)$; therefore $(n-2)(n-3)=12$; the only admissible root of this quadratic is 6.

6. Any two of the ten dice may give sixes; this then can happen in $\frac{10.9}{1.2}$ ways: next any three of the remaining eight dice may give fives; this then can happen in $\frac{8.7.6}{1.2.3}$ ways; the remaining five dice must give twos; this can happen in one way, and $1=\frac{5.4.3.2.1}{1.2.3.4.5}$. Hence the total number of ways is $\frac{10.9}{1.2}\times\frac{8.7.6}{1.2.3}\times\frac{5.4.3.2.1}{1.2.3.4.5}$.

7. The number is the number of combinations of twenty things taken eighteen at a time; and this is the same as the number taken two at a time by Art. 495; that is $\frac{20\times 19}{1.2}$. If a particular pear is to occur, the other seventeen pears can be any seventeen out of the remaining nineteen; thus there are $\frac{19\times 18}{1\times 2}$ ways in which the particular pear occurs.

8. When the particular man is included we have to select 9 men out of 95; this can be done in $\frac{|\underline{95}}{|\underline{9}\,|\underline{86}}$ ways. When the particular man is excluded we have to select 10 men out of 95: this can be done in $\frac{|\underline{95}}{|\underline{10}\,|\underline{85}}$ ways.

10. From Arts. 495 and 496 we see that the combinations must be *complementary:* thus $r-r'+r+r'=n$.

11. See the solution of Example 12 in the Algebra.

13. The answer to the first part of the question is the number of permutations of 9 things taken all together, that is $|\underline{9}$. For the second part the answer would be $|\underline{10}$ if we allowed zero to occupy the extreme left-hand place; but if we do not allow this we must take away all the cases in which this occurs; and the number of these cases by the first part is $|\underline{9}$.

14. We can select 3 conservatives in $\frac{|\underline{12}}{|\underline{3}\,|\underline{9}}$ ways; and we can select 4 reformers in $\frac{|\underline{16}}{|\underline{4}\,|\underline{12}}$ ways: the product of these two numbers is the required number of ways.

16. The combinations which are equal in number must be complementary; therefore $r+r+1=n$. And $\frac{|\underline{n}}{|\underline{r}\,|\underline{r+1}}=\frac{5}{4}\cdot\frac{|\underline{n}}{|\underline{r-1}\,|\underline{r+2}}$; therefore $\frac{r+2}{r}=\frac{5}{4}$; therefore $r=8$.

17. Out of the m things we can get $\dfrac{|\underline{m}}{|\underline{r}\,|\underline{m-r}}$ combinations of r things; out of the n things we can get $\dfrac{|\underline{n}}{|\underline{s}\,|\underline{n-s}}$ combinations of s things; then the $s+r$ things will give rise to $|\underline{s+r}$ permutations if they are all different.

18. A parallelepiped has three different edges: thus the answer is the number of combinations of n things taken three at a time.

19. The number of combinations of $4n$ things taken $2n$ at a time is $\dfrac{|\underline{4n}}{|\underline{2n}\,|\underline{2n}}$; the number of combinations of $2n$ things taken n at a time is $\dfrac{|\underline{2n}}{|\underline{n}\,|\underline{n}}$. Hence the required ratio $=\dfrac{|\underline{4n}\,|\underline{n}\,|\underline{n}}{|\underline{2n}\,|\underline{2n}\,|\underline{2n}}$.

Now $\quad |\underline{2n}=1.3.5...(2n-1).2.4.6...2n=1.3.5.(2n-1)\,2^n|\underline{n}.$

Similarly $|\underline{4n}=1.3.5...(4n-1)\,2^{2n}|\underline{2n}.$

Substitute and the ratio reduces to the required value.

20. Out of the 17 consonants we get $\dfrac{17\times16}{1.2}$ combinations of two; thus $\dfrac{17\times16}{1.2}\times5$ expresses the number of ways in which we can get two consonants and one vowel: and each combination of three letters gives rise to $|\underline{3}$ permutations. Thus we have $\dfrac{17\times16}{2}\times5\times|\underline{3}$, that is 4080 words.

21. As in the preceding Example we have $\dfrac{10\times9\times8}{|\underline{3}}\times\dfrac{4\times3}{|\underline{2}}\times|\underline{5}$, that is 86400 words.

22. If the 3 given letters are to retain the same order always they count as one letter, so that we have as it were 5 letters; and the answer is $|\underline{5}$. But if the three given letters may occur in *any order* by taking the permutations of them we obtain $|\underline{5}\times|\underline{3}$ words.

23. Suppose we use 4 flags. The number of permutations of the 10 numerals taken 4 at a time is $10.9.8.7$: but all the cases in which zero occupies the extreme left-hand place should be excluded, and thus we can form $10.9.8.7-9.8.7$ signals with 4 flags. Similarly we can form $10.9.8-9.8$ signals with 3 flags; $10.9-9$ signals with 2 flags; and 10 signals with 1 flag.

24. Take the *permutations* of 6 consonants 2 at a time: the number is 30. Then put a vowel in the middle of each permutation. Then 30×3 is the number of words.

25. Put the 3 vowels in the even places; this can be done in $|\underline{3}$ ways: then put the 3 consonants in the odd places; this can be done in $|\underline{3}$ ways. Thus on the whole we get $|\underline{3}\times|\underline{3}$ words.

26. Put the 3 men to their proper side, and the 2 men to their proper side. Out of the remaining 3 men take one to be put on the former side: this choice can be made in 3 ways. Then each set of 4 men can be arranged in $\underline{|4}$ ways. Thus on the whole there are $3 \times \underline{|4} \times \underline{|4}$ ways in which the crew can be arranged.

27. The first arm can be put into n distinct positions; so can the second: thus with these two arms we can form n^2 signals. Now take the third arm; each position may be combined with any pair of positions of the first and second arms: then we have $n^2 \times n$, that is n^3 signals. And so on.

28. We can form one set in $\dfrac{52 \,.\, 51 \ldots 40}{\underline{|13}}$ ways: then out of the remaining 39 cards we can form a set in $\dfrac{39 \,.\, 38 \ldots 27}{\underline{|13}}$ ways; and so on.

29. $\dfrac{10 \,.\, 9 \,.\, 8}{1 \,.\, 2 \,.\, 3}$.

30. Take any pair out of the $n-p$ points; thus we get $\dfrac{(n-p)(n-p-1)}{1 \,.\, 2}$ straight lines. Take any one of the $n-p$ points and any one of the p points: thus we get $(n-p)p$ straight lines. All the p points lie on one straight line. Thus the total number is $\dfrac{(n-p)(n-p-1)}{1 \,.\, 2} + (n-p)p + 1$. Or thus: if no three of the points were in the same straight line we should get $\dfrac{n(n-1)}{1 \,.\, 2}$ straight lines. Now if p of the points come into one straight line we have this straight line instead of the $\dfrac{p(p-1)}{1 \,.\, 2}$ straight lines which the p points furnished. This gives the result in the form $\dfrac{n(n-1)}{1 \,.\, 2} - \dfrac{p(p-1)}{1 \,.\, 2} + 1$. The two forms will be found to agree.

31. Take any three of the $n-p$ points; thus we get $\dfrac{(n-p)(n-p-1)(n-p-2)}{\underline{|3}}$ triangles. Take two of the $n-p$ points and one of the p points; thus we get $\dfrac{p(n-p)(n-p-1)}{1 \,.\, 2}$ triangles. Take one of the $n-p$ points and two of the p points; thus we get $\dfrac{p(p-1)(n-p)}{1 \,.\, 2}$ triangles. The sum of these three expressions is the total number. Or thus: if no three of the points were in the same straight line we should get $\dfrac{n(n-1)(n-2)}{\underline{|3}}$ triangles. Now if p of the points come into one straight line we lose the $\dfrac{p(p-1)(p-2)}{\underline{|3}}$ triangles which they would have furnished. This gives the result in the form $\dfrac{n(n-1)(n-2)}{\underline{|3}} - \dfrac{p(p-1)(p-2)}{\underline{|3}}$. The two forms will be found to agree.

32. Proceed as in Example 31. Thus we obtain for the result

$$\frac{(n-p)(n-p-1)(n-p-2)}{\underline{|3}}+\frac{p(n-p)(n-p-1)}{1.2}+\frac{p(p-1)(n-p)}{1.2}+1.$$

Or, by adopting the other method, we obtain the same result in the form $\frac{n(n-1)(n-2)}{\underline{|3}}-\frac{p(p-1)(p-2)}{\underline{|3}}+1.$

33. The number of straight lines is $\frac{n(n-1)}{2}$, or N say. Now N straight lines would give $\frac{N(N-1)}{2}$ points of intersection. But as $n-1$ straight lines pass through each of the n original points $\frac{(n-1)(n-2)}{1.2}$ points of intersection coincide at each of these n points. Thus excluding these there remain $\frac{N(N-1)}{1.2}-\frac{n(n-1)(n-2)}{1.2}$ points of intersection; that is there remain $\frac{1}{2}\times\frac{n(n-1)}{1.2}\left\{\frac{n(n-1)}{1.2}-1\right\}-\frac{n(n-1)(n-2)}{1.2}$ points of intersection: this reduces to $\frac{n(n-1)(n-2)(n-3)}{8}$.

34. This is an example of Art. 497. For as the boats which belong to one club do not undergo any permutations among themselves they are in the same case as letters which are alike. Hence the number is $\frac{\underline{|24}}{\underline{|3}\,\underline{|3}\,\underline{|2}\,\underline{|2}\,\underline{|2}\,\underline{|2}\,\underline{|2}}$.

35. The sets are to be treated as if they were single volumes.

37. I. There is 1 case in which no letter is repeated. II. 5 cases in which p occurs twice, as many in which r occurs twice, and as many in which o occurs twice; 15 in all. III. 10 cases in which o occurs three times. IV. 6 cases in which p and r each occur twice, as many in which p and o each occur twice, and as many in which r and o each occur twice; 18 in all. V. 4 cases in which o occurs three times and p twice, and as many in which o occurs three times and r twice; 8 in all. VI. 1 case in which p occurs twice, r twice, and o twice. Then $1+15+10+18+8+1=53$.

38. Out of the first set we get $\frac{\underline{|2a}}{\underline{|a}\,\underline{|a}}$ combinations; out of the second $\frac{\underline{|3a}}{\underline{|a}\,\underline{|2a}}$; out of the third $\frac{\underline{|4a}}{\underline{|a}\,\underline{|3a}}$; and so on; the product of all these expressions gives us the total number of combinations. It is obvious that by cancelling, the product takes the form given in the Example.

39. There are $\underline{|5}$ numbers in all. Consider one of the digits, as 3 for example. In $\underline{|4}$ cases 3 is in the units' place, in as many cases 3 is in the tens'

place, in as many cases 3 is in the hundreds' place; and so on. Thus the sum arising from the 3 alone is $\lfloor 4 \{3+30+300+3000+30000\}$, that is $3\times\lfloor 4\times 11111$. Proceed similarly with the other digits. Thus the final result $=(1+2+3+4+5)\lfloor 4\times 11111$.

40. The sum in the preceding Example has for one factor $1+2+3+4+5$, that is the sum of the digits: and it is obvious that a similar result will hold whatever be the digits.

XXXV.

11.
$$\{a+\sqrt{(a^2-1)}\}^6+\{a-\sqrt{(a^2-1)}\}^6$$
$$=2\left\{a^6+\frac{6.5}{1.2}a^4(a^2-1)+\frac{6.5.4.3}{1.2.3.4}a^2(a^2-1)^2+(a^2-1)^3\right\}$$
$$=2\{a^6+15a^4(a^2-1)+15a^2(a^2-1)^2+(a^2-1)^3\}=\&c.$$

12. $\left(y^2+\frac{c^3}{y}\right)^5=\frac{(y^3+c^3)^5}{y^5}$; the coefficient of y in the expansion of this expression is the same as the coefficient of y^6 in the expansion of $(y^3+c^3)^5$.

13. We have $(x+a)^n=A+B$, $(x-a)^n=A-B$; therefore by multiplication $(x^2-a^2)^n=A^2-B^2$.

14. This result may be obtained by direct work. Or we may proceed thus: $(1+x)^{n+1}(1-x)=(1+x)^n(1-x^2)$; hence we infer that the coefficient of any assigned power of x will be the same whether it is obtained from the left-hand expression or from the right-hand expression. See Chapter XLVIII. Now in the expansion of $(1+x)^{n+1}(1-x)$ the coefficient of x^{r+1} will be found by subtracting the coefficient of x^r in the expansion of $(1+x)^{n+1}$ from the coefficient of x^{r+1}, &c.

15. The middle term $=\frac{\lfloor 2n}{\lfloor n\lfloor n}x^n$; and
$$\lfloor 2n=1.3.5\ldots(2n-1)\,2.4.6\ldots 2n=1.3.5\ldots(2n-1)\,2^n\lfloor n;$$
therefore
$$\frac{\lfloor 2n}{\lfloor n}=1.3.5\ldots(2n-1)\,2^n.$$

16. Suppose that 2916 is the r^{th} term in the expansion of $(x+a)^n$: then $2916=\frac{\lfloor n}{\lfloor r-1\lfloor n-r+1}x^{n-r+1}a^{r-1}$, and therefore $4860=\frac{\lfloor n}{\lfloor r\lfloor n-r}x^{n-r}a^r$; hence by division $\frac{4860}{2916}=\frac{n-r+1}{r}\frac{a}{x}$. In the same way we obtain $\frac{4320}{4860}=\frac{n-r}{r+1}\frac{a}{x}$, $\frac{2160}{4320}=\frac{n-r-1}{r+2}\frac{a}{x}$. Thus, simplifying, we have
$$\frac{5}{3}=\frac{n-r+1}{r}\frac{a}{x},\quad \frac{8}{9}=\frac{n-r}{r+1}\frac{a}{x},\quad \frac{1}{2}=\frac{n-r-1}{r+2}\frac{a}{x}:$$

multiply the first of these three equations by r and the second by $r+1$, and subtract; thus $\frac{5r}{3}-\frac{8(r+1)}{9}=\frac{a}{x}$; and similarly $\frac{8(r+1)}{9}-\frac{r+2}{2}=\frac{a}{x}$; therefore $\frac{5r}{3}-\frac{8(r+1)}{9}=\frac{8(r+1)}{9}-\frac{r+2}{2}$, from which $r=2$. Then $\frac{a}{x}=\frac{2}{3}$; and then $n=6$. Hence putting $x=\frac{3a}{2}$ and $n=6$, and $r=2$ in the first equation we find $a=2$.

17. $\left(x+\frac{1}{x}\right)^n=\frac{(x^2+1)^n}{x^n}$; the coefficient of x^r in the expansion of this expression is the same as the coefficient of x^{n+r} in the expansion of $(x^2+1)^n$. Now all the powers of x in the expansion of $(x^2+1)^n$ are obviously *even* powers, and thus $n+r$ must be an even number. The coefficient of x^{n+r} in the expansion of $(x^2+1)^n$ is the same as the coefficient of $y^{\frac{n+r}{2}}$ in the expansion of $(y+1)^n$; and this is $\frac{\underline{|n}}{\underline{|\tfrac{1}{2}(n-r)}\,\underline{|\tfrac{1}{2}(n+r)}}$.

18. The coefficient is the same as that of $x^{2r+1+2n+1}$ in the expansion of $(x^2-1)^{2n+1}$; this is the same as the coefficient of y^{r+n+1} in the expansion of $(y-1)^{2n+1}$.

19. We must find the r^{th} term from the beginning, the r^{th} term from the end, and the middle term of $(x^2-1)^{2n}$; and divide each by x^{2n}.

20. Since $(x+a\sqrt{-1})(x-a\sqrt{-1})=x^2+a^2$ we may conclude that

$$(x+a\sqrt{-1})^n(x-a\sqrt{-1})^n=(x^2+a^2)^n.$$

Now $\qquad (x+a\sqrt{-1})^n=t_0-t_2+t_4-\ldots+(t_1-t_3+t_5-\ldots)\sqrt{-1},$

and $\qquad (x-a\sqrt{-1})^n=t_0-t_2+t_4-\ldots-(t_1-t_3+t_5-\ldots)\sqrt{-1};$

by multiplying together the two expressions on the right-hand side we obtain $(t_0-t_2+t_4-\ldots)^2+(t_1-t_3+t_5-\ldots)^2$; and therefore this $=(x^2+a^2)^n$.

XXXVI.

1. $$(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1.2}x^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{\underline{|3}}x^3+\ldots$$
$$=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\text{\&c.}$$

2. $$(1+x)^{\frac{1}{4}}=1+\frac{1}{4}x+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{1.2}x^2+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)}{\underline{|3}}x^3+\ldots$$
$$=1+\frac{1}{4}x-\frac{3}{32}x^2+\frac{7}{128}x^3-\ldots$$

3. $$(1+x)^{\frac{2}{3}}=1+\frac{2}{3}x+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{1.2}x^2+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{\underline{|3}}x^3+\dots$$
$$=1+\frac{2}{3}x-\frac{1}{9}x^2+\frac{4}{81}x^3-\dots$$

4. $$(1+x)^{-\frac{1}{2}}=1+\left(-\frac{1}{2}\right)x+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1.2}x^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{\underline{|3}}x^3+\dots$$
$$=1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+\dots$$

5. $$(1+x)^{-\frac{1}{4}}=1+\left(-\frac{1}{4}\right)x+\frac{\left(-\frac{1}{4}\right)\left(-\frac{1}{4}-1\right)}{1.2}x^2$$
$$+\frac{\left(-\frac{1}{4}\right)\left(-\frac{1}{4}-1\right)\left(-\frac{1}{4}-2\right)}{\underline{|3}}x^3=1-\frac{1}{4}x+\frac{5}{32}x^2-\frac{15}{128}x^3+\dots$$

6. $$(1+x)^{-\frac{2}{3}}=1+\left(-\frac{2}{3}\right)x+\frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)}{1.2}x^2$$
$$+\frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)\left(-\frac{2}{3}-2\right)}{\underline{|3}}x^3+\dots=1-\frac{2}{3}x+\frac{5}{9}x^2-\frac{40}{81}x^3+\dots$$

7. $$(1-x)^{\frac{1}{5}}=1-\frac{1}{5}x+\frac{\frac{1}{5}\left(\frac{1}{5}-1\right)}{1.2}x^2-\frac{\frac{1}{5}\left(\frac{1}{5}-1\right)\left(\frac{1}{5}-2\right)}{1.2.3}x^3-\dots$$
$$=1-\frac{1}{5}x-\frac{2}{25}x^2-\frac{6}{125}x^3-\dots$$

8. $$(1-2x)^{\frac{3}{4}}=1-\frac{3}{4}2x+\frac{\frac{3}{4}\left(\frac{3}{4}-1\right)}{1.2}(2x)^2-\frac{\frac{3}{4}\left(\frac{3}{4}-1\right)\left(\frac{3}{4}-2\right)}{\underline{|3}}(2x)^3-\dots$$
$$=1-\frac{3}{2}x-\frac{3}{8}x^2-\frac{5}{16}x^3-\dots$$

9. $\sqrt{(a^2-x^2)}=a\left(1-\dfrac{x^2}{a^2}\right)^{\frac{1}{2}}$

$$=a\left\{1-\frac{1}{2}\frac{x^2}{a^2}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1.2}\frac{x^4}{a^4}-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{|\underline{3}}\frac{x^6}{a^6}+\dots\right\}$$

$$=a\left\{1-\frac{1}{2}\frac{x^2}{a^2}-\frac{1}{8}\frac{x^4}{a^4}-\frac{1}{16}\frac{x^6}{a^6}-\dots\right\}.$$

10. $(3a-2x)^{\frac{2}{3}}=(3a)^{\frac{2}{3}}\left\{1-\dfrac{2x}{3a}\right\}^{\frac{2}{3}}$

$$=(3a)^{\frac{2}{3}}\left\{1-\frac{2}{3}\frac{2x}{3a}+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{1.2}\left(\frac{2x}{3a}\right)^2-\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{|\underline{3}}\left(\frac{2x}{3a}\right)^3+\dots\right\}$$

$$=(3a)^{\frac{2}{3}}\left\{1-\frac{4}{9}\frac{x}{a}-\frac{4}{81}\frac{x^2}{a^2}-\frac{32}{2187}\frac{x^3}{a^3}-\dots\right\}$$

11. $(a^2-bx)^{-\frac{2}{5}}=(a^2)^{-\frac{2}{5}}\left\{1-\dfrac{bx}{a^2}\right\}^{-\frac{2}{5}}$

$$=a^{-\frac{4}{5}}\left\{1+\frac{2}{5}\frac{bx}{a^2}+\frac{\frac{2}{5}\left(\frac{2}{5}+1\right)}{1.2}\frac{b^2x^2}{a^4}+\frac{\frac{2}{5}\left(\frac{2}{5}+1\right)\left(\frac{2}{5}+2\right)}{|\underline{3}}\frac{b^3x^3}{a^6}+\dots\right\}$$

$$=a^{-\frac{4}{5}}\left\{1+\frac{2}{5}\frac{bx}{a^2}+\frac{7}{25}\frac{b^2x^2}{a^4}+\frac{28}{125}\frac{b^3x^3}{a^6}+\dots\right\}.$$

12. $(1+5x)^{\frac{17}{5}}=1+\dfrac{17}{5}5x+\dfrac{\frac{17}{5}\left(\frac{17}{5}-1\right)}{1.2}(5x)^2+\dfrac{\frac{17}{5}\left(\frac{17}{5}-1\right)\left(\frac{17}{5}-2\right)}{|\underline{3}}(5x)^3+\dots$

$$=1+17x+102x^2+238x^3+\dots$$

13. $\dfrac{4(4+1)(4+2)\dots(4+r-1)}{|\underline{r}}x^r=\dfrac{(r+1)(r+2)(r+3)}{|\underline{3}}x^r.$

14. $\dfrac{5(5+1)(5+2)\dots(5+r-1)}{|\underline{r}}=\dfrac{(r+1)(r+2)(r+3)(r+4)}{|\underline{4}}x^r.$

15. $\dfrac{\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\dots\left\{\frac{1}{n}-(r-1)\right\}}{|\underline{r}}(-x)^r=-\dfrac{(n-1)(2n-1)\dots\{(r-1)n-1\}}{n^r|\underline{r}}x^r.$

16. $\dfrac{\frac{1}{p}\left(\frac{1}{p}-1\right)\left(\frac{1}{p}-2\right)\dots\left\{\frac{1}{p}-(r-1)\right\}}{|\underline{r}}(-px)^r$

$$==\frac{(p-1)(2p-1)(3p-1)\dots\{(r-1)p-1\}}{|\underline{r}}x^r.$$

17. $\dfrac{1}{\sqrt{(1+x)}}=(1+x)^{-\frac{1}{2}}$; the $(r+1)^{\text{th}}$ term $=$

$$\frac{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\dots\left(\frac{1}{2}+r-1\right)}{\lfloor r}(-1)^r x^r=\frac{1.3.5\dots(2r-1)}{2^r\lfloor r}(-1)^r x^r.$$

18. $$\frac{\frac{2}{3}\left(\frac{2}{3}+1\right)\left(\frac{2}{3}+2\right)\dots\left(\frac{2}{3}+r-1\right)}{\lfloor r}(x^2)^r=\frac{2.5.8\dots(3r-1)}{3^r\lfloor r}x^{2r}.$$

19. $$\frac{\frac{7}{2}\left(\frac{7}{2}+1\right)\left(\frac{7}{2}+2\right)\dots\left(\frac{7}{2}+r-1\right)}{\lfloor r}(2x)^r=\frac{7.9.11\dots(2r+5)}{\lfloor r}x^r.$$

20. $\dfrac{1}{\sqrt[4]{(1-x)}}=(1-x)^{-\frac{1}{4}}$; the $(r+1)^{\text{th}}$ term $=$

$$\frac{\frac{1}{4}\left(\frac{1}{4}+1\right)\left(\frac{1}{4}+2\right)\dots\left(\frac{1}{4}+r-1\right)}{\lfloor r}x^r=\frac{1.5.9\dots(4r-3)}{4^r\lfloor r}x^r.$$

21. $\sqrt{(24)}=(25-1)^{\frac{1}{2}}=5\left(1-\dfrac{1}{25}\right)^{\frac{1}{2}}=5\left\{1-\dfrac{1}{2}\,\dfrac{1}{25}-\dfrac{1}{8}\left(\dfrac{1}{25}\right)^2\dots\right\}.$

22. $\sqrt[3]{(999)}=(1000-1)^{\frac{1}{3}}=10\left(1-\dfrac{1}{1000}\right)^{\frac{1}{3}}$

$$=10\left\{1-\frac{1}{3}\,\frac{1}{1000}-\frac{1}{9}\left(\frac{1}{1000}\right)^2\dots\right\}.$$

23. $\sqrt[5]{(31)}=(32-1)^{\frac{1}{5}}=2\left(1-\dfrac{1}{32}\right)^{\frac{1}{5}}=2\left\{1-\dfrac{1}{5}\,\dfrac{1}{32}-\dfrac{2}{25}\left(\dfrac{1}{32}\right)^2\dots\right\}.$

24. $\sqrt[5]{(99000)}=(100000-1000)^{\frac{1}{5}}=10\left(1-\dfrac{1}{100}\right)^{\frac{1}{5}}$

$$=10\left\{1-\frac{1}{5}\,\frac{1}{100}-\frac{2}{25}\left(\frac{1}{100}\right)^2\dots\right\}.$$

25. $(1+x)^{\frac{1}{2}}=1+\dfrac{1}{2}x-\dfrac{1}{8}x^2+\dots$; $(1-x)^{\frac{2}{3}}=1-\dfrac{2}{3}x-\dfrac{1}{9}x^2-\dots$;

thus the expression $=\dfrac{1+\frac{1}{2}x-\frac{1}{8}x^2+1-\frac{2}{3}x-\frac{1}{9}x^2+\dots}{1+x+1+\frac{1}{2}x-\frac{1}{8}x^2+\dots}$

$$=\frac{2-\frac{1}{6}x-\frac{17}{72}x^2+\dots}{2+\frac{3}{2}x-\frac{1}{8}x^2+\dots}=\frac{1-\frac{1}{12}x-\frac{17}{144}x^2+\dots}{1+\frac{3}{4}x-\frac{1}{16}x^2+\dots}.$$

Now divide the numerator by the denominator and we find for the first two terms of the expression $1-\dfrac{5x}{6}$.

26. We have to shew that

$$\frac{n}{1}+\frac{n(n-1)(n-2)}{\lfloor 3}+\frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 5}+\dots$$

exceeds by unity $\quad \dfrac{n(n-1)}{1.2}+\dfrac{n(n-1)(n-2)(n-3)}{\lfloor 4}+\dots;$

that is we have to shew that

$$n-\frac{n(n-1)}{1.2}+\frac{n(n-1)(n-2)}{\lfloor 3}-\frac{n(n-1)(n-2)(n-3)}{\lfloor 4}+\dots$$

equals unity; this is obvious for the expression $=1-(1-1)^n$.

27. By Art. 523 the number $=\dfrac{n(n+1)\dots(2n-1)}{\lfloor n}$; multiply both numerator and denominator by $\lfloor n-1$.

28. The $(r+1)^{\text{th}}$ term may be found by multiplying the r^{th} term by $\left(\dfrac{n+1}{r}-1\right)x$, that is in this case by $\left(\dfrac{5}{r}-1\right)\dfrac{2}{3}$: putting this $=1$ we get $r=2$, so that the 3rd term is equal to the 2nd, and these are greater than any other term.

29. The numerical value of the $(r+1)^{\text{th}}$ term may be found by multiplying that of the r^{th} term by $\left(\dfrac{n-1}{r}+1\right)x$, that is in this case by $\left(\dfrac{11}{r}+1\right)\dfrac{1}{5}$; this multiplier first becomes less than 1 when $r=3$: thus the third term is the greatest.

30. In this case the numerical value of the $(r+1)^{\text{th}}$ term may be found by multiplying that of the r^{th} term by $\left(\dfrac{2}{r}+1\right)\dfrac{5}{7}$; this is equal to 1 when $r=5$, so that the 6th term is equal to the 5th, and these are greater than any other term.

31. In this case the numerical value of the $(r+1)^{\text{th}}$ term may be found by multiplying that of the r^{th} by $\left(\dfrac{5}{3r}+1\right)\dfrac{7}{12}$; this multiplier first becomes less than 1 when $r=3$: thus the third term is the greatest.

32. $\left(n-\dfrac{1}{n}\right)^{2n+1}=n^{2n+1}\left(1-\dfrac{1}{n^2}\right)^{2n+1}$; thus we require the greatest term in the expansion of $\left(1-\dfrac{1}{n^2}\right)^{2n+1}$. In this case the numerical value of the $(r+1)^{\text{th}}$ term may be found by multiplying that of the r^{th} by $\left(\dfrac{2n+2}{r}-1\right)\dfrac{1}{n^2}$. If $n=1$ this expression is equal to 1 when $r=2$; if $n=2$ this expression first becomes less than 1 when $r=2$; if n is greater than 2 this expression is always less than 1.

33. By Art. 524 the number $= \dfrac{4 \,.\, 5 \ldots 13}{|\underline{10}} = \dfrac{11 \,.\, 12 \,.\, 13}{|\underline{3}}$.

34. We must find when $\dfrac{11}{3} - r + 1$ first becomes negative; this is when $r = 5$; and so the 6^{th} term is the first negative term.

35. The coefficient of x^p in the expansion of $x^n(1-x)^{-2n}$ is the same as the coefficient of x^{p-n} in the expansion of $(1-x)^{-2n}$, and this is

$$\frac{2n(2n+1) \ldots\ldots (2n+p-n-1)}{|\underline{p-n}}.$$

Now multiply both numerator and denominator by $|\underline{2n-1}$; thus the coefficient becomes $\dfrac{|\underline{p+n-1}}{|\underline{p-n}\,|\underline{2n-1}}$; and then by cancelling $|\underline{p-n}$ we get

$$\frac{(p-n+1)(p-n+2) \ldots\ldots (p+n-1)}{|\underline{2n-1}};$$

the middle factor of the numerator is p; the factors before and after p are $p-1$ and $p+1$ respectively, the product of which is p^2-1^2; and so on.

36. We require the coefficient of x^{2n} in the expansion of

$$(1-6x+12x^2-8x^3)(1-3x^2)^{-4}.$$

Now $(1-3x^2)^{-4}$ when expanded contains only *even* powers of x; thus x^{2n} will occur only in $(1+12x^2)(1-3x^2)^{-4}$. The coefficient of x^{2n} in $(1-3x^2)^{-4}$ is $\dfrac{4 \,.\, 5 \,.\, 6 \ldots\ldots (n+4-1)}{|\underline{n}} 3^n$, that is $\dfrac{(n+1)(n+2)(n+3)}{|\underline{3}} 3^n$; and the coefficient of x^{2n-2} is $\dfrac{n(n+1)(n+2)}{|\underline{3}} 3^{n-1}$. Thus finally we have

$$\frac{(n+1)(n+2)(n+3)}{|\underline{3}} 3^n + \frac{12n(n+1)(n+2)}{|\underline{3}} 3^{n-1},$$

$$\text{that is } \frac{3^{n-1}(n+1)(n+2)(5n+3)}{2}.$$

37. We require the coefficient of x^n in the expansion of $(1+2x+x^2)(1-x)^{-4}$: by Example 13 this will be

$$\frac{(n+1)(n+2)(n+3)}{|\underline{3}} + \frac{2n(n+1)(n+2)}{|\underline{3}} + \frac{(n-1)n(n+1)}{|\underline{3}};$$

it will be found that this is $\dfrac{n+1}{3}(2n^2+4n+3)$.

38. $\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{2}} = \dfrac{a+x}{(a^2-x^2)^{\frac{1}{2}}} = (a+x)(a^2-x^2)^{-\frac{1}{2}} = \left(1+\dfrac{x}{a}\right)\left(1-\dfrac{x^2}{a^2}\right)^{-\frac{1}{2}}$; we should therefore expand $\left(1-\dfrac{x^2}{a^2}\right)^{-\frac{1}{2}}$ in powers of $\dfrac{x^2}{a^2}$, and multiply the result

by $1+\frac{x}{a}$. Hence to find the coefficient of x^{2r} in the original expression we must find the coefficient of x^{2r} in $\left(1-\frac{x^2}{a^2}\right)^{-\frac{1}{2}}$: and to find the coefficient of x^{2r+1} in the original expression we must find the coefficient of x^{2r} in $\left(1-\frac{x^2}{a^2}\right)^{-\frac{1}{2}}$, and multiply it by $\frac{1}{a}$.

39. The n^{th} coefficient $=\dfrac{n(n+1)\dots(n+n-2)}{\underline{|n-1}}=\dfrac{2n(n+1)\dots(n+n-3)}{\underline{|n-2}}$.

40. The middle term in the expansion of $(1+x)^{2r}$

$$=\frac{2r(2r-1)\dots(2r-r+1)}{\underline{|r}}x^r=\frac{\underline{|2r}}{\underline{|r}\,\underline{|r}}x^r=\frac{1.3.5\dots(2r-1)\,2^r\,\underline{|r}}{\underline{|r}\,\underline{|r}}x^r$$

$$=\frac{1.3.5\dots(2r-1)}{\underline{|r}}2^rx^r.$$

And the coefficient of x^r in the expansion of $(1-4x)^{-\frac{1}{2}}$

$$=\frac{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\dots\left(\frac{1}{2}+r-1\right)}{\underline{|r}}(4x)^r=\frac{1.3.5\dots(2r-1)}{\underline{|r}}2^r.$$

42. Suppose we expand $(1+x)^n$ and also $\left(1+\frac{1}{x}\right)^n$, and multiply the two series together; then the term which does not contain x will be

$$1+\left\{\frac{n}{1}\right\}^2+\left\{\frac{n(n-1)}{1.2}\right\}^2+\left\{\frac{n(n-1)(n-2)}{\underline{|3}}\right\}^2+\dots\dots$$

This shews that the sum of the squares of the coefficients in the expansion of $(1+x)^n$ is equal to the coefficient of the term which does not contain x in the expansion of $(1+x)^n\times\left(1+\frac{1}{x}\right)^n$, that is in the expansion of $\frac{(1+x)^{2n}}{x^n}$. Now the coefficient of the term which does not contain x in the expansion of this expression is the coefficient of x^n in the expansion of $(1+x)^{2n}$; and is therefore $\dfrac{\underline{|2n}}{\underline{|n}\,\underline{|n}}$.

43. It is easily seen that p_r is the coefficient of x^r in the expansion of $(1-x)^{-\frac{1}{2}}$. Thus $(1-x)^{-\frac{1}{2}}=1+p_1x+p_2x^2+p_3x^3+\dots+p_rx^r+p_{r+1}x^{r+1}+\dots\dots$ Hence we see that if we multiply $(1-x)^{-\frac{1}{2}}$ by itself the coefficient of x^{2n+1} in the product will be $2\{p_{2n+1}+p_1p_{2n}+\dots+p_{n-1}p_{n+2}+p_np_{n+1}\}$.

But $\quad (1-x)^{-\frac{1}{2}}\times(1-x)^{-\frac{1}{2}}=(1-x)^{-1}=1+x+x^2+\dots+x^{2n+1}+\dots\dots$

Hence we infer that we must have $2\{p_{2n+1}+p_1p_{2n}+\dots+p_{n-1}p_{n+2}+p_np_{n+1}\}=1$.

44. The coefficient of x^m in the expansion of $(1-x)^{-n-1}$

$$=\frac{(n+1)(n+2)\dots(n+m)}{\underline{|m}}=\frac{\underline{|n+m}}{\underline{|n}\,\underline{|m}};$$

the coefficient of x^n in the expansion of $(1-x)^{-m-1}$ will be found to have the same value.

45. $1+2x+3x^2+4x^3+\dots=(1-x)^{-2}$; thus we require the coefficient of x^r in the expansion of $(1-x)^{-2n}$; this is $\dfrac{2n(2n+1)\dots(2n+r-1)}{\underline{|r}}$.

XXXVII.

1. $q+2r=4,\quad p+q+r=3.$

The solutions are $r=2,\ q=0,\ p=1;\ r=1,\ q=2,\ p=0.$

$$\frac{\underline{|3}}{\underline{|2}}+\frac{\underline{|3}}{\underline{|2}}=3+3=6.$$

In stating the solutions in future we shall not explicitly record the *zero* values which some of the letters may take.

2. $q+2r=5,\quad p+q+r=4.$

$$r=2,\ q=1,\ p=1;\quad r=1,\ q=3.$$

$$\frac{\underline{|4}}{\underline{|2}}(-1)^1+\frac{\underline{|4}}{\underline{|3}}(-1)^3=-12-4=-16.$$

3. $q+2r+3s=8,\ p+q+r+s=4.$

$$s=2,\ r=1,\ p=1;\quad s=2,\ q=2;\quad s=1,\ r=2,\ q=1;\quad r=4.$$

$$\frac{\underline{|4}}{\underline{|2}}3(-4)^2+\frac{\underline{|4}}{\underline{|2}\,\underline{|2}}(-2)^2(-4)^2+\frac{\underline{|4}}{\underline{|2}}(-2)3^2(-4)+\frac{\underline{|4}}{\underline{|4}}3^4$$

$$=2^6.3^2+2^7.3+2^5.3^3+3^4=1905.$$

4. $q+2r+3s+4t+5u=14,\ p+q+r+s+t+u=3.$

$$u=2,\ t=1.\quad \frac{\underline{|3}}{\underline{|2}}=3.$$

5. $q+2r=6,\ p+q+r=5.$

$$r=3,\ p=2;\ r=2,\ q=2,\ p=1;\ r=1,\ q=4.$$

$$\frac{\underline{|5}}{\underline{|3}\,\underline{|2}}2^3(-4)^3+\frac{\underline{|5}}{\underline{|2}\,\underline{|2}}2(-3)^2(-4)^2+\frac{\underline{|5}}{\underline{|4}}(-3)^4(-4)$$

$$=-2^9.5+2^6.3^3.5-2^2.3^4.5.$$

6. $q+2r=8,\ p+q+r=12.$

$r=4,\ p=8;\ r=3,\ q=2,\ p=7;\ r=2,\ q=4,\ p=6,\ r=1,\ q=6,\ p=5;\ q=8,\ p=4.$

$$\frac{\underline{|12}}{\underline{|8}\,\underline{|4}}2^4+\frac{\underline{|12}}{\underline{|7}\,\underline{|2}\,\underline{|3}}2^3+\frac{\underline{|12}}{\underline{|6}\,\underline{|4}\,\underline{|2}}2^2+\frac{\underline{|12}}{\underline{|5}\,\underline{|6}}2+\frac{\underline{|12}}{\underline{|4}\,\underline{|8}}.$$

7. $q+2r=4,\ p+q+r=5.$

$r=2,\ p=3;\ r=1,\ q=2,\ p=2;\ q=4,\ p=1.$

$$\frac{\underline{|5}}{\underline{|3}\,\underline{|2}}2^3.7^2-\frac{\underline{|5}}{\underline{|2}\,\underline{|2}}2^2.5^2.7+\frac{\underline{|5}}{\underline{|4}}2.5^4=2^4.5.7^2-2^3.3.5^3.7+2.5^5.$$

8. $2r+4t=8,\ p+r+t=-2.$

$t=2,\ p=-4;\ t=1,\ r=2,\ p=-5;\ r=4,\ p=-6.$

$$\frac{(-2)(-3)}{\underline{|2}}4^2+\frac{(-2)(-3)(-4)}{\underline{|2}}2^2.4+\frac{(-2)(-3)(-4)(-5)}{\underline{|4}}.2^4$$
$$=48-192+80=-64.$$

9. $q+2r=4,\ p+q+r=-5.$

$r=2,\ p=-7;\ r=1,\ q=2,\ p=-8;\ q=4,\ p=-9.$

$$\frac{(-5)(-6)}{\underline{|2}}+\frac{(-5)(-6)(-7)}{\underline{|2}}+\frac{(-5)(-6)(-7)(-8)}{\underline{|4}}=15-105+70=-20.$$

10. $q+2r=5,\ p+q+r=-\frac{1}{2}.$

$r=2,\ q=1,\ p=-3\frac{1}{2};\ r=1,\ q=3,\ p=-4\frac{1}{2};\ q=5,\ p=-5\frac{1}{2}.$

$$\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{\underline{|2}}2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{\underline{|3}}2^3(-1)$$
$$+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\left(-\frac{9}{2}\right)}{\underline{|5}}2^5=-\frac{15}{8}-\frac{35}{4}-\frac{63}{8}=-\frac{37}{2}.$$

11. $2r+4t=8,\ p+r+t=-2.$

$t=2,\ p=-4;\ t=1,\ r=2,\ p=-5;\ r=4,\ p=-6.$

$$\frac{(-2)(-3)}{\underline{|2}}\left(\frac{1}{4}\right)^2+\frac{(-2)(-3)(-4)}{\underline{|2}}.\frac{1}{4}\left(\frac{1}{2}\right)^2+\frac{(-2)(-3)(-4)(-5)}{\underline{|4}}\left(\frac{1}{2}\right)^4$$
$$=\frac{3}{16}-\frac{3}{4}+\frac{5}{16}=-\frac{1}{4}.$$

12. $q+2r+3s=4,\ p+q+r+s=-\dfrac{1}{2}.$

$s=1,\ q=1,\ p=-2\tfrac{1}{2};\ r=2,\ p=-2\tfrac{1}{2};\ r=1,\ q=2,\ p=-3\tfrac{1}{2};\ q=4,\ p=-4\tfrac{1}{2}.$

$$\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)2(-2)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{\underline{|2}}(-4)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{\underline{|2}}2^2(-4)$$

$$+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{\underline{|4}}2^4=-3+6+15+\frac{35}{8}.$$

13. $q+4t=6,\ p+q+t=\dfrac{1}{4}.$

$$t=1,\ q=2,\ p=-\frac{11}{4};\ q=6,\ p=-\frac{23}{4}.$$

$$\frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{\underline{|2}}(-2)^2+\frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)\left(-\frac{11}{4}\right)\left(-\frac{15}{4}\right)\left(-\frac{19}{4}\right)}{\underline{|6}}(-2)^6$$

$$=\frac{3.7}{2^5}-\frac{7.11.19}{2^{10}}.$$

14. Put $x=y^2$: then we require the coefficient of y^8 in the expansion of $(1+y+y^3+y^5-y^7)^5$. $q+3s+5u+7w=8,\ p+q+s+u+w=5.$

$w=1,\ q=1,\ p=3;\ u=1,\ s=1,\ p=3;\ u=1,\ q=3,\ p=1;\ s=2,\ q=2,\ p=1.$

$$\frac{\underline{|5}}{\underline{|3}}(-1)+\frac{\underline{|5}}{\underline{|3}}+\frac{\underline{|5}}{\underline{|3}}+\frac{\underline{|5}}{\underline{|2}\,\underline{|2}}=-20+20+20+30=50.$$

15. $q+2r=4,\ p+q+r=n.$

$r=2,\ p=n-2;\ r=1,\ q=2,\ p=n-3;\ q=4,\ p=n-4.$

$$\frac{n(n-1)}{\underline{|2}}+\frac{n(n-1)(n-2)}{\underline{|2}}+\frac{n(n-1)(n-2)(n-3)}{\underline{|4}}=\frac{n^4+6n^3-13n^2+6n}{24}.$$

16. It is easy to see that $1+3x+5x^2+7x^3+9x^4+\ldots$
$=(1+x)(1+2x+3x^2+4x^3+5x^4+\ldots)=(1+x)(1-x)^{-2}.$

Hence we require the coefficient of x^4 in $(1+x)^7(1-x)^{-14}$; and we have only to expand the two factors by the Binomial Theorem and multiply the results together. Or proceeding as usual we have

$$q+2r+3s+4t=4,\ \ p+q+r+s+t=7.$$

$t=1,\ p=6;\ s=1,\ q=1,\ p=5;\ r=2,\ p=5;\ r=1,\ q=2,\ p=4;\ q=4,\ p=3.$

$$\frac{\underline{|7}}{\underline{|6}}9+\frac{\underline{|7}}{\underline{|5}}3.7+\frac{\underline{|7}}{\underline{|5}\,\underline{|2}}5^2+\frac{\underline{|7}}{\underline{|4}\,\underline{|2}}3^2.5+\frac{\underline{|7}}{\underline{|3}\,\underline{|4}}.3^4.$$

The result will be found to be 9030.

17. $1+x+x^2+x^3+\ldots=(1-x)^{-1}$; thus we require the coefficient of x^m in the expansion of $(1-x)^{-2}$; and this is easily found to be $m+1$. If we proceed as usual we have $q+2r+3s+4t+5u+\ldots=m$, $p+q+r+\ldots=2$. It is easy to verify that if $m=3$ the coefficient will be 4, if $m=4$ the coefficient will be 5, if $m=5$ the coefficient will be 6, and so on.

18. $q+2r=8$, $p+q+r=n$;

$$r=4,\ p=n-4;\quad r=3,\ q=2,\ p=n-5;$$

$$r=2,\ q=4,\ p=n-6;\quad r=1,\ q=6,\ p=n-7;\quad q=8,\ p=n-8.$$

Hence the result.

19. $q+2r+3s+4t=4$, $p+q+r+s+t=-\dfrac{1}{2}$.

$$t=1,\ p=-1\tfrac{1}{2};\quad s=1,\ q=1,\ p=-2\tfrac{1}{2};$$

$$r=2,\ p=-2\tfrac{1}{2};\quad r=1,\ q=2,\ p=-3\tfrac{1}{2};\quad q=4,\ p=-4\tfrac{1}{2}.$$

$$\left(-\frac{1}{2}\right)5+\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)2.4+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{\lfloor 2}3^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{\lfloor 2}2^2.3$$

$$+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{\lfloor 4}.2^4=-\frac{5}{2}+6+\frac{27}{8}-\frac{45}{4}+\frac{35}{8}=0.$$

In fact we require the coefficient of x^4 in the expansion of $\{(1-x)^{-2}\}^{-\frac{1}{2}}$, that is in the expansion of $1-x$; and thus the result should be zero.

20. $q+2r+3s=12$, $p+q+r+s=5$.

$$s=4,\ p=1;\quad s=3,\ r=1,\ q=1;\quad s=2,\ r=3.$$

$$\frac{\lfloor 5}{\lfloor 4}a_3^4+\frac{\lfloor 5}{\lfloor 3}a_1a_2a_3^3+\frac{\lfloor 5}{\lfloor 2\,\lfloor 3}a_2^3a_3^2.$$

21. $q+2r=5$, $p+q+r=n$.

$$r=2,\ q=1,\ p=n-3;\quad r=1,\ q=3,\ p=n-4;\quad q=5,\ p=n-5.$$

$$\frac{n(n-1)(n-2)}{\lfloor 2}a_0^{n-3}a_1a_2^2+\frac{n(n-1)(n-2)(n-3)}{\lfloor 3}a_0^{n-4}a_1^3a_2$$

$$+\frac{n(n-1)\ldots(n-4)}{\lfloor 5}a_0^{n-5}a_1^5.$$

22. $2r+3s+5u=8$, $p+r+s+u=4$.

$$u=1,\ s=1,\ p=2;\quad s=2,\ r=1,\ p=1;\quad r=4.$$

$$\frac{\lfloor 4}{\lfloor 2}(-1)+\frac{\lfloor 4}{\lfloor 2}(-1)+\frac{\lfloor 4}{\lfloor 4}(-1)^4=-12-12+1=-23.$$

23. $q+2r=2,\ p+q+r=-\frac{1}{2}.$

$$r=1,\ p=-\frac{3}{2};\quad q=2,\ p=-\frac{5}{2}.$$

$$\left(-\frac{1}{2}\right)b+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{\underline{|2}}a^2=-\frac{b}{2}+\frac{3a^2}{8}.$$

24. $q+2r+3s=3,\ p+q+r+s=m.$

$s=1,\ p=m-1;\quad r=1,\ q=1,\ p=m-2;\quad q=3,\ p=m-3.$

$$ma_3+m(m-1)a_1a_2+\frac{m(m-1)(m-2)}{\underline{|3}}a_1^3.$$

25. $\dfrac{\underline{|5}}{\underline{|3}}=20.$

26. $\dfrac{\underline{|7}}{\underline{|2}\,\underline{|3}\,\underline{|2}}(-1)^2(-1)^3=-210.$

27. $\dfrac{\underline{|9}}{\underline{|2}\,\underline{|4}\,\underline{|3}}=1260.$

28. $\dfrac{\underline{|10}}{\underline{|2}\,\underline{|3}\,\underline{|4}}(-1)^2(-1)^4=12600.$

31. Every coefficient is of the form $\dfrac{\underline{|10}}{\underline{|p}\,\underline{|q}\,\underline{|r}}$ where $p+q+r=10$. It is obvious that the *least* value of $\underline{|p}\,\underline{|q}\,\underline{|r}$ is $1.1.1.2.2.2.3.3.3.4$ for it is thus made up of the 10 least factors which are admissible: for as there are only *three* quantities p, q, r, we cannot have more than 3 *ones*, we cannot have more than 3 *twos*, we cannot have more than 3 *threes*. Thus the greatest coefficient $=\dfrac{\underline{|10}}{\{\underline{|3}\}^3 4}$.

32. Every coefficient is of the form $\dfrac{\underline{|14}}{\underline{|p}\,\underline{|q}\,\underline{|r}\,\underline{|s}}$ where $p+q+r+s=14$. It is obvious that the *least* value of $\underline{|p}\,\underline{|q}\,\underline{|r}\,\underline{|s}$ is $1.1.1.1.2.2.2.2.3.3.3.3.4.4$ for it is thus made up of the 14 least factors which are admissible: for as there are only *four* quantities p, q, r, s, we cannot have more than 4 *ones*, we cannot have more than 4 *twos*, we cannot have more than 4 *threes*. Thus the greatest coefficient $=\dfrac{\underline{|14}}{\{\underline{|3}\}^4 4^2}$.

33. This follows easily from considering such special cases as those in Examples 31 and 32.

34. Write down $a_0+a_1x+a_2x^2+\ldots$ and underneath it write the same expression; then multiply them together and arrange the result: by examining the coefficients of $x^3, x^5, x^7, \ldots$ we easily see that the proposed result is true. Example 17 is a particular case of this Example. See also Example 43 of Chapter XXXVI.

35. The first term of the expansion is 1.

To find the coefficient of x:

$$q+2r=1,\ p+q+r=-\frac{1}{2}.\quad q=1,\ p=-1\tfrac{1}{2}.\quad \left(-\frac{1}{2}\right)(-2b)=b.$$

To find the coefficient of x^2:

$$q+2r=2,\ p+q+r=-\frac{1}{2}.$$

$$r=1,\ p=-1\tfrac{1}{2};\ q=2,\ p=-2\tfrac{1}{2}.$$

$$\left(-\frac{1}{2}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{\lfloor 2}(-2b)^2=-\frac{1}{2}+\frac{3b^2}{2}.$$

To find the coefficient of x^3:

$$q+2r=3,\ p+q+r=-\frac{1}{2}.$$

$$r=1,\ q=1,\ p=-2\tfrac{1}{2};\ q=3,\ p=-3\tfrac{1}{2}.$$

$$\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-2b)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{\lfloor 3}(-2b)^3=-\frac{3b}{2}+\frac{5b^3}{2}.$$

To find the coefficient of x^4:

$$q+2r=4,\ p+q+r=-\frac{1}{2}.$$

$$r=2,\ p=-2\tfrac{1}{2};\ r=1,\ q=2,\ p=-3\tfrac{1}{2};\ q=4,\ p=-4\tfrac{1}{2}.$$

$$\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{\lfloor 2}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{\lfloor 2}(-2b)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{\lfloor 4}(-2b)^4$$

$$=\frac{3}{8}-\frac{15b^2}{4}+\frac{35b^4}{8}.$$

36. The first term of the expansion is a^{-1}.

To find the coefficient of x:

$$q+2r=1,\quad p+q+r=-1.$$

$$q=1,\ p=-2.\quad (-1)\,a^{-2}b=-a^{-2}b.$$

To find the coefficient of x^2:

$$q+2r=2,\ p+q+r=-1.$$
$$r=1,\ p=-2;\ \ q=2,\ p=-3.$$
$$(-1)a^{-2}c+\frac{(-1)(-2)}{\lfloor 2}a^{-3}b^2=-a^{-2}c+a^{-3}b^2.$$

To find the coefficient of x^3:

$$q+2r=3,\ p+q+r=-1.$$
$$r=1,\ q=1,\ p=-3;\ \ q=3,\ p=-4.$$
$$(-1)(-2)a^{-3}bc+\frac{(-1)(-2)(-3)}{\lfloor 3}a^{-4}b^3=2a^{-3}bc-a^{-4}b^3.$$

To find the coefficient of x^4:

$$q+2r=4,\ p+q+r=-1.$$
$$r=2,\ p=-3;\ r=1,\ q=2,\ p=-4;\ \ q=4,\ p=-5.$$
$$\frac{(-1)(-2)}{\lfloor 2}a^{-3}c^2+\frac{(-1)(-2)(-3)}{\lfloor 2}a^{-4}b^2c+\frac{(-1)(-2)(-3)(-4)}{\lfloor 4}a^{-5}b^4$$
$$=a^{-3}c^2-3a^{-4}b^2c+a^{-5}b^4.$$

Or we might proceed thus: $\quad (a+bx+cx^2)^{-1}=a^{-1}\left(1+\frac{bx}{a}+\frac{cx^2}{a}\right)^{-1}$

$$=a^{-1}\left\{1-\left(\frac{bx}{a}+\frac{cx^2}{a}\right)+\left(\frac{bx}{a}+\frac{cx^2}{a}\right)^2-\left(\frac{bx}{a}+\frac{cx^2}{a}\right)^3+\left(\frac{bx}{a}+\frac{cx^2}{a}\right)^4-\dots\right\};$$

then expand the various powers of $\frac{bx}{a}+\frac{cx^2}{a}$ and collect the terms.

37. The first term of the expansion is 1.

To find the coefficient of x:

$$q+2r+3s=1,\ p+q+r+s=n.$$
$$q=1,\ p=n-1.\quad n(-1)=-n.$$

To find the coefficient of x^2:

$$q+2r+3s=2,\ p+q+r+s=n.$$
$$r=1,\ p=n-1;\ \ q=2,\ p=n-2.$$
$$n(-1)+\frac{n(n-1)}{\lfloor 2}(-1)^2=-n+\frac{n(n-1)}{2}=\frac{n(n-3)}{2}.$$

To find the coefficient of x^3:

$$q+2r+3s=3,\ p+q+r+s=n.$$
$$s=1,\ p=n-1;\ r=1,\ q=1,\ p=n-2;\ \ q=3,\ p=n-3.$$
$$n(-1)+n(n-1)(-1)(-1)+\frac{n(n-1)(n-2)}{\lfloor 3}(-1)^3=-n+n(n-1)-\frac{n(n-1)(n-2)}{\lfloor 3}$$
$$=-\frac{n}{6}(n-2)(n-7).$$

38. These results are easily established by induction. For simplicity it is well to take a particular value of r; it will be found that the reasoning would apply to any other value of r. We may shew then by actual multiplication that the laws which are stated hold when $n=2$ or when $n=3$. Assume that they hold for any assigned value of n. Let $r=3$, and suppose that

$$(1+x+x^2+x^3)^n=a_0+a_1x+a_2x^2+\ldots\ldots+a_{3n-1}x^{3n-1}+a_{3n}x^{3n};$$

multiply both sides by $1+x+x^2+x^3$; then

$$(1+x+x^2+x^3)^{n+1}=a_0+(a_1+a_0)x+(a_2+a_1+a_0)x^2+(a_3+a_2+a_1+a_0)x^3 \\ +(a_4+a_3+a_2+a_1)x^4+\ldots$$

Then we see that the coefficient of x^4 is greater than that of x^3 if a_4 is greater than a_0; and this is the case by supposition: and so on.

XXXVIII.

1. Let x denote the required logarithm; then $144=(2\sqrt{3})^x$; that is $2^4 3^2=(2\sqrt{3})^x$; that is $(2\sqrt{3})^4=(2\sqrt{3})^x$: therefore $x=4$.

2. 7 lies between 2^2 and 2^3; thus the required characteristic is 2.

3. 5 lies between 3^1 and 3^2; thus the required characteristic is 1.

4. Let x denote the required logarithm; then $3125=5^x$; that is $5^5=5^x$: therefore $x=5$.

5. 1230 lies between 10^3 and 10^4: thus the required characteristic is 3. ·0123 lies between 10^{-1} and 10^{-2}: thus the required characteristic is -2.

6. $\text{Log}\ {\cdot}05=\log\frac{5}{100}=\log 5-\log 100=\log 5-2$; and $\log 5=\log\frac{10}{2}=1-\log 2$.

$\text{Log}\ 5{\cdot}4=\log\frac{54}{10}=\log 54-1$; and $54=2\times 3^3$, so that $\log 54=\log 2+3\log 3$.

7. $\text{Log}\ {\cdot}006=\log\frac{6}{1000}=\log 6-3$; and $\log 6=\log 2+\log 3$.

8. $\text{Log}\ 36=\log(2^2\times 3^2)=2\log 2+2\log 3$; $\log 27=\log 3^3=3\log 3$;

$$\log 16=\log 2^4=4\log 2.$$

9. Divide 648 by 2; the quotient by 2; and so on as long as possible; then divide by 3; the quotient by 3; and so on. Thus we find $648=2^3\times 3^4$. Similarly $864=2^5\times 3^3$. Therefore we have

$$3\log 2+4\log 3=2{\cdot}81157501,\quad 5\log 2+3\log 3=2{\cdot}93651374.$$

From these two equations find $\log 2$ and $\log 3$. Then $\log 5=\log\frac{10}{2}=1-\log 2$.

10. $$\text{Log}\sqrt{(1{\cdot}25)}=\log\left(\frac{125}{100}\right)^{\frac{1}{2}}=\frac{1}{2}(\log 125-\log 100)=\frac{3}{2}\log 5-1 \\ =\frac{3}{2}(1-\log 2)-1.$$

11. $\text{Log}\,\cdot 0025 = \log \dfrac{25}{10000} = \log 25 - 4 = 2\log 5 - 4 = 2(1 - \log 2) - 4.$

12. $\text{Log}\sqrt[3]{\cdot 0125} = \log\left(\dfrac{125}{10000}\right)^{\frac{1}{3}} = \dfrac{1}{3}(\log 125 - \log 10000)$

$$= \log 5 - \frac{4}{3} = 1 - \log 2 - \frac{4}{3}.$$

13. $\text{Log}\,1080 = \log(10 \times 2^2 \times 3^3) = 1 + 2\log 2 + 3\log 3.$

$$\text{Log}\,(\cdot 0045)^{\frac{1}{9}} = \frac{1}{9}\log\frac{45}{10000} = \frac{1}{9}\left\{\log(5 \times 3^2) - 4\right\}$$

$$= \frac{1}{9}\log 5 + \frac{2}{9}\log 3 - \frac{4}{9} = \frac{1}{9}(1 - \log 2) + \frac{2}{9}\log 3 - \frac{4}{9}.$$

14. First find the logarithm to the base 10. $\text{Log}\left(\dfrac{4}{343}\right)^{\frac{1}{2}} = \dfrac{1}{2}\log\dfrac{4}{343}$ $= \dfrac{1}{2}\log\dfrac{2^2}{7^3} = \dfrac{1}{2}(\log 2^2 - \log 7^3) = \log 2 - \dfrac{3}{2}\log 7 = -\cdot 966617.$ And the logarithm to the base 1000 will be found by multiplying this by $\dfrac{1}{\log_{10}1000}$, that is by $\dfrac{1}{3}$: see Art. 538. Hence the required logarithm is $-\dfrac{\cdot 966617}{3}$.

15. $\text{Log}\,2^{64} = 64\log 2 = 19\cdot 26592$: thus 2^{64} lies between 10^{19} and 10^{20}; and so has 20 digits.

16. $\text{Log}\,(\cdot 0625)^{\frac{1}{5}} = \dfrac{1}{5}\log\dfrac{625}{10000} = \dfrac{1}{5}(\log 5^4 - \log 10000) = \dfrac{4}{5}\log 5 - \dfrac{4}{5}$ $= \dfrac{4}{5}(1 - \log 2) - \dfrac{4}{5} = \cdot 559176 - \dfrac{4}{5} = \cdot 559176 + \dfrac{1}{5} - 1 = \cdot 759176 - 1 = \log\dfrac{5\cdot 743491}{10}$ $= \log\cdot 5743491.$ Therefore $(\cdot 0625)^{\frac{1}{5}} = \cdot 5743491.$

17. The numbers from 10^p to $10^{p+1} - 1$, both inclusive, have the characteristic p; thus $P = 10^p(10 - 1)$. The reciprocals of numbers between 10^q and $10^{q-1} + 1$, both inclusive, have the characteristic $-q$; thus $Q = 10^{q-1}(10 - 1)$. Therefore $\log P - \log Q = p - (q - 1)$.

18. Take the logarithms of the given equations; thus $\log y = \dfrac{1}{1 - \log x}$; $\log z = \dfrac{1}{1 - \log y}$; substitute in the latter the value of $\log y$ from the former; thus $\log z = \dfrac{1 - \log x}{-\log x}$; therefore $\log x = \dfrac{1}{1 - \log z}$; therefore $x = 10^{\frac{1}{1 - \log z}}$.

19. Suppose $n = a^x = b^y = c^z$; so that $x = \log_a n$, $y = \log_b n$, $z = \log_c n$. Then $a = n^{\frac{1}{x}}$, $b = n^{\frac{1}{y}}$, $c = n^{\frac{1}{z}}$. But $b^2 = ac$; therefore $n^{\frac{2}{y}} = n^{\frac{1}{x} + \frac{1}{z}}$; therefore $\dfrac{2}{y} = \dfrac{1}{x} + \dfrac{1}{z}$.

20. Let P denote the number of the population at the beginning of a certain year; then the number at the end of the year will be $P+\frac{P}{45}-\frac{P}{60}$, that is $\frac{181}{180}P$. In a similar manner the number at the end of two years will be $\frac{181}{180}\times\frac{181}{180}P$, that is $\left(\frac{181}{180}\right)^2 P$; and the number at the end of x years will be $\left(\frac{181}{180}\right)^x P$. Suppose that in x years the population is doubled; then $\left(\frac{181}{180}\right)^x P=2P$; therefore $\left(\frac{181}{180}\right)^x=2$; therefore $x\log\frac{181}{180}=\log 2$; therefore $x=\frac{\log 2}{\log 181-\log 180}=\frac{\cdot 301030}{\cdot 002407}$.

XXXIX.

1. In equation (1) of Art. 545 put x^2-1 for m, and x^2 for n; thus we obtain the proposed result. Multiply by $\frac{1}{\log_e 10}$ and we have

$$\log_{10}(x+1)=2\log_{10}x-\log_{10}(x-1)-\frac{2}{\log_e 10}\left\{\frac{1}{2x^2-1}+\frac{1}{3}\left(\frac{1}{2x^2-1}\right)^3+\dots\right\}.$$

Put 10 for x: thus $\log_{10}11=2-2\log_{10}3-\frac{2}{\log_e 10}\left\{\frac{1}{199}+\frac{1}{3\,(199)^3}+\dots\right\}$.

4. We have $\lambda+\mu=-\frac{b}{a}$, $\lambda\mu=\frac{c}{a}$ by Art. 336: thus

$$a-bx+cx^2=a\{1+(\lambda+\mu)x+\lambda\mu x^2\}=a(1+\lambda x)(1+\mu x);$$

therefore $\qquad \log(a-bx+cx^2)=\log a+\log(1+\lambda x)+\log(1+\mu x)$:

then expand $\log(1+\lambda x)$ and $\log(1+\mu x)$ by Art. 544.

7. $\text{Log}_e\left\{(1+x)^{\frac{1+x}{2}}(1-x)^{\frac{1-x}{2}}\right\}=\frac{1+x}{2}\log_e(1+x)+\frac{1-x}{2}\log_e(1-x)$;

then expand $\log_e(1+x)$ and $\log_e(1-x)$.

8. $$\frac{501}{499}=\frac{1+\frac{1}{500}}{1-\frac{1}{500}}=\frac{1+\cdot 002}{1-\cdot 002}; \text{ therefore}$$

$$\log_e\frac{501}{499}=2\left\{\cdot 002+\frac{(\cdot 002)^3}{3}+\frac{(\cdot 002)^5}{5}+\dots\right\}.$$

Now $\frac{(\cdot 002)^3}{3}$ has no significant figure in the first *eight* places of decimals, so if we stop at $\cdot 002$ the result is true to at least seven places; and $\frac{(\cdot 002)^5}{5}$

has no significant figure in the first *fourteen* places of decimals, so if we stop at $\frac{(\cdot 002)^3}{3}$ the result is true to at least thirteen places: and so on.

9. We have in fact to find when n is very large an expression in powers of x for $\left(1+\frac{x}{n}\right)^n \div e^x$, that is for $\left(1+\frac{x}{n}\right)^n e^{-x}$.

Now $\log\left\{\left(1+\frac{x}{n}\right)^n e^{-x}\right\} = n\log\left(1+\frac{x}{n}\right) - x = -\frac{x^2}{2n}+\frac{x^3}{3n^2}-\dots;$

therefore $\quad \left(1+\frac{x}{n}\right)^n e^{-x} = e^{-\frac{x^2}{2n}+\frac{x^3}{3n^2}-\dots} = 1-\frac{x^2}{2n}+\frac{x^3}{3n^2}+\frac{x^4}{4n^2}+\dots;$

here the terms which are not expressed involve higher powers of $\frac{1}{n}$; and are very small when n is very large.

10. We require the coefficient of x^n in the expansion of $(a+bx+cx^2)e^{-x}$. Now $e^{-x} = 1-x+\frac{x^2}{\lfloor 2}-\dots+\frac{(-1)^n x^n}{\lfloor n}+\dots$; thus the coefficient

$$=\frac{a(-1)^n}{\lfloor n}+\frac{b(-1)^{n-1}}{\lfloor n-1}+\frac{c(-1)^{n-2}}{\lfloor n-2}=\frac{(-1)^n}{\lfloor n-2}\left\{\frac{a}{n(n-1)}-\frac{b}{n-1}+c\right\}.$$

11. In the series for $\log_e(1+x)$ put 1 for x: thus

$$\log_e 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots$$

By combining terms we get $\log_e 2 = \frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\dots$; and by combining them in another way we get $\log_e 2 = 1-\frac{1}{2.3}-\frac{1}{4.5}-\frac{1}{6.7}-\dots$; hence by addition we get $2\log_e 2 = 1+\frac{2}{1.2.3}+\frac{2}{3.4.5}+\frac{2}{5.6.7}+\dots$

12. We must find the term on the right-hand side of (1) in Art. 549, which involves x^{n+2}.

$$\text{Now } \left(x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\dots\right)^n = x^n\left(1+\frac{x}{\lfloor 2}+\frac{x^2}{\lfloor 3}+\frac{x^3}{\lfloor 4}+\dots\right)^n$$
$$= x^n\left\{1+ny+\frac{n(n-1)}{\lfloor 2}y^2+\dots\right\},$$

where y stands for $\quad \frac{x}{\lfloor 2}+\frac{x^2}{\lfloor 3}+\frac{x^3}{\lfloor 4}+\dots$

Thus the term which involves x^{n+2} is $\frac{n}{\lfloor 3}+\frac{n(n-1)}{\lfloor 2}\left(\frac{1}{\lfloor 2}\right)^2 = \frac{n}{6}+\frac{n(n-1)}{8}$.

Hence the required result is obtained like the others in Art. 549.

XL.

2. The ratio of the $(r+1)^{\text{th}}$ term to the $r^{\text{th}} = \frac{2r+3}{2r+1} \cdot \frac{r^2+1}{(r+1)^2+1} x$; the factor $\frac{2r+3}{2r+1} \cdot \frac{r^2+1}{(r+1)^2+1}$ approaches continually to unity as r increases: hence by Arts. 559, 560 the series is divergent if x is numerically greater than unity, and convergent if x is numerically less than unity.

8. If p is greater than unity the series is convergent; for it is obviously less than that in Art. 562. If p is equal to unity the series is divergent; for $\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots$ is greater than $\frac{1}{2}\left\{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\right\}$, as we see by comparing term with term; that is the series is greater than the half of one which is known to be divergent. If p is less than unity or negative, the series is divergent, each term being greater than the corresponding term when $p=1$.

10. The ratio of the $(r+1)^{\text{th}}$ term to the $r^{\text{th}} = \left(\frac{a+rb}{a+rb+b}\right)^p x$; the factor $\left(\frac{a+rb}{a+rb+b}\right)^p$ approaches continually to unity as r increases; hence by Arts. 559, 560 the series is divergent if x is numerically greater than unity, and convergent if x is numerically less than unity. If $x=1$ the series

$$=\frac{1}{b^p}\left\{\frac{1}{(c+1)^p}+\frac{1}{(c+2)^p}+\frac{1}{(c+3)^p}+\ldots\right\}, \text{ where } c \text{ stands for } \frac{a}{b}.$$

The series last given is of the same character with respect to convergence and divergence as that in Art. 562. For instance, if $c=4$ the series is the same as that in Art. 562 omitting the first four terms; if c lies between 4 and 5 the series is less than that of Art. 562 omitting the first four terms, and greater than that of Art. 562 omitting the first five terms.

11. $u_2+u_3>2u_3$, $u_4+u_5+u_6+u_7>2^2u_7$, $u_8+u_9\ldots+u_{15}>2^3u_{15}$, and so on. Hence we find that the first series is greater than half the second, and so if the second is divergent the first is also divergent.

Again, $u_1+u_2<2u_1$, $u_3+u_4+u_5+u_6<2^2u_3$, $u_7+u_8\ldots+u_{14}<2^3u_7$, and so on. Hence we see that the first series is less than the second, and so if the second is convergent the first is also convergent.

12. The series $=1+\frac{2-1}{2^n}+\frac{3-1}{3^n}+\frac{4-1}{4^n}+\ldots$

$$=1+\frac{1}{2^{n-1}}+\frac{1}{3^{n-1}}+\ldots-\left\{\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\ldots\right\}.$$

If n is greater than 2 each of these two series is convergent, and so therefore is the proposed series. If $n=2$ the first series is divergent, and the second convergent; and so the proposed series is divergent. If n is less than 2 each term of the proposed series is greater than it would be if $n=2$; and so the proposed series is divergent.

XLI.

1. Let M be the sum due, D the discount. Then $D=\frac{Mnr}{1+nr}$. Again let I be the interest on the sum due: then $I=Mnr$. Half the harmonic mean between M and $I=\frac{M.I}{M+I}=\frac{M^2nr}{M+Mnr}=\frac{Mnr}{1+nr}$.

2. With the notation of Art. 577, $Mnr=180$. $\frac{Mnr}{1+nr}=150$; therefore $\frac{180}{1+nr}=150$; therefore $\frac{18}{15}=1+nr$; therefore $nr=\frac{1}{5}$; therefore $M=180\times 5=900$.

3. Let r denote the rate of interest; then $Ar=\frac{Br}{1+r}$; therefore $r=\frac{B-A}{A}$.

4. Let r denote the rate of interest; then $1+40r=2$; therefore $r=\frac{1}{40}$.

5. Let M denote the price of any article for a credit of six months; P the ready money price of the same article: then, by Art. 577, $\frac{P}{M}=\frac{1}{1+nr}$. But $r=\frac{5}{100}=\frac{1}{20}$, and $n=\frac{1}{2}$; therefore $\frac{P}{M}=\frac{1}{1+\frac{1}{40}}=\frac{40}{41}$.

6. Let n be the number of years; then by Art. 577

$$1050=100R^n=100\left(1+\frac{5}{100}\right)^n; \text{ therefore } \left(\frac{105}{100}\right)^n=\frac{105}{10};$$

$$\text{therefore } \left(\frac{210}{200}\right)^n=\frac{210}{20}; \text{ therefore } n\log\frac{210}{200}=\log\frac{210}{20};$$

$$\text{therefore } n=\frac{\log 210-\log 20}{\log 210-\log 200}=\frac{\log 210-\log 2-1}{\log 210-\log 2-2}.$$

Now $\log 210=\log 14+\log 15$; and $\log 2=\log(16)^{\frac{1}{4}}=\frac{1}{4}\log 16$. Thus $n=\frac{1\cdot 02119}{\cdot 02119}$.

7. Let n be the number of years; then by Art. 577

$$P\left(1+\frac{3\frac{1}{2}}{100}\right)^n=3P; \text{ therefore } (1\cdot 035)^n=3;$$

$$\text{therefore } n\log 1\cdot 035=\log 3; \text{ therefore } n=\frac{\log 3}{\log 1\cdot 035}=\frac{\cdot 47712}{\cdot 01494}.$$

8. With the notation of Art. 577, $pP=PR^m$, $qP=PR^n$;

$$\text{therefore } p=R^m,\ q=R^n; \text{ therefore } p^{\frac{1}{m}}=q^{\frac{1}{n}} \text{ for each }=R;$$

$$\text{therefore } p^{\frac{n}{m}}=q; \text{ therefore } \log_p q=\frac{n}{m}.$$

XLII.

1. With the notation of Art. 580 we have $P_1=400$, $P_2=2100$, $t_1=2$, $t_2=8$, $r=\frac{5}{100}$; hence $\dfrac{2100(8-x)}{1+\frac{8-x}{20}}=400(x-2)$; therefore

$$2100(8-x)=400(x-2)+20(8-x)(x-2), \text{ \&c.}$$

2. Here with the notation of Art. 580 we have $P_1=20$, $P_2=16\frac{1}{4}$, $t_1=0$, $t_2=270$, *the time being expressed in days;* also $r=\dfrac{2\frac{1}{2}}{100\times 240}=\dfrac{1}{9600}$: hence

$$\frac{16\frac{1}{4}(270-x)}{1+\frac{270-x}{9600}}=20x;\text{ therefore } \frac{65}{4}(270-x)=20x+\frac{x(270-x)}{480}, \text{ \&c.}$$

3. By Arts. 574 and 577 we see that when interest is due every moment the discount on P_2 for t_2-x years is $P_2\{1-e^{-r(t_2-x)}\}$; and the interest on P_1 for $x-t_1$ years is $P_1\{e^{r(x-t_1)}-1\}$: then we must equate these two expressions.

4. Let x, y, z denote the numbers of pounds in the three parts; then $xR^a=yR^b=zR^c$. We may write these equations thus: $\dfrac{x}{R^{-a}}=\dfrac{y}{R^{-b}}=\dfrac{z}{R^{-c}}$; then by Art. 384, each of the fractions $=\dfrac{x+y+z}{R^{-a}+R^{-b}+R^{-c}}$. But $x+y+z=$ the given sum, which is known: thus x, y, and z are found.

5. As in Art. 526, suppose I to denote the *integral* part of $\{a+\sqrt{(a^2-1)}\}^n$, and $I+F$ its complete value, so that F is a proper fraction; then

$$I+F=a^n+na^{n-1}\sqrt{(a^2-1)}+\frac{n(n-1)}{\lfloor 2}a^{n-2}(a^2-1)+\dots.$$

And $a-\sqrt{(a^2-1)}$ is a proper fraction; and therefore so also is $\{a-\sqrt{(a^2-1)}\}^n$, which we will denote by F': thus

$$F'=a^n-na^{n-1}\sqrt{(a^2-1)}+\frac{n(n-1)}{\lfloor 2}a^{n-2}(a^2-1)-\dots.$$

Hence by addition we have $I+F+F'=$ an *even* integer; therefore $F+F'=1$, and $I=$ an *odd* integer.

6. As in the preceding Example,

$$I+F=\{\sqrt{(a^2+1)}+a\}^n=(a^2+1)^{\frac{n}{2}}+n(a^2+1)^{\frac{n-1}{2}}a+\frac{n(n-1)}{\lfloor 2}(a^2+1)^{\frac{n-2}{2}}a^2+\dots,$$

$$F'=\{\sqrt{(a^2+1)}-a\}^n=(a^2+1)^{\frac{n}{2}}-n(a^2+1)^{\frac{n-1}{2}}a+\frac{n(n-1)}{\lfloor 2}(a^2+1)^{\frac{n-2}{2}}a^2-\dots.$$

If n be *even* we add these two results, as in Ex. 5, and arrive at a similar conclusion. If n be *odd* we subtract the second result from the first, and then we get $I+F-F'=$ an *even* integer; hence we must have $F-F'=0$, and $I=$ an *even* integer.

7. $\left(\frac{a}{a+x}\right)^2 = \left(\frac{1}{1+\frac{x}{a}}\right)^2 = \left(1+\frac{x}{a}\right)^{-2} = 1 - 2\frac{x}{a} + 3\left(\frac{x}{a}\right)^2 - \ldots.$

Thus we must find the sum of $1 - 2\frac{x}{a} + 3\left(\frac{x}{a}\right)^2 \ldots + (n+1)(-1)^n\left(\frac{x}{a}\right)^n$, and subtract it from $\left(\frac{a}{a+x}\right)^2$, and we shall have the required remainder.

In Art. 473 put 1 for a, 1 for b, and $-\frac{x}{a}$ for r: thus the sum is

$$\frac{1-n\left(-\frac{x}{a}\right)^n}{1+\frac{x}{a}} - \frac{\frac{x}{a}\left\{1-\left(-\frac{x}{a}\right)^{n-1}\right\}}{\left(1+\frac{x}{a}\right)^2}, \text{ that is } \frac{1-(n+1)\left(-\frac{x}{a}\right)^n - n\frac{x}{a}\left(-\frac{x}{a}\right)^n}{\left(1+\frac{x}{a}\right)^2}.$$

Subtract this from $\frac{1}{\left(1+\frac{x}{a}\right)^2}$; the remainder $=\left(-\frac{x}{a}\right)^n \frac{n+1+\frac{nx}{a}}{\left(1+\frac{x}{a}\right)^2}$.

10. The whole coefficient of α will be found to be

$$1 - n + \frac{n(n-1)}{\underline{|2}} - \frac{n(n-1)(n-2)}{\underline{|3}} + \ldots, \text{ that is } (1-1)^n, \text{ that is } 0.$$

The whole coefficient of β will be found to be

$$-n\left\{1-(n-1)+\frac{(n-1)(n-2)}{\underline{|2}} - \ldots\right\}, \text{ that is } -n(1-1)^{n-1}, \text{ that is } 0.$$

11. $$s = a\frac{(1+x)^n - 1}{1+x-1}; \text{ therefore } (1+x)^n = 1 + \frac{sx}{a}.$$

Take the logarithms; $\quad n\log(1+x) = \log\left(1+\frac{sx}{a}\right)$;

therefore $$n = \frac{\frac{sx}{a} - \frac{s^2x^2}{2a^2} + \frac{s^3x^3}{3a^3} - \ldots}{x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots} = \frac{\frac{s}{a}\left\{1 - \frac{sx}{2a} + \frac{s^2x^2}{3a^2} - \ldots\right\}}{1 - \frac{x}{2} + \frac{x^2}{3} - \ldots}.$$

Divide $1 - \frac{sx}{2a} + \frac{s^2x^2}{3a^2} - \ldots$ by $1 - \frac{x}{2} + \frac{x^2}{3} - \ldots$; thus we obtain $1 - \frac{(s-a)x}{2a} + \ldots$

12. By Art. 574 the value at any time t will be ae^{rt} where r is some constant; but we know that b is the value at the time t_1; therefore $b = ae^{rt_1}$. Thus $e^r = \left(\frac{b}{a}\right)^{\frac{1}{t_1}}$, and $e^{rt} = \left(\frac{b}{a}\right)^{\frac{t}{t_1}}$.

XLIII.

1. In Art. 589 put $P=600\frac{1}{4}$, $n=35$, $r=\frac{4}{100}$:

thus $600\frac{1}{4}=\dfrac{35\left(1+\frac{68}{100}\right)A}{1+\frac{140}{100}}$; hence we find $A=\frac{49}{2}=24\frac{1}{2}$.

2. By Art. 589 we must have $\dfrac{n+\frac{1}{2}n(n-1)r}{1+nr}A=\dfrac{nA}{2}$;

therefore $$1+\frac{1}{2}(n-1)r=\frac{1}{2}(1+nr).\quad\text{Hence } r=1.$$

3. In Art. 596 put $A=100$, $P=2500$; then $r=\dfrac{A}{P}=\dfrac{1}{25}=\dfrac{4}{100}$.

4. By Art. 597 the present value $=\dfrac{AR^{-2}}{R-1}$; where

$A=168{\cdot}1$, $R=1+\dfrac{2\frac{1}{2}}{100}=1+\dfrac{5}{200}=\dfrac{41}{40}$. Thus the present value

$$=\frac{168{\cdot}1\times\left(\frac{41}{40}\right)^{-2}}{\frac{1}{40}}=168{\cdot}1\times\left(\frac{40}{41}\right)^2\times 40=\frac{168{\cdot}1\times(40)^3}{1681}=6400.$$

5. By Art. 595, if n denote the number of years the annuity continues which is worth 20 years' purchase $20A=\dfrac{A(1-R^{-n})}{R-1}$, $26A=\dfrac{A(1-R^{-2n})}{R-1}$;

therefore, by division, $\dfrac{26}{20}=\dfrac{1-R^{-2n}}{1-R^{-n}}=1+R^{-n}$; therefore $R^{-n}=\dfrac{6}{20}$.

Substitute in the first equation; $20=\dfrac{1-\frac{6}{20}}{R-1}$; therefore $R-1=\dfrac{14}{400}=\dfrac{3\frac{1}{2}}{100}$.

6. By Art. 597 the sum $=\dfrac{320\left(1+\frac{3\frac{1}{5}}{100}\right)^{-10}}{\frac{3\frac{1}{5}}{100}}=10000\,(1{\cdot}032)^{-10}$. Denote this by S. Then $\log S=\log 10000-10\log 1{\cdot}032=4-{\cdot}136797=3{\cdot}863203$. Therefore $S=7297{\cdot}98$.

7. Since an annuity to continue for ever is worth 25 years' purchase by Arts. 595, 596, we have $25=\frac{1}{r}$; therefore $r=\frac{1}{25}$. Next we have by Art. 595,

$$625=A\frac{1-\left(\frac{26}{25}\right)^{-3}}{\frac{1}{25}}=A\left\{1-\left(\frac{25}{26}\right)^3\right\}25;\ \text{therefore } A=\frac{25\,(26)^3}{(26)^3-(25)^3}=\frac{25\times 17576}{1951}.$$

8. Let m stand for $\frac{130}{100}$; then we have to find the present value of A due at the end of 1 year, of mA due at the end of 2 years, of m^2A due at the end of 3 years; thus, as in Art. 595, we get

$$P=\frac{A}{R}+\frac{mA}{R^2}+\frac{m^2A}{R^3}\ldots+\frac{m^{n-1}A}{R^n}=\frac{A}{R}\,\frac{\frac{m^n}{R^n}-1}{\frac{m}{R}-1}.$$

Here $\qquad P=1000,\ A=40,\ R=\frac{104}{100},\ \frac{m}{R}=\frac{130}{104}=\frac{5}{4}.$

Hence we obtain $\left(\frac{5}{4}\right)^n=\frac{15}{2}$; and taking logarithms $n=\frac{\log 15-\log 2}{\log 5-\log 4}$; and $\log 4=2\log 2$, $\log 5=1-\log 2$, $\log 15=\log 5+\log 3$: thus we find $n=\frac{\cdot 87506}{\cdot 09691}$.

9. Let A denote the first payment; then its present value is $\frac{A}{R^p}$. The second payment is mA; and its present value is $\frac{mA}{R^{p+1}}$. The third payment is m^2A; and its present value is $\frac{m^2A}{R^{p+2}}$. And so on. Thus the entire present value $=\frac{A}{R^p}\left\{1+\frac{m}{R}+\frac{m^2}{R^2}+\frac{m^3}{R^3}+\ldots\right\}$, *and with the limitation that* $\frac{m}{R}$ *is less than unity*, this is $\frac{A}{R^p}\cdot\frac{1}{1-\frac{m}{R}}$.

10. Let x denote the sum in pounds; then $xe^{nr}=1$, where $n=20$ and $r=\frac{5}{100}=\frac{1}{20}$; therefore $xe=1$.

11. The amount of 1 pound in 1 year $=e^r$; therefore the interest of 1 pound in 1 year $=e^r-1=\frac{1}{m}$ by supposition; therefore $e^r=\frac{1+m}{m}$. The amount of P pounds in n years $=Pe^{nr}=P\left(\frac{1+m}{m}\right)^n$.

12. Suppose P the sum borrowed; at the end of a year the debt is $P+Pr$; of this $2Pr$ is paid, so that $P-Pr$ is still due, that is $P(1-r)$. At the end of the second year we find in like manner that $P(1-r)\times(1-r)$ is still due, that is $P(1-r)^2$. At the end of the third year $P(1-r)^3$ is still due. And so on.

13. By Art. 598 we have to find the value of an annuity of A pounds to begin at the end of 13 years and to continue for 7 years; and by Art. 597

this is $\frac{A}{R-1}\left(R^{-13}-R^{-20}\right)$, where $R=\frac{106}{100}=1{\cdot}06$. Now $\log R^{-13}=-13\log 1{\cdot}06$ $=-{\cdot}8289767={\cdot}6710233-1$; therefore $R^{-13}={\cdot}4688385$. $\operatorname{Log} R^{-20}=-20\log 1{\cdot}06$ $=-{\cdot}5061180={\cdot}4938820-1$; therefore $R^{-20}={\cdot}3118042$.

Hence the fine $=A\times\frac{100}{6}\left({\cdot}4688385-{\cdot}3118042\right)$.

14. The present value of an annuity of b pounds per year is $\frac{b(1-R^{-n})}{R-1}$: it is obvious that the man is ruined if this be greater than a; that is if $b(1-R^{-n})$ is greater than $a(R-1)$. In this case $b=90$, $R=\frac{105}{100}$, $a=1000$; thus the man will be ruined if $90(1-R^{-n})$ is greater than 50, that is if $1-R^{-n}$ is greater than $\frac{5}{9}$, that is if R^{-n} is less than $\frac{4}{9}$. We have then to shew that $\frac{1}{R^n}$ is less than $1\div\frac{9}{4}$, that is that R^n is greater than $\frac{9}{4}$, when $n=17$. $\operatorname{Log} R^{17}=17\log R=17\log\frac{21}{20}=17(\log 21-\log 20)=17(\log 7+\log 3-1-\log 2)$; and $\log\frac{9}{4}=\log 9-\log 4=2\log 3-2\log 2$. It will be found that the former logarithm is the greater.

XLIV.

7. By Art. 605, $p_1q_2-p_2q_1=\pm 1$, $p_2q_3-p_3q_2=\mp 1$; therefore $p_1q_2-p_2q_1=-(p_2q_3-p_3q_2)$; therefore $(p_3-p_1)q_2=(q_3-q_1)p_2$.

8. Suppose $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ to be consecutive convergents. Then q_1 and q_2 have no common measure greater than unity; for $p_1q_2-p_2q_1=\pm 1$, so that if any number greater than 1 could divide both q_1 and q_2 it would also divide ± 1, which is absurd. Similarly no number greater than 1 can divide both p_1 and p_2.

9. $p_1q_2-p_2q_1=-1$, for the first convergent is too small, and is therefore less than the second convergent. Then $p_2q_3-p_3q_2=-(p_1q_2-p_2q_1)=(-1)^2$. Then $p_3q_4-p_4q_3=-(p_2q_3-p_3q_2)=(-1)^3$; and so on.

10. $$\frac{p_2}{q_2}-\frac{p_1}{q_1}=\frac{1}{q_1q_2},\quad \frac{p_3}{q_3}-\frac{p_2}{q_2}=-\frac{1}{q_2q_3},\ \ldots\ \frac{p_n}{q_n}-\frac{p_{n-1}}{q_{n-1}}=\frac{(-1)^n}{q_{n-1}q_n};$$ add all these results together.

11. As in Example 7, we have

$$(p_{n+1}-p_{n-1})\,q_n=(q_{n+1}-q_{n-1})\,p_n,\quad (p_{n+2}-p_n)\,q_{n+1}=(q_{n+2}-q_n)\,p_{n+1};$$

therefore $(p_{n+1}-p_{n-1})(p_{n+2}-p_n)\,q_nq_{n+1}=(q_{n+1}-q_{n-1})(q_{n+2}-q_n)\,p_np_{n+1}$; divide by $p_np_{n+1}q_nq_{n+1}$.

12. We know that $\dfrac{p_n}{q_n}=\dfrac{p_{n-1}\mu_n+p_{n-2}}{q_{n-1}\mu_n+q_{n-2}}$; therefore
$p_nq_{n-2}-p_{n-2}q_n=\mu_n(p_{n-1}q_n-p_nq_{n-1})=-\mu_n(-1)^n$ by Example 9, $=(-1)^{n-1}\mu_n$.

13. Assume that $p_n=a_np_{n-1}+\beta_np_{n-2}$, $q_n=a_nq_{n-1}+\beta_nq_{n-2}$. The next convergent may be obtained from $\dfrac{p_n}{q_n}$ by changing a_n into $a_n+\dfrac{\beta_{n+1}}{a_{n+1}}$, so that it

$$=\frac{\left(a_n+\dfrac{\beta_{n+1}}{a_{n+1}}\right)p_{n-1}+\beta_np_{n-2}}{\left(a_n+\dfrac{\beta_{n+1}}{a_{n+1}}\right)q_{n-1}+\beta_nq_{n-2}}=\frac{a_{n+1}p_n+\beta_{n+1}p_{n-1}}{a_{n+1}q_n+\beta_{n+1}q_{n-1}}.$$

If therefore we suppose $p_{n+1}=a_{n+1}p_n+\beta_{n+1}p_{n-1}$ and $q_{n+1}=a_{n+1}q_n+\beta_{n+1}q_{n-1}$, the next convergent to $\dfrac{p_n}{q_n}$ will be $\dfrac{p_{n+1}}{q_{n+1}}$; thus the convergent $\dfrac{p_{n+1}}{q_{n+1}}$ may be formed by the same law that was supposed to hold for $\dfrac{p_n}{q_n}$. Now the law may be seen to be true for trial for the third convergent and therefore is applicable for every subsequent convergent.

Again, since $p_n=a_np_{n-1}+\beta_np_{n-2}$, and $q_n=a_nq_{n-1}+\beta_nq_{n-2}$, we have

$$p_nq_{n-1}-p_{n-1}q_n=\beta_n(p_{n-2}q_{n-1}-p_{n-1}q_{n-2})=-\beta_n(p_{n-1}q_{n-2}-p_{n-2}q_{n-1}).$$

Then by actual work we have $p_2q_1-p_1q_2=-\beta_1\beta_2$;
therefore $p_3q_2-p_2q_3=-\beta_3(p_2q_1-p_1q_2)=(-1)^2\beta_1\beta_2\beta_3$; and so on.

14. This may be shewn by Induction. Assume that $P=p_nR_{n-1}+p_{n-1}R_n$. Now if R_{n-1} be divided by R_n the quotient will be the $(n+1)^{\text{th}}$ quotient of the continued fraction, so that $R_{n-1}=\mu_{n+1}R_n+R_{n+1}$, where μ_{n+1} denotes this quotient. Hence we get

$$P=p_n(\mu_{n+1}R_n+R_{n+1})+p_{n-1}R_n=(p_n\mu_{n+1}+p_{n-1})R_n+p_nR_{n+1}=p_{n+1}R_n+p_nR_{n+1}.$$

Thus we see that if the result holds for a specific value of n it holds for the next greater value. Now it may be shewn to hold when $n=2$; for if we start with $\dfrac{P}{Q}$ in the manner of Art. 601, we get $\dfrac{P}{Q}=a+\dfrac{R_1}{Q}$, $\dfrac{Q}{R_1}=b+\dfrac{R_2}{R_1}$; so that $P=aQ+R_1=(bR_1+R_2)a+R_1=(ab+1)R_1+aR_2$, and this is what we had to shew.

In like manner the result $Q=q_nR_{n-1}+q_{n-1}R_n$ may be established.

15. $\dfrac{P}{Q}-\dfrac{p_n}{q_n}=\dfrac{Pq_n-p_nQ}{Qq_n}=\dfrac{(q_np_{n-1}-q_{n-1}p_n)R_n}{Qq_n}$ by Example 14; and as $q_np_{n-1}-q_{n-1}p_n=\pm1$, the required result is obtained.

16. If R_n and R_{n-1} had any common measure greater than unity it would divide both P and Q by reason of the result given in Example 14; but this is impossible, since P and Q by supposition have no common measure greater than unity.

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15. The quotients are 3, 1, 1, 1, 1, 6, 1, 1, 1, 1, 6, ...

16. The quotients are 5, 1, 1, 3, 5, 3, 1, 1, 10, 1, 1, 3, 5, 3, 1, 1, 10, ...

17. The quotients are 5, 1, 2, 1, 10, 1, 2, 1, 10, ...; the 9th convergent is $\frac{11357}{1977}$: the error by Art. 608 is less than $\frac{1}{(1977)^2}$, which is less than ·000001.

18. The quotients are 4, 1, 3, 1, 8, 1, 3, 1, 8, ...; the convergents are $\frac{4}{1}, \frac{5}{1}, \frac{19}{4}, \frac{24}{5}, \frac{211}{44}, \frac{235}{49}, \frac{916}{191}, \frac{1151}{240}, \frac{10124}{2111} \ldots$ Hence by Art. 608 the error lies between $\frac{1}{44 \times 49}$ and $\frac{1}{44(44+49)}$.

19. See the preceding Example and Art. 608.

20. See Example 18 and Art. 608.

21. The 8th convergent is $\frac{1520}{273}$; the 9th is $\frac{16063}{2885}$; then see Art. 608.

22. Let $x = 1 + \frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2+} \ldots$; thus $x = 1 + \cfrac{1}{3 + \cfrac{1}{2 + x - 1}}$, that is

$$x = 1 + \cfrac{1}{\cfrac{3x+4}{x+1}} = 1 + \frac{x+1}{3x+4} = \frac{4x+5}{3x+4};\ \text{therefore } 3x^2 + 4x = 4x + 5;\ \&c.$$

23. Let $x = \frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+\ldots}$; therefore $x = \cfrac{1}{b + \cfrac{1}{a+x}} = \frac{a+x}{ab+1+bx}$; therefore $bx^2 + abx = a$; therefore $x(x+a) = \frac{a}{b}$, which is what was to be shewn.

24. Let $x = 2a + \frac{1}{a+}\,\frac{1}{4a+}\,\frac{1}{a+}\,\frac{1}{4a+}$; therefore $x = 2a + \cfrac{1}{a + \cfrac{1}{4a + x - 2a}}$

$$= 2a + \cfrac{1}{\cfrac{2a^2+ax+1}{2a+x}} = 2a + \frac{2a+x}{2a^2+ax+1} = \frac{4a^3 + (2a^2+1)x + 4a}{2a^2+ax+1};$$

therefore $ax^2 = 4a(a^2+1)$; &c. Then see Art. 608.

25. $\sqrt{(a^2+a+1)} = a + \sqrt{(a^2+a+1)} - a = a + \frac{a+1}{\sqrt{(a^2+a+1)}+a}$,

$$\frac{\sqrt{(a^2+a+1)}+a}{a+1}=1+\frac{\sqrt{(a^2+a+1)}-1}{a+1}=1+\frac{a}{\sqrt{(a^2+a+1)}+1},$$
$$\frac{\sqrt{(a^2+a+1)}+1}{a}=1+\frac{\sqrt{(a^2+a+1)}-(a-1)}{a}=1+\frac{3}{\sqrt{(a^2+a+1)}+a-1}.$$

Thus the convergents are $\frac{a}{1}, \frac{a+1}{1}, \frac{2a+1}{2}, \ldots$

26. $$\frac{\sqrt{3}}{4}=0+\cfrac{1}{\frac{4}{\sqrt{3}}}=0+\cfrac{1}{\frac{4\sqrt{3}}{3}}=\cfrac{1}{\frac{\sqrt{(48)}}{3}},$$
$$\frac{\sqrt{(48)}}{3}=2+\frac{\sqrt{(48)}-6}{3}=2+\frac{4}{\sqrt{(48)}+6},$$
$$\frac{\sqrt{(48)}+6}{4}=3+\frac{\sqrt{(48)}-6}{4}=3+\frac{3}{\sqrt{(48)}+6},$$
$$\frac{\sqrt{(48)}+6}{3}=4+\frac{\sqrt{(48)}-6}{3}=4+\frac{4}{\sqrt{(48)}+6}.$$

Thus the quotients are 0, 2, 3, 4, 3, 4, ...; the convergents are

$\frac{1}{2}, \frac{3}{7}, \frac{13}{30}, \frac{42}{97}, \frac{181}{418}, \ldots$; and $2910=30\times 97$: see Art. 608.

27. $\sqrt{\frac{3}{2}}=\frac{\sqrt{6}}{2}=1+\frac{\sqrt{6}-2}{2}=1+\frac{1}{\sqrt{6}+2}$: and so on.
The quotients are 1, 4, 2, 4, 2, ...

28. The root is $\frac{3+\sqrt{(57)}}{4}$;
the quotients are 2, 1, 1, 1, 3, 7, 3, 1, 1, 1, 3, 7, 3, ...

29. For the root $\frac{5+\sqrt{(13)}}{2}$ the quotients are 4, 3, 3, 3, ...;

for the root $\frac{5-\sqrt{(13)}}{2}$ the quotients are 0, 1, 2, 3, 3, ...

30. The root is $\frac{7+\sqrt{(17)}}{4}$; the quotients are 2, 1, 3, 1, 1, 3, 1,. .

31. $\sqrt{(45)}=6+\frac{\sqrt{(45)}-6}{1}=6+\frac{9}{\sqrt{(45)}+6}$; the quotients for $\sqrt{(45)}$ will be found to be 6, 1, 2, 2, 2, 1, 12, ...; therefore the quotients for $\frac{1}{\sqrt{(45)}}$ will be these preceded by 0. The convergents to $\frac{1}{\sqrt{(45)}}$ are $\frac{1}{6}, \frac{1}{7}, \frac{3}{20}, \frac{7}{47}, \frac{17}{114}$.

32. Let $x=1+\frac{1}{2+}\,\frac{1}{2+}\ldots$; then $x=1+\frac{1}{2+x-1}$;
therefore $x-1=\frac{1}{x+1}$; therefore $x^2-1=1$; therefore $x^2=2$.

33. Let $x=\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\dots$; then $x=\dfrac{1}{1+\dfrac{1}{2+x}}=\dfrac{2+x}{3+x}$;

therefore $x^2+3x=x+2$; therefore $x^2+2x=2$; &c.

34. Let $x=1+\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\dots$; then

$$x=1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+x-1}}}=1+\cfrac{1}{2+\cfrac{x}{3x+1}}=1+\frac{3x+1}{7x+2}=\frac{10x+3}{7x+2};$$

therefore $7x^2+2x=10x+3$; &c.

35. Let $x=\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\dots$; then

$$x=\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{1+x}}}=\cfrac{1}{3+\cfrac{1+x}{3+2x}}=\frac{3+2x}{10+7x};$$

therefore $7x^2+10x=3+2x$; &c.

36. Let $x=2+\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+}\dots$;

and let $\quad y=\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+}\dots$; then

$$x=2+\cfrac{1}{1+\cfrac{1}{3+y}}=2+\frac{3+y}{4+y}=\frac{11+3y}{4+y},\ \text{and}\ y=\cfrac{1}{5+\cfrac{1}{1+y}}=\frac{1+y}{6+5y}.$$

From the first equation we get $y=\dfrac{11-4x}{x-3}$; substitute in the second: thus $\dfrac{11-4x}{x-3}=\dfrac{8-3x}{37-14x}$; therefore $(11-4x)(37-14x)=(x-3)(8-3x)$; &c.

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Examples 1...10 may be solved by the method used in Art. 628 or by the aid of Arts. 631, 633. For instance take Example 2. Here $17x+23y=183$; divide by 17: thus $x+y+\dfrac{6y}{17}=10+\dfrac{13}{17}$. Hence $\dfrac{13-6y}{17}$ must be an integer; denote it by p so that $13-6y=17p$; divide by 6: thus $2+\dfrac{1}{6}-y=2p+\dfrac{5p}{6}$. Hence $\dfrac{5p-1}{6}$ must be an integer; denote it by q so that $5p-1=6q$; divide by 5; thus $p-\dfrac{1}{5}=q+\dfrac{q}{5}$. Hence $\dfrac{q+1}{5}$ must be an integer; denote it by r so that $q=5r-1$. Then we find in succession $p=6r-1$, $y=5-17r$, $x=4+23r$. Hence $x=4$ and $y=5$ are the only corresponding positive integral values.

If we use Art. 633 we convert $\frac{23}{17}$ into a continued fraction; the convergent immediately preceding $\frac{23}{17}$ will be found to be $\frac{4}{3}$; and $4\times17-3\times23=-1$. Hence we obtain for the general solution $x=23t-4\times183$, $y=-17t+3\times183$: and if we put 32 for t we obtain $x=4$, $y=5$.

11. Let x denote the number of guineas, and y the number of five-pound notes required for any one way of payment; then, expressing all in shillings, we have $21x+100y=10000$; the general solution of this equation will be found to be $x=100t$, $y=100-21t$. Thus there are 4 ways if we exclude a zero value of x, and 5 ways if we admit this zero value.

12. Let x denote the number of guineas, and y the number of crowns required for any one way of payment; then $21x+5y=2000$; the general solution will be found to be $x=5t$, $y=400-21t$.

13. Let x denote the number of half-guineas, and y the number of sovereigns required for any one way of payment; then $21x+40y=4000$; the general solution will be found to be $x=40t$, $y=100-21t$.

14. Let x denote the number of florins, and y the number of half-crowns required for any one way of payment; then $4x+5y=39$. The general solution will be found to be $x=1+5t$, $y=7-4t$.

15. Let x denote the number of five-franc pieces, and y the number of dollars required for any one way of payment; then $8x+7y=887$. The general solution will be found to be $x=5+7t$, $y=121-8t$.

16. Let x denote the number of 7 shilling coins, and y the number of 17 shilling coins required for any one way of payment; then $7x+17y=600$. The general solution will be found to be $x=76-17t$, $y=4+7t$.

17. Suppose that he gives x guineas and takes y half-crowns; then $42x-5y=21$. The general solution will be found to be $x=3+5t$, $y=21+42t$.

18. Suppose that to pay the debt x sovereigns are given and y francs taken; then $20x-\frac{20}{25}y=44$; that is $20x-\frac{4y}{5}=44$; therefore $25x-y=55$. The general solution will be found to be $x=3+t$, $y=20+25t$.

19. Let x denote the quotient in the part which is divided by 6; then this part is $6x+5$: let y denote the quotient in the part which is divided by 11; then this part is $11y+4$. Hence $6x+5+11y+4=200$; therefore $6x+11y=191$. The general solution will be found to be $x=30-11t$, $y=1+6t$. Therefore the one part must be $185-66t$, and the other part $15+66t$.

20. Suppose there are x crowns and y half-crowns; then $\frac{81}{100}x+\frac{666}{1000}y=36$; therefore $90x+74y=4000$; therefore $45x+37y=2000$. The general solution will be found to be $x=28-37t$, $y=20+45t$.

21. Let x denote the first term, and y the common difference; then $\frac{n}{2}\{2x+(n-1)y\}=n^2$; therefore $2x+(n-1)y=2n$. First try $y=1$; then $x=\frac{n+1}{2}$ which is admissible if n be odd. Next try $y=2$; then $x=1$. If we were to suppose y greater than 2 we should have x negative which is inadmissible.

22. Suppose that x is the quotient when the number is divided by 28; then the number is $28x+21$: suppose that y is the quotient when the number is divided by 19; then the number is $19y+17$. Thus $28x+21=19y+17$; therefore $28x-19y=-4$. The general solution of this equation will be found to be $x=8+19t$, $y=12+28t$; and the general form of the number will be $28x+21$, that is $245+28\times 19t$: the least number is obtained by putting $t=0$.

23. Suppose that x is the quotient when the number is divided by 3, y the quotient when the number is divided by 5, and z the quotient when the number is divided by 7. Then the number $=3x+2=5y+4=7z+6$.

Take $3x+2=5y+4$; therefore $3x-5y=2$. The general solution will be found to be $x=4+5\tau$, $y=2+3\tau$. Now take $3x+2=7z+6$: substitute for x the expression just obtained; thus $15\tau-7z=-8$. The general solution will be found to be $\tau=6+7t$, $z=14+15t$. Hence $7z+6=105t+104$.

24. Suppose x, y, z to denote the quotients when the number is divided by 28, 19, 15. Then the number $=28x+13=19y+2=15z+7$.

Take $28x+13=19y+2$; the general solution will be found to be $x=3+19\tau$, $y=5+28\tau$. Now take $19y+2=15z+7$: substitute for y the expression just obtained; thus $19\times 28\tau+90=15z$. The general solution will be found to be $\tau=15t$, $z=6+19\times 28t$. Hence $15z+7=97+15\times 19\times 28t$; this is the general form of the required number: the least number is obtained by putting $t=0$.

25. The value of y cannot be greater than 8, for 23×9 is greater than 200; ascribe to y in succession the values 1, 2, ... 8, and find the corresponding values of x and z. For instance if $y=1$ we have $17x+3z=177$. The general solution will be found to be $x=3t$, $z=59-17t$; thus excluding the zero value of x there are three solutions in this case.

26. Eliminate z between the equations; thus we get $7x+3y=160$. The general solution will be found to be $x=1+3t$, $y=51-7t$; and substituting for x and y in either of the given equations we obtain $z=63+13t$.

27. Let x denote the number of shillings, and y the number of sixpences required for any one way of payment; then $x+y$ denotes the number of half-crowns. Thus $5(x+y)+2x+y=600$; that is $7x+6y=600$. The general solution will be found to be $x=6t$, $y=100-7t$.

28. Let x denote the number of guineas, y the number of crowns, z the number of shillings required for any one way of payment. Then

$$21x+5y+z=96,\quad x+y+z=16.$$

Hence by eliminating z we have $20x+4y=80$; therefore $5x+y=20$. The general solution will be found to be $x=4-t$, $y=5t$; hence $z=12-4t$.

29. Let x denote the number of half-crowns, and y the number of florins required for any one way of payment; then $x+y$ denotes the number of crowns. Thus $10(x+y)+5x+4y=88$; that is $15x+14y=88$. The general solution will be found to be $x=4-14t$, $y=2+15t$; so that the only solution in positive integers is $x=4$ and $y=2$.

30. Let x denote the digit in the hundreds' place, and y the digit in the tens' place; the digit in the units' place is zero since the number is divisible by 10. Then $100x+10y-(x+y)=99$; that is $99x+9y=99$. Hence the only admissible solution is $x=1$ and $y=0$.

31. Suppose that the number is represented in the undenary scale by $(11)^2x+y$; then it is represented in the septenary scale by $(7)^2y+x$: thus $(11)^2x+y=(7)^2y+x$; that is $120x=48y$; therefore $5x=2y$. The only admissible solution is $x=2$ and $y=5$, since neither x nor y can be greater than 6.

32. Let x denote the digit in the hundreds' place, y the digit in the tens' place, and z the digit in the units' place. Then $x+y+z=20$. Also $\dfrac{100x+10y+z-16}{2}=100z+10y+x$; therefore $98x-10y-199z=16$. Eliminating y we have $108x-189z=216$; therefore $4x-7z=8$. The only admissible solution is $x=9$ and $z=4$; and then $y=7$.

33. Let x denote the digit in the thousands' place, y the digit in the hundreds' place, z the digit in the tens' place, and u the digit in the units' place; then

$$10^3x+10^2y+10z+u=9^3u+9^2y+9z+x;$$

therefore $$999x+19y+z-728u=0.$$

Then we must proceed to solve this by trial; ascribe to u in succession the values 1, 2, 3, ... 8. If $u=1$ it is obvious that there is no solution, for $x=1$ would be too great. If $u=2$ there is no solution, for $x=1$ would be too small and $x=2$ would be too great. If $u=3$ there is no solution, for $x=2$ would be too small and $x=3$ would be too great. In this way we find that $u=7$ is alone admissible; and this gives $x=5$, $y=5$, $z=6$.

34. Let x denote the number of oxen, y the number of sheep, z the number of ducks. Then $x+y+z=100$. Also $100x+20y+z=2000$. Eliminating z we have $99x+19y=1900$. The general solution will be found to be $x=19t$, $y=100-99t$; therefore $z=80t$.

35. Let the three fractions be denoted by $\dfrac{x}{6}$, $\dfrac{y}{9}$, and $\dfrac{z}{18}$,

then $$\frac{x}{6}+\frac{y}{9}+\frac{z}{18}=2\tfrac{2}{3}; \text{ and } \frac{x}{6}+\frac{z}{18}=\frac{2y}{9}.$$

Hence $\dfrac{3y}{9}=\dfrac{8}{3}$; so that $y=8$. Therefore $\dfrac{x}{6}+\dfrac{z}{18}=\dfrac{16}{9}$; therefore $3x+z=32$. The general solution will be found to be $x=5-t$, $z=17+3t$. If the fractions are to be *proper* fractions the only admissible solution is that obtained by making $t=0$.

36. Let x denote the number of times the first bell tolled, y the number of times the second bell tolled, and z the number of times the third bell

tolled: then as the second bell ceased 18 seconds after the first and the third bell 21 seconds after the first, $25(x-1)=29(y-1)-18=33(z-1)-21$. Take $25(x-1)=29(y-1)-18$. The general solution of this equation will be found to be $x=20+29\tau$, $y=18+25\tau$. Now take $25(x-1)=33(z-1)-21$; substitute for x the expression just obtained: thus $25.29.\tau-33z+529=0$. The general solution of this equation will be found to be

$$\tau=1+33t, \quad z=38+25.29.t.$$

Hence $x=49+29.33t$; and the general expression in seconds for the time during which the first bell tolled is $25(48+29.33.t)$. As this time is less than half an hour we must have $t=0$.

37. Let us find the distance between the x^{th} division on the first rod and the y^{th} on the second; the former division is $\frac{xc}{m}$ inches from the common end, and the latter is $\frac{yc}{n}$ inches from the common end: hence the distance is $\frac{xc}{m} \sim \frac{yc}{n}$ inches, that is $\frac{c}{mn}(xn \sim ym)$ inches. Now we cannot have $xn \sim ym=0$, for then $\frac{m}{n}=\frac{x}{y}$ so that $\frac{m}{n}$ would be reduced to lower terms which is impossible. But we can have $xn \sim ym=1$; in fact we can solve both $xn-ym=1$ and $xn-ym=-1$; see Art. 631.

For instance if $m=250$ and $n=243$ we find that $x=107$ and $y=104$ is a solution of $xn-ym=1$; and $x=250-107$ and $y=243-104$ is a solution of $xn-ym=-1$.

38. Suppose that on one shelf there are x sets of 5 volumes each, y sets of 4 volumes each, and z sets of 3 volumes each. Then $5x+4y+3z=20$. We must then find all the solutions of this equation, and try if there are three solutions such that the sum of the values of x is 3, the sum of the values of y is 6, and the sum of the values of z is 7.

39. Suppose the sum is c sixpences, and that it is paid by x half-crowns and y shillings; so that $5x+2y=c$. Now we know from Art. 634 that the number of solutions cannot differ by more than unity from $\frac{c}{5\times 2}$, so that we have only to examine values for c beginning with $c=100$. Put $c=100$; then by Case IV. of Art. 634 there are 11 solutions. Put $c=101$; then by Case I. there are 10 solutions. Put $c=102$; then by Case III. there are 11 solutions. Put $c=103$; then by Case I. there are 10 solutions. Put $c=104$; then by Case III. there are 11 solutions. Put $c=105$; then by Case II. there are 11 solutions. Thus we shall find that 103 is the greatest admissible value of c.

40. Use the same notation as in the solution of Example 39. By examining the four cases of Art. 634, we see that Case IV. will furnish the greatest value of c corresponding to a given number of solutions. If we take $c=110$ we have exactly 10 solutions. Any greater value of c would be inadmissible; for instance if $c=111$, there would be 11 solutions by Case I.

XLVII.

1. $y(3x-4)=14-3x$; therefore $y=\frac{-3x+14}{3x-4}=-1+\frac{10}{3x-4}$; therefore $3x-4=\pm 1$, or ± 2, or ± 5, or ± 10. We find on trial that the only cases in which both x and y are positive integers are when $3x-4=2$ or 5.

2. $y(x-3)=29+2x-x^2$; therefore $y=\frac{-x^2+2x+29}{x-3}=-x-1+\frac{26}{x-3}$; therefore $x-3=\pm 1$, or ± 2, or ± 13, or ± 26. We find on trial that we must take $x-3=1$ or 2.

3. The quotients for $\sqrt{(13)}$ are 3, 1, 1, 1, 1, 6,......: the convergents are $\frac{3}{1}, \frac{4}{1}, \frac{7}{2}, \frac{11}{3}, \frac{18}{5}, \ldots\ldots$

4. The quotients are 10, 20,......; see Example XLV. 11: the first convergent is $\frac{10}{1}$.

5. Let M denote one of the numbers; then $M=n^2-1=10m^2$; therefore $n^2-10m^2=1$: to find values of m and n which satisfy this equation we must form the convergents to $\sqrt{(10)}$; the second convergent is $\frac{19}{6}$: thus $n=19$ and $m=6$ is the least solution we can obtain in this way. And thus $M=360$.

6. Suppose that the paddock contained x^2+3 square yards; and that the brother's paddock contained y^2 square yards; then $y^2+1=\frac{1}{2}(x^2+3)$; therefore $x^2-2y^2=-1$. The convergents to $\sqrt{2}$ are $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \ldots\ldots$ Now as the paddock is between one and two roods x^2+3 must lie between 1210 and 2420; therefore we must take $x=41$ and $y=29$.

7. Let x denote the integral part of the square root; and let x^2+y denote the number; then $x^2+y=3x+1$. Solve the quadratic in x; thus $x=\frac{3\pm\sqrt{(13-4y)}}{2}$. Then ascribe to y in succession the values 1, 2, 3: it will be found that $y=1$ and $y=3$ are admissible, y cannot be greater than 3, for then x would be impossible.

8. We have $y^2=b-ax^2$; thus x cannot be greater than $\sqrt{\frac{b}{a}}$; and so the number of solutions in positive integers is limited.

9. Solve the quadratic in y; thus $y=\frac{1}{3}\left\{x\pm\sqrt{(81-20x^2)}\right\}$. Hence we find that the only admissible values of x are 0 and 2.

10. $2x^2-9xy+7y^2=(2x-7y)(x-y)$; thus $(2x-7y)(x-y)=38$. Hence we have the following cases for trial:

$$2x-7y=\pm19,\ x-y=\pm2;\quad 2x-7y=\pm2,\ x-y=\pm19;$$
$$2x-7y=\pm38,\ x-y=\pm1;\quad 2x-7y=\pm1,\ x-y=\pm38.$$

It will be found on examination that the only admissible cases are

$$2x-7y=-19,\ x-y=-2,\ \text{and}\ 2x-7y=1,\ x-y=38.$$

11. In the formulæ of Art. 643 put 24 for p, 5 for q, and 23 for N.

12. In the formulæ of Art. 645 put 3 for p, 1 for q, and 2 for N. Also put 3 for m, and 2 for n: see Example 6.

XLVIII.

1. $\dfrac{1}{3-2x}=\dfrac{1}{3}\left(1-\dfrac{2x}{3}\right)^{-1}=\dfrac{1}{3}\left\{1+\dfrac{2x}{3}+\left(\dfrac{2x}{3}\right)^2+\ldots\ldots+\left(\dfrac{2x}{3}\right)^n+\ldots\ldots\right\}.$

2. $2-x-3x^2=(1+x)(2-3x)$. Assume then that

$$\frac{5-10x}{(1+x)(2-3x)}=\frac{A}{1+x}+\frac{B}{2-3x};$$

therefore $\quad 5-10x=A(2-3x)+B(1+x)=2A+B-(3A-B)x.$

Thus $5=2A+B$, $\quad 10=3A-B$; therefore $A=3$, $B=-1$. Then

$$\frac{3}{1+x}=3\left\{1-x+x^2-\ldots\ldots+(-1)^nx^n+\ldots\ldots\right\}$$

$$-\frac{1}{2-3x}=-\frac{1}{2}\left(1-\frac{3x}{2}\right)^{-1}=-\frac{1}{2}\left\{1+\frac{3x}{2}+\left(\frac{3x}{2}\right)^2+\ldots\ldots+\left(\frac{3x}{2}\right)^n+\ldots\right\}.$$

3. Assume $\dfrac{3x-2}{(x-1)(x-2)(x-3)}=\dfrac{A}{x-1}+\dfrac{B}{x-2}+\dfrac{C}{x-3}$; therefore

$$3x-2=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$$
$$=x^2(A+B+C)-x(5A+4B+3C)+6A+3B+2C.$$

Thus $0=A+B+C$, $3=-(5A+4B+3C)$, $-2=6A+3B+2C$; therefore

$$A=\frac{1}{2},\ B=-4,\ C=\frac{7}{2};\ \text{\&c.}$$

4. Assume $\dfrac{x}{(1-x)(1-px)}=\dfrac{A}{1-x}+\dfrac{B}{1-px}$; therefore

$$x=A(1-px)+B(1-x)=-x(Ap+B)+A+B;$$

therefore $1=-(Ap+B)$, $0=A+B$; therefore $A=\dfrac{1}{1-p}$, $B=-\dfrac{1}{1-p}$; &c.

5. $\dfrac{1}{1-2x+x^2}=\dfrac{1}{(1-x)^2}=(1-x)^{-2}$; then see Art. 521.

6. $\dfrac{5+6x}{(1-3x)^2}=(5+6x)(1-3x)^{-2}$. By Art. 521 we have

$$(1-3x)^{-2}=1+2(3x)+\ldots\ldots+n(3x)^{n-1}+(n+1)(3x)^n+\ldots.$$

Multiply this by $5+6x$, and take the coefficient of x^n; thus we get

$$5(n+1)3^n+6n\,3^{n-1},\ \text{that is}\ (7n+5)3^n.$$

7. $\dfrac{1+4x+x^2}{(1-x)^4}=(1+4x+x^2)(1-x)^{-4}$. As in Art. 521, we find that

$$(1-x)^{-4}=1+4x+\ldots\ldots+\frac{(n+1)(n+2)(n+3)}{\lfloor 3}x^n+\ldots\ldots$$

Multiply this by $1+4x+x^2$, and take the coefficient of x^n; thus we get

$$\frac{(n+1)(n+2)(n+3)}{\lfloor 3}+\frac{4n(n+1)(n+2)}{\lfloor 3}+\frac{(n-1)n(n+1)}{\lfloor 3},$$

which reduces to $(n+1)^3$.

8. As in Art. 653 the first term is 1, the second term is x; the coefficients of the other powers of x are found in succession by the law $u_n-u_{n-1}+u_{n-2}=0$.

9. As in Art. 653 the first term is 1, the second term is $2x$; the coefficients of the other powers of x are found in succession by the law $u_n-2u_{n-1}+3u_{n-2}=0$.

10. By actual division $\dfrac{1-x^3}{2-2x-x^2}=x-2+\dfrac{5-6x}{2-2x-x^2}=x-2+\dfrac{\frac{5}{2}-3x}{1-x-\frac{1}{2}x^2}$.

The latter fraction can be expanded by Art. 653; the first term is $\frac{5}{2}$, the second term is $-\frac{1}{2}x$; the coefficients of the other powers of x are found in succession by the law $u_n-u_{n-1}-\frac{1}{2}u_{n-2}=0$.

11. $\dfrac{1}{a^2+ax+x^2}=\dfrac{1}{a^2}\,\dfrac{1}{1+\frac{x}{a}+\frac{x^2}{a^2}}$. By Art. 653 the first term of $\dfrac{1}{1+\frac{x}{a}+\frac{x^2}{a^2}}$ is 1, the second term is $-\frac{x}{a}$; the coefficients of the other powers of x are found in succession by the law $u_n+\frac{1}{a}u_{n-1}+\frac{1}{a^2}u_{n-2}=0$.

12. Assume for the required expansion $u_0+u_1x+u_2x^2+u_3x^3+\ldots$; then $1=(1-px+px^2-x^3)(u_0+u_1x+u_2x^2+u_3x^3+\ldots)$. To find u_0, u_1 and u_2 we have $1=u_0$, $0=u_1-pu_0$, $0=u_2-u_1p+u_0p$; and the higher coefficients are determined in succession by the law $u_n-pu_{n-1}+pu_{n-2}-u_{n-3}=0$.

13. The r^{th} term is $\dfrac{a^{r-1}x}{(1+a^{r-1}x)(1+a^rx)}$; assume it $=\dfrac{A}{1+a^{r-1}x}+\dfrac{B}{1+a^rx}$: then $\quad a^{r-1}x=A(1+a^rx)+B(1+a^{r-1}x)=A+B+a^{r-1}(Aa+B)x$.

Therefore $A+B=0$, $Aa+B=1$; thus $A=\dfrac{1}{a-1}$, $B=-\dfrac{1}{a-1}$. When each term of the series is thus resolved into two, we find they disappear by cancelling, except one at the beginning and one at the end.

14. The r^{th} term is $\dfrac{a^{r-1}x(1-a^r x)}{(1+a^{r-1}x)(1+a^r x)(1+a^{r+1}x)}$; assume that this $=\dfrac{A}{1+a^{r-1}x}+\dfrac{B}{1+a^r x}+\dfrac{C}{1+a^{r+1}x}$: then, proceeding as before, $a^{r-1}x(1-a^r x)$ $=A+B+C+a^{r-1}x\{A(a+a^2)+B(1+a^2)+C(1+a)\}+a^{2r-1}x^2\{Aa^2+Ba+C\}$. Therefore $A+B+C=0$, $A(a+a^2)+B(1+a^2)+C(1+a)=1$, $Aa^2+Ba+C=-1$; thus $A=C=-\dfrac{1}{(a-1)^2}$, $B=\dfrac{2}{(a-1)^2}$. When each term of the series is thus resolved into three, we find they disappear by cancelling except two at the beginning and two at the end.

15. As in Example 7, we find that the n^{th} term in the expansion is $\{au_{n-1}+bu_{n-2}+cu_{n-3}+du_{n-4}+eu_{n-5}\}x^{n-1}$, where $u_{n-1}=\dfrac{n(n+1)(n+2)(n+3)}{\lfloor 4}$; u_{n-2} is obtained from u_{n-1} by changing n into $n-1$; u_{n-3} is obtained in a similar way from u_{n-2}; and so on. Hence multiplying by $\lfloor 4$ we find that the following relation must hold for all positive integral values of n:

$$n^4\lfloor 4=a(n^4+6n^3+11n^2+6n)+b(n^4+2n^3-n^2-2n)$$
$$+c(n^4-2n^3-n^2+2n)+d(n^4-6n^3+11n^2-6n)+e(n^4-10n^3+35n^2-50n+24);$$

equate the coefficients of the various powers of n; thus we see at once that $e=0$; and to find a, b, c, d, we have $0=6a-2b+2c-6d$, $0=11a-b-c+11d$, $0=6a+2b-2c-6d$; $\lfloor 4=a+b+c+d$.

16. Assume that the proposed fraction is equal to $\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dfrac{C}{x-c}+\dots$; then we have $x^p=A(x-b)(x-c)\dots+B(x-a)(x-c)\dots+C(x-a)(x-b)\dots+\dots$ Now this being *identically* true we may give to x any value we please. Put a for x; thus $a^p=A(a-b)(a-c)\dots$; the terms involving B, $C\dots$ vanish. This finds A. Similarly if we put b for x we find B. And so on.

The original fraction being thus identically equal to $\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dots$ we may give any value to x; put then $x=0$; thus

$$0=\frac{a^{p-1}}{(a-b)(a-c)\dots}+\frac{b^{p-1}}{(b-a)(b-c)\dots}+\dots$$

The condition that p is to be less than n is required by Art. 648.

XLIX.

1. To find p and q we have $21=9p+4q$, $51=21p+9q$; thus $p=5$, $q=-6$. The required expression, by Art. 656, is $\dfrac{4-11x}{1-5x+6x^2}$. Assume that this $=\dfrac{A}{1-2x}+\dfrac{B}{1-3x}$; then $A=3$, $B=1$; &c.

2. Here $89=11p+q$, $659=89p+11q$; thus $p=10$, $q=-21$. The required expression is $\dfrac{1+x}{1-10x+21x^2}$; and this we find $=\dfrac{2}{1-7x}-\dfrac{1}{1-3x}$; &c.

3. Here $11=3p+q$, $43=11p+3q$; thus $p=5$, $q=-4$. The required expression is $\frac{1-2x}{1-5x+4x^2}$; and this we find $=\frac{1}{3}\cdot\frac{1}{1-x}+\frac{2}{3}\cdot\frac{1}{1-4x}$; &c.

4. The expansion of $\frac{1}{1-x}$ is convergent if x is less than 1, and the expansion of $\frac{1}{1-4x}$ is convergent if $4x$ is less than 1; hence both expansions are convergent if $4x$ is less than 1.

5. Here $32=11p+3q$, $84=32p+11q$; thus $p=4$, $q=-4$. Hence the general term is the coefficient of x^n in the expansion of $\frac{3-x}{1-4x+4x^2}$, that is in the expansion of $(3-x)(1-2x)^{-2}$; this coefficient is $3(n+1)2^n-n2^{n-1}$. See Example XLVIII. 6.

6. We first find the general term. We assume that the series is a recurring series with the scale of relation $1-p-q$. To find p and q we have $17=5p+q$, $53=17p+5q$; thus $p=4$, $q=-3$. We see that the other two given terms follow this law. Hence the general term is the coefficient of x^n in the expansion of $\frac{1+x}{1-4x+3x^2}$; and this we find $=\frac{2}{1-3x}-\frac{1}{1-x}$. Thus the $(n+1)^{\text{th}}$ term is 2.3^n-1; and the sum of the first n terms can be immediately found.

7. Here $10=14p+10q$, $6=10p+14q$; therefore $p=\frac{5}{6}$, $q=-\frac{1}{6}$. Hence the general term is the coefficient of x^n in the expansion of $\dfrac{10+\frac{17x}{3}}{1-\frac{5}{6}x+\frac{1}{6}x^2}$;

and this we find $=\dfrac{64}{1-\frac{1}{2}x}-\dfrac{54}{1-\frac{1}{3}x}$. Hence we can get the general term.

The sum to infinity is obtained by putting $x=1$; so it is $128-81$, that is 47.

8. Assume for the scale of relation $1-px-qx^2-rx^3$; then

$$-5=2p-q+2r,\quad 10=-5p+2q-r,\quad -17=10p-5q+2r:$$

thus we obtain $p=-3$, $q=-3$, $r=-1$. Suppose then that the series represents the development of $\frac{a+bx+cx^2}{1+3x+3x^2+x^3}$, that is of $\frac{a+bx+cx^2}{(1+x)^3}$; then, as in Art. 653, we have $a+bx+cx^2=(1+3x+3x^2+x^3)(2-x+2x^2-5x^3+\dots)$: hence equating coefficients we have $a=2$, $b=6-1=5$, $c=6-3+2=5$. By Art. 521 the coefficient of x^n in the development of $(2+5x+5x^2)(1+x)^{-3}$ is $\left\{2(n+1)(n+2)-5n(n+1)+5(n-1)n\right\}\frac{(-1)^n}{1.2}$, that is $(n^2-2n+2)(-1)^n$,

L.

1. This is a particular case of Art. 661, namely that in which $a=1$ and $b=0$; so that the C of that Article vanishes.

2. Put $a=1$ and $b=0$ in the result of Art. 663; thus we find that the sum is $\dfrac{1}{m-1}\left\{\dfrac{1}{\underline{|m-1}} - \dfrac{1}{(n+1)(n+2)\ldots(n+m-1)}\right\}$.

3. $\dfrac{1}{1.2}=\dfrac{1}{1}-\dfrac{1}{2}$, $\dfrac{1}{2.3}=\dfrac{1}{2}-\dfrac{1}{3}$, ..., $\dfrac{1}{n(n+1)}=\dfrac{1}{n}-\dfrac{1}{n+1}$; hence by addition we obtain $1-\dfrac{1}{n+1}$: when n is infinite $\dfrac{1}{n+1}$ vanishes.

4. The n^{th} term $=\dfrac{1}{2n(2n+2)(2n+4)}=\dfrac{1}{8n(n+1)(n+2)}$; hence the sum is found by putting $m=3$ in Example 2, and dividing the result by 8.

5. The n^{th} term $=\dfrac{1}{n(n+3)}=\dfrac{1}{3}\left(\dfrac{1}{n}-\dfrac{1}{n+3}\right)$; decompose each term into two in this manner, and add them; then we find that they disappear by cancelling except three at the beginning, and three at the end.

6. The n^{th} term $=\dfrac{1}{n(n+2)(n+4)}=\dfrac{(n+1)(n+3)}{n(n+1)(n+2)(n+3)(n+4)}$

$$=\frac{n(n+4)+3}{n(n+1)(n+2)(n+3)(n+4)}=\frac{1}{(n+1)(n+2)(n+3)}+\frac{3}{n(n+1)(n+2)(n+3)(n+4)}.$$

The sum by Art. 663 is $\dfrac{1}{12}-\dfrac{1}{2(n+2)(n+3)}+\dfrac{3}{96}-\dfrac{3}{4(n+1)(n+2)(n+3)(n+4)}$.

7. The n^{th} term $=\dfrac{3n+1}{(n+1)(n+2)(n+3)}=\dfrac{3}{(n+2)(n+3)}-\dfrac{2}{(n+1)(n+2)(n+3)}$.

The sum is $1-\dfrac{3}{n+3}-2\left\{\dfrac{1}{12}-\dfrac{1}{2(n+2)(n+3)}\right\}$, that is $\dfrac{5}{6}-\dfrac{3}{n+3}+\dfrac{1}{(n+2)(n+3)}$.

8. The first term is 1, the second term is $1+2$, the third term is $1+2+3$, the fourth term is $1+2+3+4$, and so on; hence the n^{th} term is $1+2+3+ \ldots\ldots +n$, that is $\dfrac{n(n+1)}{2}$; therefore the sum of n terms is $\dfrac{n(n+1)(n+2)}{2\times 3}$ by Example 1.

9. Put $2m$ for n; then we have to find the sum of

$$1\,.\,2m+2(2m-1)+3(2m-2)\ldots+m\{2m-(m-1)\},$$

that is of $2m\{1+2+3+m\}-\{1\,.\,2+2\,.\,3+\ldots+(m-1)\,m\}$. Thus the sum

$$=2m\frac{m(m+1)}{2}-\frac{(m-1)\,m\,(m+1)}{3}=\frac{m(m+1)(2m+1)}{3}=\frac{n(n+1)(n+2)}{12}.$$

10. The series $=na^2+2a\{1+2+\ldots+(n-1)\}+1^2+2^2+\ldots+(n-1)^2$,

that is $$na^2+(n-1)\,na+\frac{(n-1)\,n\,(2n-1)}{6}.$$

11. Let $s=1^2+2^2x+3^2x^2+\ldots+n^2x^{n-1}$; multiply by x and subtract; thus $s(1-x)=1+3x+5x^2+\ldots+(2n-1)x^{n-1}-n^2x^n$; therefore by Art. 473, $s=\dfrac{1-(2n-1)x^n}{(1-x)^2}+\dfrac{2x(1-x^{n-1})}{(1-x)^3}-\dfrac{n^2x^n}{1-x}$. Collect the terms.

12. The first term is na^nr, the second term is $n(n-1)a^{n-1}br^2$, the third term is $\dfrac{n(n-1)}{1.2}a^{n-2}b^2r^3$, and so on; the sum $=nar(a+br)^{n-1}$.

13. $\dfrac{x}{(1-x)^2-cx}=\dfrac{x}{(1-x)^2}\left\{1-\dfrac{cx}{(1-x)^2}\right\}^{-1}=\dfrac{x}{(1-x)^2}\left\{1+\dfrac{cx}{(1-x)^2}+\dfrac{c^2x^2}{(1-x)^4}+\ldots\right\}$.

We must now expand each of these terms separately, and select the coefficient of x^n in each. The coefficient of x^n in the expansion of $\dfrac{x}{(1-x)^2}$ is the same as that of x^{n-1} in the expansion of $(1-x)^{-2}$; this is n by Art. 521. The coefficient of x^n in the expansion of $\dfrac{cx^2}{(1-x)^4}$ is the same as that of x^{n-2} in the expansion of $c(1-x)^{-4}$; this is $\dfrac{c(n-1)n(n+1)}{\lfloor 3}$ by Art. 521. And so on.

14. $\dfrac{x(1-ax)}{(1-x)(1-ax-by)}=\dfrac{x}{1-x}\left(1-\dfrac{by}{1-ax}\right)^{-1}$. Expand the second factor in powers of $\dfrac{by}{1-ax}$; thus the term which contains y^n is the expression $\dfrac{x}{1-x}\,\dfrac{b^ny^n}{(1-ax)^n}$. Thus the coefficient of x^my^n is the coefficient of x^{m-1} in the expansion of $\dfrac{b^n}{1-x}\times\dfrac{1}{(1-ax)^n}$, that is in the expansion of $b^n(1-x)^{-1}(1-ax)^{-n}$, that is in the product $b^n\left\{1+x+x^2+x^3+\ldots\right\}\left\{1+nax+\dfrac{n(n+1)}{1.2}a^2x^2+\ldots\right\}$.

16. $\dfrac{p_1}{p_0}=n$, $\dfrac{2p_2}{p_1}=n-1$, $\dfrac{3p_3}{p_2}=n-2$, ...; the sum of these terms is $\dfrac{n(n+1)}{2}$.

Also
$$p_0+p_1=\frac{p_1}{n}+p_1=\frac{n+1}{n}p_1,$$
$$p_1+p_2=p_2+\frac{2p_2}{n-1}=\frac{n+1}{n-1}p_2,$$
$$p_2+p_3=p_3+\frac{3p_3}{n-2}=\frac{n+1}{n-2}p_3,$$
and so on. Hence the product is $(n+1)^n\dfrac{p_1p_2\cdots p_n}{\lfloor n}$.

Let $S_n=p_1-\dfrac{p_2}{2}+\dfrac{p_3}{3}-\ldots+\dfrac{(-1)^{n-1}p_n}{n}=n-\dfrac{n(n-1)}{2\lfloor 2}+\dfrac{n(n-1)(n-2)}{3\lfloor 3}-\ldots$

Change n into $n+1$; thus $S_{n+1}=n+1-\frac{(n+1)n}{2\lfloor 2}+\frac{(n+1)n(n-1)}{3\lfloor 3}-\ldots$ Hence by subtraction $S_{n+1}-S_n=1-\frac{n}{\lfloor 2}+\frac{n(n-1)}{\lfloor 3}-\ldots=\frac{1}{n+1}\left\{1-(1-1)^{n+1}\right\}=\frac{1}{n+1}$.

Thus $S_{n+1}=S_n+\frac{1}{n+1}$. But $S_1=1$; thus $S_2=1+\frac{1}{2}$; $S_3=1+\frac{1}{2}+\frac{1}{3}$; and so on.

17. $$\left(\frac{1}{1-x}-1\right)^n=\frac{1}{(1-x)^n}-\frac{n}{1}\frac{1}{(1-x)^{n-1}}+\frac{n(n-1)}{1\,.\,2}\frac{1}{(1-x)^{n-2}}-\ldots$$
Expand each of the terms on the right-hand side and pick out the coefficient of x^p; thus we obtain the given expression.

Again in the expansion of $\frac{x^n}{(1-x)^n}$ if n is greater than p there will be no term involving x^p; and so the proposed expression will be zero.

18. Put $n=9$ in the formula of (1) of Art. 667; thus we have $\frac{9\times 10\times 11}{6}$.

19. This is the difference of the numbers in two piles; one having 15 shots in each side of the base, and the other having 10 shots: the required number is therefore $\frac{15\,.\,16\,.\,17}{6}-\frac{10\,.\,11\,.\,12}{6}$.

20. In the top layer there are mn balls, in the next layer $(m+1)(n+1)$ balls, in the next layer $(m+2)(n+2)$ balls, and so on. The sum of p terms of this series is $pmn+(m+n)\{1+2+\ldots+(p-1)\}+1^2+2^2+\ldots+(p-1)^2$, that is $pmn+(m+n)\frac{p(p-1)}{2}+\frac{(p-1)p(2p-1)}{6}$; collect the terms.

21. This is an example of Art. 671; we have $r=4$: thus we obtain immediately the first formula given for the sum, and by actual multiplication we can shew that the second formula is equivalent to the first.

23. Suppose that $(1+x)(1+cx)(1+c^2x)+\ldots=1+A_1x+A_2x^2+A_3x^3+\ldots$, where A_1, A_2, A_3,... do not contain x. Now change x into cx; then we can infer that $(1+x)(1+A_1cx+A_2c^2x^2+A_3c^3x^3+\ldots)=1+A_1x+A_2x^2+\ldots$

Equate the coefficients of x^r on the two sides; thus $A_rc^r+A_{r-1}c^{r-1}=A_r$; therefore $A_r(1-c^r)=A_{r-1}c^{r-1}$. Put in succession 1, 2, 3, ... for r; thus we can determine in succession A_1, A_2, A_3, ..., observing that A_0 stands for 1.

24. Suppose that $(1+x)^2\left(1+\frac{x}{2}\right)^2\left(1+\frac{x}{2^2}\right)^2\ldots=1+A_1x+A_2x^2+\ldots$

Change x into $\frac{x}{2}$; thus we can infer that

$$(1+x)^2\left\{1+A_1\frac{x}{2}+A_2\frac{x^2}{2^2}+\ldots\right\}=1+A_1x+A_2x^2+\ldots$$

Equate the coefficients of x^r on the two sides; thus if r be not less than 2 we have $\frac{A_r}{2^r}+\frac{2A_{r-1}}{2^{r-1}}+\frac{A_{r-2}}{2^{r-2}}=A_r$. And by equating the coefficient of x we have $\frac{A_1}{2}+2=A_1$, so that $A_1=4$. Then we can find in succession A_2, A_3,...

LI.

1. Suppose $a+b>c$, $b+c>a$, $c+a>b$; then

$$c(a+b)>c^2, \quad a(b+c)>a^2, \quad b(c+a)>b^2: \text{ add};$$

thus
$$2(ab+bc+ca)>a^2+b^2+c^2.$$

2. $2(ll'+mm'+nn')=l^2+m^2+n^2+l'^2+m'^2+n'^2-(l-l')^2-(m-m')^2-(n-n')^2$;

thus
$$2(ll'+mm'+nn') \text{ is } <2.$$

3. By simplifying this reduces to $a^2+b^2+c^2>ab+bc+ca$; now

$$a^2+b^2>2ab, \quad b^2+c^2>2bc, \quad c^2+a^2>2ca: \text{ add};$$

thus
$$2(a^2+b^2+c^2)>2(ab+bc+ca).$$

4. We have to shew that $a^{\frac{3}{2}}+b^{\frac{3}{2}}>(ab)^{\frac{1}{2}}\{a^{\frac{1}{2}}+b^{\frac{1}{2}}\}$; divide by $a^{\frac{1}{2}}+b^{\frac{1}{2}}$, then we have to shew that $a-(ab)^{\frac{1}{2}}+b>(ab)^{\frac{1}{2}}$, that is $a+b>2(ab)^{\frac{1}{2}}$: and this is evident.

5. $a^2+b^2>2ab$; therefore $c(a^2+b^2)>2abc$: similarly $a(b^2+c^2)>2abc$, and $b(c^2+a^2)>2abc$. Then add. Again, as in Example 4, we have

$$ab(a+b)<a^3+b^3, \quad bc(b+c)<b^3+c^3, \quad ca(c+a)<c^3+a^3.$$

6. This follows by adding $2abc$ to each side of the first inequality in Example 5. Or thus, $a+b>2\sqrt{ab}$, $b+c>2\sqrt{bc}$, $c+a>2\sqrt{ca}$: then multiply.

7. $x^2-8x+22=(x-4)^2+6$; this cannot be less than 6.

8. $2x^3-x-1=2(x^3-x)+x-1=(x-1)\{2x(x+1)+1\}$; this is positive or negative according as x is $>$ or $<$ 1.

9. $x+\dfrac{1}{nx}-\left(1+\dfrac{1}{n}\right)=x-1+\dfrac{1}{n}\left(\dfrac{1}{x}-1\right)=\dfrac{1}{x}(x-1)\left(x-\dfrac{1}{n}\right)$; this is positive if x is >1 or $<\dfrac{1}{n}$.

10. $$\frac{(a+x)(b+x)}{x}=\frac{ab}{x}+a+b+x=\left(\sqrt{\frac{ab}{x}}-\sqrt{x}\right)^2+a+b+2\sqrt{ab};$$

thus the least value is when $\left(\sqrt{\dfrac{ab}{x}}-\sqrt{x}\right)^2=0$, that is when $x=\sqrt{ab}$.

11. Let $2n+1$ denote the odd integer, and x one of the two parts; then the other part is $2n+1-x$, and the product is $x(2n+1-x)$: denote this by y. Then $x^2-(2n+1)x+y=0$; therefore $x=\dfrac{2n+1\pm\sqrt{(2n+1)^2-4y}}{2}$. The quantity under the square root is $4(n^2+n-y)+1$; and thus the greatest possible value of y is n^2+n, since y is an integer. Hence we find $x=n$ or $n+1$; and thus the two parts are n and $n+1$.

12. $\sqrt{(a^2-b^2)}+\sqrt{(2ab-b^2)}$ is $>a$ if $\sqrt{(2ab-b^2)}$ is $>a-\sqrt{(a^2-b^2)}$, that is if $2ab-b^2$ is $>a^2+a^2-b^2-2a\sqrt{(a^2-b^2)}$, that is if $\sqrt{(a^2-b^2)}$ is $>a-b$, that is if $a^2-b^2>(a-b)^2$, that is if $a+b$ is $>a-b$: and this is obviously the case.

13. Since a, b, c are in H.P. we have $a=\dfrac{bc}{2c-b}$; and since b, c, d are in H.P. we have $d=\dfrac{bc}{2b-c}$. Thus $a+d-(b+c)=\dfrac{bc}{2c-b}+\dfrac{bc}{2b-c}-(b+c)$

$$=\frac{bc(b+c)}{(2c-b)(2b-c)}-(b+c)=\frac{(b+c)\{bc-(2c-b)(2b-c)\}}{(2c-b)(2b-c)}=\frac{2(b+c)(b-c)^2}{(2c-b)(2b-c)}:$$

this is positive, for $2c-b$ and $2b-c$ are positive since a and d are positive by supposition.

14. Of the two quantities a and c suppose a the greater.

$a^n+c^n-2b^n=a^n-b^n-(b^n-c^n)=(a-b)\{a^{n-1}+a^{n-2}b+\ldots\}-(b-c)\{c^{n-1}+c^{n-2}b+\ldots\}$.

Now this expression is positive: for $a-b$ is $>b-c$ by Art. 479; and $a^{n-1}+a^{n-2}b+\ldots$ is obviously greater than $c^{n-1}+c^{n-2}b+\ldots$

15. $\dfrac{x+a}{\sqrt{(x^2+a^2)}}$ is $>$ or $<\dfrac{x+b}{\sqrt{(x^2+b^2)}}$ according as $\dfrac{(x+a)^2}{x^2+a^2}$ is $>$ or $<\dfrac{(x+b)^2}{x^2+b^2}$, that is according as $1+\dfrac{2ax}{x^2+a^2}$ is $>$ or $<1+\dfrac{2bx}{x^2+b^2}$, that is according as $a(x^2+b^2)$ is $>$ or $<b(x^2+a^2)$, that is according as $(a-b)x^2$ is $>$ or $<ab(a-b)$.

16. It will be found that $a^2b+b^2c+c^2a-(a^2c+b^2a+c^2b)=(b-a)(c-b)(a-c)$. Now if a, b, c are in *descending* order of magnitude $b-a$ is negative, $c-b$ is negative, and $a-c$ is positive; thus the product is positive. Similarly the other cases may be treated.

17. Multiply out and bring all the terms to the left-hand side; thus we have $A^2b^2+B^2a^2-2AaBb+A^2c^2+C^2a^2-2AcCa+\ldots$; that is we have the squares $(Ab-Ba)^2+(Ac-Ca)^2+\ldots$: this expression is therefore positive.

18. $2(a^3+b^3+c^3)>ab(a+b)+bc(b+c)+ca(c+a)$ by Example 5, and $\dfrac{a^3+b^3+c^3}{3}>(a^3b^3c^3)^{\frac{1}{3}}$ by Art. 681; so that $a^3+b^3+c^3>3abc$. Add, and we obtain the result.

19. $6abc<ab(a+b)+bc(b+c)+ca(c+a)$ by Example 5; also by the preceding solution $3abc<a^3+b^3+c^3$. Add.

20. Multiply by 2 and bring all the terms to the left-hand side; we thus obtain a set of squares $(\sqrt{a_1}-\sqrt{a_2})^2+(\sqrt{a_1}-\sqrt{a_3})^2+\ldots+(\sqrt{a_2}-\sqrt{a_3})^2+\ldots$; there being one square for every pair of quantities. This expression is therefore positive.

21. Let a and b denote the two numbers, a being the greater: we have to shew that $\dfrac{a+b}{2}-\sqrt{ab}$ is less than $\dfrac{(a-b)^2}{8b}$ and greater than $\dfrac{(a-b)^2}{8a}$. Now $\dfrac{a+b}{2}-\sqrt{ab}$ is $<\dfrac{(a-b)^2}{8b}$ if $4b(\sqrt{a}-\sqrt{b})^2$ is $<(a-b)^2$, that is if $4b$ is $<(\sqrt{a}+\sqrt{b})^2$, that is if $2\sqrt{b}$ is $<\sqrt{a}+\sqrt{b}$; and this is obviously the case. Similarly the second part of the Example may be treated.

22. Take n quantities $1, 2, 3, \ldots n$; their sum is $\dfrac{n(n+1)}{2}$, thus their arithmetical mean is $\dfrac{n+1}{2}$: therefore $\dfrac{n+1}{2}$ is $>\{\underline{|n}\}^{\frac{1}{n}}$ by Art. 681.

24. Take n quantities $1, 3, 5, \ldots 2n-1$; their sum is n^2, thus their arithmetical mean is n: then apply Art. 681.

25. By Example 23 we have $\lfloor 2n > (2n)^n$; that is

$$(2n-1)(2n-3)\ldots 3.1 \times 2^n \lfloor n > (2n)^n.$$

Divide both sides by $2^n \lfloor n\, n^n$, and we obtain the result.

26. $a^2(b^2+c^2) > 2a^2bc,\ b^2(c^2+a^2) > 2b^2ca,\ c^2(a^2+b^2) > 2c^2ab$; therefore $a^2b^2+b^2c^2+c^2a^2 > (a+b+c)\,abc$. Moreover, as in Example 3, we have $a^4+b^4+c^4 > a^2b^2+b^2c^2+c^2a^2$.

27. $6(a^3+b^3+c^3) > 3\{ab(a+b)+bc(b+c)+ca(c+a)\}$ by Example 5;
$2(a^3+b^3+c^3) > 6abc$ as in Example 18. Add.

28. Multiply up; thus the inequality reduces to
$2(a^3+b^3+c^3) > ab(a+b)+bc(b+c)+ca(c+a)$; this is Example 5.

29. See Example 3 of Chapter xxv. Or thus: $(a+b+c)^3 > 27abc$; this is a case of Art. 681. Again $(a+b+c)^3 = a^3+b^3+c^3+3(a+b)(b+c)(c+a)$; and this is $< 9(a^3+b^3+c^3)$ by Example 27.

30. $\dfrac{\log_a(1-p)}{\log_a(1-q)} = \dfrac{\log_e(1-p)}{\log_e(1-q)} = \dfrac{p+\dfrac{p^2}{2}+\dfrac{p^3}{3}+\ldots}{q+\dfrac{q^2}{2}+\dfrac{q^3}{3}+\ldots}$; this is $< \dfrac{p+p^2+p^3+\ldots}{q}$

and $> \dfrac{p}{q+q^2+q^3+\ldots}$, that is $< \dfrac{p}{q(1-p)}$ and $> \dfrac{p}{q \times \dfrac{1}{1-q}}$.

31. Multiply up; then the inequality is a case of Art. 681, there being n quantities the first of which is $a_1{}^2a_3a_4\ldots a_n$, and the others like this.

32. This is solved in the Algebra. We may also proceed thus; we have to shew that $1+ax > x+a^x$; put $\dfrac{p}{q}$ for x, where p and q are integers: thus we have to shew that $\dfrac{q-p+pa}{q} > a^{\frac{p}{q}}$. Now this is obvious by Art. 681: for on the left-hand side we have the arithmetical mean of q quantities, p of which are equal to a, and the rest equal to unity; and on the right-hand side we have the geometrical mean of the same q quantities.

LII.

1. $p-q = p+q-2q$, this is the difference of two even numbers, and is therefore even.

2. Resolve 3234 into its prime factors; thus we find that $3234 = 2.3.7^2.11$. Hence to obtain a perfect square we must multiply by $2.3.11$.

3. Resolve 1845 into its prime factors; thus we find that $1845 = 3^2.5.41$. Hence to obtain a perfect cube we must multiply by $3.5^2.41^2$.

4. $6480=2^4.3^4.5$. Multiply by $2^2.3^2.5^2$.

5. $13168=2^4.823$; and 823 is a prime number. Multiply by $2^2.(823)^2$.

6. The odd square number must be the square of some odd number, say of $2n+1$; the even square number must be the square of some even number, say of $2m$. Thus $(2n+1)^2+4m^2$ is a square, and must therefore be the square of some odd number, say equal to $(2p+1)^2$. Hence $n(n+1)+m^2=p(p+1)$. But $n(n+1)$ is an even number, and so is $p(p+1)$; hence m^2 is even; therefore m is even, say $=2q$: therefore $(2m)^2=(4q)^2=16q^2$.

7. By Fermat's Theorem N^4-1 is a multiple of 5 if N be not a multiple of 5; thus either N^2-1 or N^2+1 is a multiple of 5 if N be not. That is if N be not a multiple of 5 we have $N^2=5n\pm1$.

8. By Fermat's Theorem N^6-1 is a multiple of 7 if N be not a multiple of 7; thus either N^3-1 or N^3+1 is a multiple of 7 if N be not. That is if N be not a multiple of 7 we have $N^3=7n\pm1$.

9. If a number is both a square and a cube it must be perfect sixth power; and by Fermat's Theorem N^6-1 is a multiple of 7 if N be not a multiple of 7.

10. If a number is divisible by 3, so also is its square, and therefore is not of the form $3n-1$. If a number is not divisible by 3 it is of the form $3m\pm1$; and so its square is $9m^2\pm6m+1$ which is of the form $3n+1$.

11. Let $\dfrac{m(m+1)}{2}$ denote the triangular number; if either m or $m+1$ is divisible by 3 this is of the form $3n$. If neither m nor $m+1$ is divisible by 3 then m must be of the form $3p+1$; thus the triangular number becomes $\dfrac{9p(p+1)}{2}+1$, which is of the form $3n+1$.

13. Suppose a prime to b, and $a-b$ an odd number; then will $a+b$ be prime to $a-b$. For $a-b=a+b-2b$, and so if any number divides both $a-b$ and $a+b$ it will divide $2b$, and therefore b, since $a-b$ is odd. Similarly since $a-b=2a-(a+b)$ we see that if any number divides both $a-b$ and $a+b$ it will divide a. Thus any number which divides $a-b$ and $a+b$ will divide both a and b; and as a is prime to b there can be no such divisor. Then see Art. 704.

14. Let a, b, and $a+b$ be the three numbers. It may be shewn that $2\{a^4+b^4+(a+b)^4\}=4(a^2+ab+b^2)^2$. This establishes the statement.

15. $x^n-1=(x-1+1)^n-1=(x-1)^n+n(x-1)^{n-1}+\ldots+n(x-1)$; thus all the terms on the right-hand side except the first are obviously divisible by n; so that the remainder when x^n-1 is divided by n is the same as the remainder when $(x-1)^n$ is divided by n.

16. If possible suppose that m^2 and n^2 when divided by $2p+1$ give the same remainder r, where neither m nor n is greater than p. Let m' and n' denote the quotients. Thus $m^2=m'(2p+1)+r$, $n^2=n'(2p+1)+r$; therefore $m^2-n^2=(m'-n')(2p+1)$. Therefore $2p+1$ divides $(m-n)(m+n)$. But this is impossible for $2p+1$ is a prime number, and is greater than $m+n$.

17. Let the power have the exponent $2p$; let the odd number be raised to the power p; the result will be some odd number, say $2m+1$; we have

then to square this. Now $(2m+1)^2=4m(m+1)+1$; this is of the form $8n+1$ because $m(m+1)$ is an even number.

18. $7^p=(8-1)^p=8^p-p8^{p-1}+...+(-1)^p$. Every term is divisible by 8 except the last; if p is odd the last term is -1; thus if p is odd 7^p is of the form $8n-1$.

21. $ax-2x^2=\frac{a^2}{8}-2\left(x-\frac{a}{4}\right)^2$; thus the greatest value is obtained by making $\left(x-\frac{a}{4}\right)^2$ as small as possible. If $a=4m$ we can take $x=m$; and the result is $\frac{a^2}{8}$. If $a=4m+1$ we can take $x=m$; and the result is $\frac{a^2}{8}-\frac{1}{8}$. If $a=4m+2$ we can take $x=m$; and the result is $\frac{a^2}{8}-\frac{1}{2}$. If $a=4m+3$ we can take $x=m+1$; and the result is $\frac{a^2}{8}-\frac{1}{8}$.

22. $n(n+1)(2n+1)=(n-1)n(n+1)+n(n+1)(n+2)$; then apply Art. 710.

23. One of the three consecutive numbers $n-1$, n, and $n+1$ must be divisible by 3; and since $n-1$ and $n+1$ are both even one must be divisible by 2 and the other by 4: thus $(n-1)n(n+1)$ must be divisible by 24.

24. Since n is not divisible by 3 it is of one of the forms $3m\pm 1$; and m is even since n is odd. Then $n^2+5=9m^2\pm 6m+6$; and this is divisible by 6.

25. n^4-1 is divisible by 5 by Fermat's Theorem, since n is prime to 5; also $n^4-1=(n-1)(n+1)(n^2+1)$; now $(n-1)(n+1)$ is divisible by 8, and n^2+1 is divisible by 2. Again $(n-1)(n+1)$ is divisible by 3, for $(n-1)n(n+1)$ is divisible by 3, and n is a prime number not equal to 3. Thus n^4-1 is divisible by $5\times 8\times 2\times 3$.

26. $\dfrac{m^5-5m^3+4m}{120}=\dfrac{(m-2)(m-1)m(m+1)(m+2)}{120}$; then apply Art. 710.

27. We see, from the demonstration given of Fermat's Theorem, that $n(n^6-1)$ is divisible by 7. Also n^6-1 is divisible by n^2-1, so that $n(n^6-1)$ is divisible by $(n-1)n(n+1)$, and therefore by 6; see Art. 710. Hence $n(n^6-1)$ is divisible by 7×6, that is by 42.

28. $x^n-x=x(x^{n-1}-1)$; now this is a multiple of n as we see from the demonstration given of Fermat's Theorem: hence x^n and x when divided by n must leave the same remainder.

29. By Fermat's Theorem $N^{n-1}=1+pn$; raise both sides to the power n; thus $N^m=(1+pn)^n$; therefore $N^m-1=n(pn)+\frac{n(n-1)}{2}(pn)^2+......$; every term on the right-hand side is divisible by n^2.

30. n^6-1 is divisible by 7 by Fermat's Theorem; also $n^6-1=(n^3+1)(n^3-1)$. These two factors are both even, and therefore one is divisible by 2 and the other by 4.

31. $N^{n-1}-1$ is divisible by n by Fermat's Theorem; and since n is a prime number greater than 2 it is an odd number: suppose $n=2m+1$; then $N^{n-1}-1=(N^m+1)(N^m-1)$. These two factors are both even, and therefore one is divisible by 2 and the other by 4.

32. $N^n - N = N(N^{n-1} - 1)$. The second factor is divisible by n by Fermat's Theorem. The first or the second factor is divisible by 2, according as N is even or odd. If N is a multiple of 3 the first factor is divisible by 3; if N is not a multiple of 3 it must be of one of the forms $3m \pm 1$; and in both cases $N^{n-1} - 1$ is divisible by 3. Hence the product is divisible by $6n$. The n which we obtain as a factor of $N^{n-1} - 1$ cannot coincide with the 2 or the 3, because by supposition n is greater than 3.

33. $N^{n-1} - 1$ is divisible by n by Fermat's Theorem. And since $n - 1$ is an even number $N^{n-1} - 1$ is divisible by 8: see Example 31. And since N is a prime number greater than 3 it must be of one of the forms $3m \pm 1$; and thus $N^{n-1} - 1$ is divisible by 3.

34. Let x^n represent any term of the series; then by Fermat's Theorem $x^n = x +$ a multiple of n. Hence by addition we find that the given series is equal to $\frac{rn(rn+1)}{2}$ increased by some multiple of n. And $\frac{rn(rn+1)}{2}$ is also a multiple of n if n be greater than 2.

35. Let N denote any number; if N is a multiple of 11 so also is N^{10}; if N is not a multiple of 11 then, by Fermat's Theorem, $N^{10} - 1$ is a multiple of 11.

36. Let N denote any number; if N is a multiple of 13 so also is N^{12}; if N is not a multiple of 13 then, by Fermat's Theorem, $N^{12} - 1$ is a multiple of 13.

37. Let N denote any number; if N is a multiple of 19 so also is N^9; if N is not a multiple of 19 then, by Fermat's Theorem, $N^{18} - 1$ is a multiple of 19: therefore either $N^9 + 1$ or $N^9 - 1$ is a multiple of 19.

38. Let N denote any number; if N is a multiple of 23 so also is N^{11}; if N is not a multiple of 23 then, by Fermat's Theorem, $N^{22} - 1$ is a multiple of 23: therefore either $N^{11} + 1$ or $N^{11} - 1$ is a multiple of 23.

39. Let N denote any number; if N is a multiple of 5 then the square or any higher power of N is a multiple of 25; if N is not a multiple of 5 then, by Fermat's Theorem, $N^4 = 1 + 5p$: therefore $N^{20} = (1+5p)^5 = 1 + 5(5p) + \ldots$; this is of the form $1 + 25n$.

40. $140 = 2^2 . 5 . 7$; $2^2 . 5 . 7 \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{7}\right) = 48.$

41. $360 = 2^3 . 3^2 . 5$; $2^3 . 3^2 . 5 \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) = 96.$

42. $1000 = 2^3 5^3$; $2^3 5^3 \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right) = 400.$

43. $3^4 . 7^2 . 11 \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{7}\right)\left(1 - \frac{1}{11}\right) = 22680.$

44. $2^n . 5^n \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right) = 2^{n+1} 5^{n-1}.$

45. $140 = 2^2 . 5 . 7$; $(2+1)(1+1)(1+1) = 12.$

46. $1845 = 3^2 . 5 . 41$; $(2+1)(1+1)(1+1) = 12.$

47. $\lfloor 9 = 2^7 . 3^4 . 5 . 7$; $(7+1)(4+1)(1+1)(1+1) = 160$.

$$\frac{(2^8-1)(3^5-1)(5^2-1)(7^2-1)}{(2-1)(3-1)(5-1)(7-1)} = 1481040.$$

48. By Art. 723 and Example 46 the number $= \frac{12}{2} = 6$.

49. $100800 = 2^6 . 3^2 . 5^2 . 7$. The number of divisors of this number $= (6+1)(2+1)(2+1)(1+1) = 126$.

50. Four right angles contain 360 degrees; $360 = 2^3 . 3^2 . 5$; the number of divisors of this number $= (3+1)(2+1)(1+1) = 24$.

Four right angles contain 400 grades; $400 = 2^4 . 5^2$; the number of divisors of this number $= (4+1)(2+1) = 15$.

51. $10^n = 2^n . 5^n$. In the equation $xy = 10^n$ we may put x equal to any divisor of 10^n, and then the value of y can be found: thus the number of solutions by Art. $722 = (n+1)(n+1)$.

52. Let $N = a^p b^q c^r \ldots$ where a, b, c, ... are prime numbers. We shall determine the power of a which will occur in P. Let ϖ denote the number of divisors of $b^q c^r \ldots$; then it is obvious that N has ϖ divisors in which a does not occur, ϖ divisors in which a occurs, ϖ divisors in which a^2 occurs, and so on down to ϖ divisors in which a^p occurs. Hence the exponent of a in P is $\varpi(1+2+3+\ldots+p)$, that is $\frac{\varpi p(p+1)}{2}$, that is $\frac{np}{2}$. Similarly the exponent of b is $\frac{nq}{2}$. And so on. Thus $P = N^{\frac{n}{2}}$. And n is even except in the case in which p, q, r, ... are *all* even: thus $N^{\frac{n}{2}}$ is always rational, and so N^n is a perfect square.

53. Suppose $a^p b^q c^r \ldots$ to be a number with 30 divisors, where a, b, c, ... are prime numbers; then $(p+1)(q+1)(r+1) = 30$. This admits of various solutions, as $p=4$, $q=2$, $r=1$; $p=5$, $q=4$, $r=0$; ... Thus there cannot be more than three different prime factors; though there may be fewer. We now take $a=2$, $b=3$, $c=5$; for these are the lowest prime numbers; then by trial we find that $2^4 . 3^2 . 5$ is the least number with 30 divisors.

54. This like Example 53 must be solved by trial. $64 = 2^6$; thus we see that a number which has 64 divisors cannot have more than 6 different prime factors; though it may have fewer. The least number which has 6 different prime factors and 64 divisors is $2 . 3 . 5 . 7 . 11 . 13$. The least number which has 5 different prime factors and 64 divisors is $2^3 . 3 . 5 . 7 . 11$: this is smaller than the former number. Then taking 4 different prime factors we see that we have to choose between $2^7 . 3 . 5 . 7$ and $2^3 . 3^3 . 5 . 7$; and taking 3 different prime factors we have to choose between $2^7 . 3^3 . 5$ and $2^3 . 3^3 . 5^3$: of these four numbers $2^3 . 3^3 . 5 . 7$ is the smallest and it is smaller than $2^3 . 3 . 5 . 7 . 11$. And then we may shew that $2^3 . 3^3 . 5 . 7$ is also smaller than any number with 64 divisors which has two different prime factors, or only one prime factor.

55. When pa is divided by b let the quotient be n and the remainder r; thus $\frac{pa}{b}=n+\frac{r}{b}$. When $(b-p)n$ is divided by b let the quotient be n' and the remainder r'; thus $\frac{(b-p)a}{b}=n'+\frac{r'}{b}$. Hence by addition $a=n+n'+\frac{r+r'}{b}$. Neither r nor r' can be zero, for a is prime to b; therefore $r+r'$ must be equal to b, and $n+n'$ equal to $a-1$.

56. Every number will be of one of the following n forms: pn, $pn+1$, $pn+2$, ... $pn+(n-1)$. Now $(pn+1)^2$ and $\{pn+(n-1)\}^2$ will leave the same remainder when divided by n; for $(pn+1)^2=n(p^2n+2p)+1$, and $\{pn+(n-1)\}^2=n\{(p+1)^2n-2(p+1)\}+1$. Similarly $(pn+2)^2$ and $\{pn+(n-2)\}^2$ will leave the same remainder when divided by n. And so on. Thus there cannot be more than $\frac{n}{2}$ different remainders.

57. $2^2 . 5 . 7x=y^3$. Thus in order that y^3 may be a perfect cube x must be of the form $2 . 5^2 . 7^2t^3$; and then $y=2 . 5 . 7 . t$.

58. Every number will be of one of the following r forms; nr, $nr+1$, $nr+2$,...$nr+(r-1)$. Then $(nr)^2$ will terminate with the digit 0; and $(nr+1)^2$, $(nr+2)^2$,... will terminate respectively with the same digits as 1^2, 2^2,... Thus by Example 16 there are $\frac{r-1}{2}$ different digits without counting 0; and counting 0 we have $\frac{r+1}{2}$ digits on the whole.

59. Suppose for example that $p=4$; let $n=a^2+b^2+c^2+d^2$: then we have to shew that $6n$ can be resolved into 12 squares. In fact we have

$$6n=(a+b)^2+(a-b)^2+(a+c)^2+(a-c)^2+(a+d)^2+(a-d)^2$$
$$+(b+c)^2+(b-c)^2+(b+d)^2+(b-d)^2+(c+d)^2+(c-d)^2.$$

In this way we may establish the result for any value of p.

60. $$2^{2n}+15n-1=15n-1+(1+3)^n=15n-1+1+3n+\frac{n(n-1)}{2}3^2+...$$
$$=18n+\frac{n(n-1)}{2}3^2+\frac{n(n-1)(n-2)}{\underline{|3}}3^3+...$$

Every term is divisible by 9.

61. Multiply out $(x+1)(x+2)......(x+n)$. By Art. 504, the product will be $x^n+P_1x^{n-1}+P_2x^{n-2}+......+P_n$. Put $x=1$; then the expression becomes equal to $\underline{|n+1}$: this is divisible by $\underline{|n}$, and after subtracting P_n which is equal to $\underline{|n}$ the remainder will also be divisible by $\underline{|n}$.

62. Let N denote any number; if N is a multiple of 5, then the cube or any higher power of N is a multiple of 125; if N is not a multiple of 5 then, by Fermat's Theorem, $N^4=1+5p$: therefore $N^{100}=(1+5p)^{25}=1+25.5p+..$; this is of the form $1+125n$.

LIII.

1. The probability of the first event is $\frac{2}{5}$, that of the second $\frac{4}{7}$; the probability of the happening of both is therefore $\frac{8}{35}$: thus the odds are 27 to 8 against their happening together.

2. The probability that both will be dead is $\frac{8}{15} \times \frac{2}{3}$, that is $\frac{16}{45}$; therefore the probability that one at least will be alive is $\frac{29}{45}$.

3. Let A and B denote the specified individuals; the probability that B is on the right-hand side of A is $\frac{1}{22}$, and the probability that B is on the left-hand side of A is also $\frac{1}{22}$: hence the probability that B is either on the right-hand side of A or on the left-hand side is $\frac{1}{11}$. Thus it is 10 to 1 against A and B being next to each other.

4. The chance that they will both fail is $\frac{3}{4} \times \frac{1}{3}$, that is, $\frac{1}{4}$: hence the chance that they will not both fail is $\frac{3}{4}$.

5. Two black balls can be drawn in $\frac{5 \times 4}{\lfloor 2}$ ways, and one red ball in 3 ways; thus the number of favourable cases is $\frac{5 \times 4 \times 3}{\lfloor 2}$: the whole number of cases, since there are ten balls, is $\frac{10 \times 9 \times 8}{\lfloor 3}$. Divide the former by the latter, and we obtain $\frac{1}{4}$.

6. The probability of throwing an ace at the first trial and missing an ace at the second is $\frac{1}{6} \times \frac{5}{6}$, that is $\frac{5}{36}$; the probability of missing an ace at the first trial and throwing an ace at the second is also $\frac{5}{36}$; hence the whole probability is $\frac{10}{36}$.

7. The probability of failing to throw ace in two trials is $\left(\frac{5}{6}\right)^2$, that is $\frac{25}{36}$; hence the probability of not failing is $1-\frac{25}{36}$, that is, $\frac{11}{36}$.

8. Since $7=6+1=5+2=4+3=3+4=2+5=1+6$ there are six cases favourable to throwing 7; there are two cases favourable to throwing 11, for $11=6+5=5+6$: thus on the whole there are 8 favourable cases, where the

whole number of cases is 36. Therefore the probability of success is $\frac{8}{36}$, that is $\frac{2}{9}$. Thus the odds are 7 to 2 against.

9. Suppose that there are a sovereigns in each purse, b shillings in the first purse and c shillings in the second. The probability of taking either purse is $\frac{1}{2}$; and thus the probability of drawing a sovereign is $\frac{1}{2}\left(\frac{a}{a+b}+\frac{a}{a+c}\right)$. If all the coins are put into one purse the probability of drawing a sovereign is $\frac{2a}{2a+b+c}$. If we subtract the second expression from the first we shall obtain after reduction $\frac{a(b-c)^2}{2(a+b)(a+c)(2a+b+c)}$: thus the first expression is the greater.

10. It is equally probable that B, or C, or D will be in the same boat with A; if A is with B or C he loses, if with D it is an even chance whether A wins or loses. Similarly we may consider the cases of B, C, and D.

11. There are 36 cases; of these 16 are unfavourable, namely the 6 doublets, and the 10 cases in which ace on one die occurs with not-ace on the other. Thus there are left 20 favourable cases. Therefore the chance is $\frac{20}{36}$.

12. The whole number of cases is the number of the combinations of 30 things taken 4 at a time, that is $\frac{30.29.28.27}{\lfloor 4}$. The favourable cases are those in which the tickets marked 1 and 2 are drawn with any pair of tickets from the remaining 28; so the number of favourable cases is $\frac{28.27}{\lfloor 2}$. Hence dividing this number by the former we obtain for the chances $\frac{3.4}{30.29}$.

13. In the first lottery we must suppose that there are 9 tickets of which 6 are blanks. The number of ways in which 3 tickets can be drawn is $\frac{9.8.7}{\lfloor 3}$, that is 84. The number of ways in which A gets no prize at all is the number of ways in which 3 tickets can be drawn from the 6 blanks; that is $\frac{6.5.4}{\lfloor 3}$, that is 20. Hence there are 64 favourable cases in which A gets one or more prizes. Thus A's chance $=\frac{64}{84}=\frac{16}{21}$. Or if we suppose each ticket drawn singly we may proceed thus: A's chance of drawing a blank at first is $\frac{6}{9}$; there are now left 8 tickets of which 5 are blanks, so that A's chance of drawing a second blank is $\frac{5}{8}$; similarly A's chance of drawing a

third blank is $\frac{4}{7}$; thus A's chance of complete failure is $\frac{6}{9}\times\frac{5}{8}\times\frac{4}{7}$, that is, $\frac{5}{21}$. Hence A's chance of getting one or more prizes is $1-\frac{5}{21}$, that is, $\frac{16}{21}$. And B's chance $=\frac{1}{3}=\frac{7}{21}$.

14. The chance of drawing a white ball from the first bag is $\frac{3}{7}$; if a white ball is drawn there will be in the second bag 4 white balls and 4 black balls: thus the chance of drawing a white ball is $\frac{4}{8}$. Therefore the chance of drawing two white balls $=\frac{3}{7}\times\frac{4}{8}=\frac{3}{14}$.

15. We require the head to occur once, or three times, or five times, ... Thus the chance is the sum of the second, fourth, sixth, ... terms in the expansion of $\left(\frac{1}{2}+\frac{1}{2}\right)^n$; and is therefore $\frac{1}{2}\left\{\left(\frac{1}{2}+\frac{1}{2}\right)^n-\left(\frac{1}{2}-\frac{1}{2}\right)^n\right\}$, that is $\frac{1}{2}$.

16. The number of cases is 2^n; the number of favourable cases is n; for the solitary head may be on any one of the coins. Hence the chance is $\frac{n}{2^n}$.

17. Let a denote the number of the combinations of $m+n$ things taken $p+q$ at a time; let b denote the number of the combinations of m things taken p at a time; let c denote the number of the combinations of n things taken q at a time. Then the whole number of cases is a, and the number of favourable cases is bc. Thus the probability is $\frac{bc}{a}$.

18. In Art. 740 put $p=\frac{1}{36}$, $q=\frac{35}{36}$, $n=6$, $r=4$.

Thus we have $\frac{1}{(36)^6}\left\{1+6.35+15.(35)^2\right\}$, that is $\frac{18586}{(36)^6}$.

19. In Art. 740 put $p=\frac{1}{6}$, $q=\frac{5}{6}$, $n=6$, $r=1$.

Thus we have $\frac{1}{6^6}\left\{1+6.5+15.5^2+20.5^3+15.5^4+6.5^5\right\}$, that is $\frac{31031}{6^6}$.

20. In Art. 740 put $p=\frac{9}{10}$, $q=\frac{1}{10}$, $n=5$, $r=3$.

Thus we have $\frac{1}{10^5}\left\{9^5+5.9^4+10.9^3\right\}$, that is $\frac{12393}{12500}$.

21. The chance of throwing no doublets is $\left(\frac{5}{6}\right)^3$; thus the chance of throwing one or more doublets is $1-\left(\frac{5}{6}\right)^3$.

22. The chance of throwing no double sixes is $\left(\frac{35}{36}\right)^3$; thus the chance of throwing one or more double sixes is $1-\left(\frac{35}{36}\right)^3$.

23. In Art. 740 put $p=\frac{1}{7}$, $q=\frac{6}{7}$, $n=5$, $r=2$.

Thus we have $\frac{1}{7^5}\left\{1+5.6+10.6^2+10.6^3\right\}$, that is, $\frac{2551}{7^5}$. We must observe that the number of the tickets is supposed *so large* that p remains practically equal to $\frac{1}{7}$ after one or two tickets have been drawn out. Compare this with the second way in which A's chance is calculated in the solution of Example 13.

24. The whole number of cases is the number of the combinations of 52 things taken 4 at a time, that is $\frac{52.51.50.49}{\underline{|4}}$. The number of favourable cases is 13^4; for any one card may be taken from any suit. Divide the latter number by the former, and we obtain the probability.

25. The whole number of cases is $\frac{52.51.50.49}{\underline{|4}}$; the number of favourable cases is 4, since there are 4 suits. Divide 4 by the whole number of cases and we obtain the probability.

26. In Art. 741 we put $p=\frac{2}{3}$, $q=\frac{1}{3}$, $m=4$, $n=1$; thus the probability of A's winning 4 games out of 5 is $\left(\frac{2}{3}\right)^4\left\{1+4.\frac{1}{3}\right\}$, that is $\frac{112}{243}$; so that the odds against it are 131 to 112.

27. In Art. 740 put $p=\frac{2}{5}$, $q=\frac{3}{5}$, $n=5$, $r=2$.

Thus we have $\frac{1}{5^5}\left\{2^5+5.2^4.3+10.2^3.3^2+10.2^2.3^3\right\}$, that is $\frac{2072}{5^5}$.

28. Call the persons A, B, C. Then A's chance of drawing a white ball at first is $\frac{3}{8}$; the chance of his failing is $\frac{5}{8}$, and then there are left 3 white balls and 4 black balls: thus the chance of B's drawing a white ball is $\frac{5}{8}\times\frac{3}{7}$. The chance that both A and B fail is $\frac{5}{8}\times\frac{4}{7}$, and then there are left 3 white balls and 3 black balls: thus the chance of C's drawing a white ball is $\frac{5}{8}\times\frac{4}{7}\times\frac{3}{6}$. Then if A, B, C all fail A has a second draw; and so on. Thus we find on the whole that A's chance is $\frac{3}{8}+\frac{5}{8}\times\frac{4}{7}\times\frac{3}{6}\times\frac{3}{5}$, that is $\frac{27}{56}$; B's chance is $\frac{5}{8}\times\frac{3}{7}+\frac{5}{8}\times\frac{4}{7}\times\frac{3}{6}\times\frac{2}{5}\times\frac{3}{4}$, that is $\frac{18}{56}$; and C's chance is $\frac{5}{8}\times\frac{4}{7}\times\frac{3}{6}+\frac{5}{8}\times\frac{4}{7}\times\frac{3}{6}\times\frac{2}{5}\times\frac{1}{4}\times\frac{3}{3}$, that is $\frac{11}{56}$.

29. To win three or more games out of the next four, either all four games must be won, or the first, the second, the third, or the fourth game must be lost. The chances for these five cases are respectively

$$\frac{2.2.2.2}{3^4},\quad \frac{1.1.2.2}{3^4},\quad \frac{2.1.1.2}{3^4},\quad \frac{2.2.1.1}{3^4},\quad \frac{2.2.2.1}{3^4};$$

the sum of these fractions is $\frac{36}{81}$, that is $\frac{4}{9}$.

30. In order that the r^{th} person may have a throw the preceding $r-1$ persons must all fail: thus $\left(\frac{n-1}{n}\right)^{r-1}\frac{1}{n}$ is the chance that he will win the stake at a first throw. If he fails and then all the other persons fail also he gets a second throw; so that his chance of winning the stake at a second throw is $\left(\frac{n-1}{n}\right)^{p+r-1}\frac{1}{n}$. And so on. Thus his whole chance is an infinite geometrical progression of which the first term is $\frac{1}{n}\left(\frac{n-1}{n}\right)^{r-1}$, and the common ratio is $\left(\frac{n-1}{n}\right)^{p}$.

31. The chance that the particular parcel is brought is $\frac{2}{3}$; the chance that the required book is obtained from the particular parcel is $\frac{2}{3}\times\frac{3}{5}$, that is $\frac{2}{5}$. There are two other parcels; the chance that one of the other parcels is brought is $\frac{1}{6}$, and the chance that the required book is obtained from that parcel is $\frac{1}{6}\times\frac{1}{5}$: thus on account of the two other parcels we have the chance $\frac{2}{30}$, that is $\frac{1}{15}$. Hence the whole chance $=\frac{2}{5}+\frac{1}{15}=\frac{7}{15}$.

32. We must find the probability that the sovereign is in the second purse. When 9 coins are taken from the first purse the probability that the sovereign is taken out is $\frac{9}{10}$. If the sovereign is taken out we have then in the second purse 18 shillings and 1 sovereign. Let 9 coins be taken out: then there are 9 cases favourable to the drawing out of the sovereign, and 10 cases favourable to its remaining behind: thus the probability that the sovereign remains is $\frac{10}{19}$. Therefore the probability that the sovereign is finally in the second purse is $\frac{9}{10}\times\frac{10}{19}$, that is $\frac{9}{19}$. Hence the probability that the sovereign is finally in the first purse is $\frac{10}{19}$.

33. The following four arrangements are equally probable for the first urn after the first drawing: 4 black balls and 5 white, 5 black balls and 4

white, 6 black balls and 5 white, 5 black balls and 6 white. Thus the chance of drawing a white ball from this urn is $\frac{1}{4}\left\{\frac{4}{9}+\frac{5}{9}+\frac{5}{11}+\frac{6}{11}\right\}$, that is $\frac{1}{2}$. Similarly the chance is $\frac{1}{2}$ for the second urn. And thus the chance is $\frac{1}{2}$ whichever urn be chosen.

34. The number of cases is 6^3; the number of favourable cases is the coefficient of x^{15} in the expansion of $(x+x^2+x^3+x^4+x^5+x^6)^3$; see Art. 742. This coefficient is the same as that of x^{12} in the expansion of $\left(\frac{1-x^6}{1-x}\right)^3$, that is in the expansion of $(1-x^6)^3(1-x)^{-3}$. Thus the coefficient is $\frac{13.14}{1.2}-3\frac{7.8}{1.2}+3$, that is 10.

35. Proceed as in Example 34. The number of favourable cases is the coefficient of x^{14} in the expansion of $(1-x^6)^3(1-x)^{-3}$; this coefficient is $\frac{15.16}{1.2}-3.\frac{9.10}{1.2}+3.\frac{3.4}{1.2}$, that is 3.

36. By the method of Examples 34 and 35 we find that the numbers of cases favourable to throwing 3, 4, 5, 6, 7, 8, 9, 10 respectively are 1, 3, 6, 10, 15, 21, 25, 27; the sum is 108; the whole number of cases is 6^3, that is 216. Therefore the chance is $\frac{1}{2}$.

37. The number most likely to be turned up is the index of that power of x which has the greatest coefficient in the expansion of $(x+x^2+x^3+x^4+x^5+x^6)^{2n}$, that is in the expansion of $x^{2n}(1+x+x^2+x^3+x^4+x^5)^{2n}$: and by Example 38 of Chapter XXXVII. the greatest coefficient is that of $x^{2n}\times x^{5n}$, that is the coefficient of x^{7n}.

38. By the same method as in the preceding Example we find that the chance of turning up any number, m, is measured by the coefficient of x^m in the expansion of $x^{2n+1}(1+x+x^2+x^3+x^4+x^5)^{2n+1}$: then by Example 38 of Chapter XXXVII. we see that the coefficient is the same for $m=7n+3$ as for $m=7n+4$, and greater than for any other value of m.

39. By the method of Art. 742 we must find the coefficient of x^{24} in the expansion of $(x+x^2+...+x^{10})^{10}$, and divide it by 10^{10}. The coefficient is the same as that of x^{14} in the expansion of $\left(\frac{1-x^{10}}{1-x}\right)^{10}$; which will be found to be $\frac{\underline{|23}}{\underline{|9}\,\underline{|14}}-\frac{10\,\underline{|13}}{\underline{|4}\,\underline{|9}}$.

40. The probability of failing to obtain s is $\frac{n}{n+1}$ at each trial; and thus the probability of failing in all the m trials is $\left(\frac{n}{n+1}\right)^m$: subtract this from unity and we have the probability of success.

41. By the method of Art. 742 we must find the coefficient of x^{10} in the expansion of $(1+1+1+1+1+x+x^2+x^3+x^4+x^5)^3$; and divide it by 10^3. Now $(5+x+x^2+x^3+x^4+x^5)^3=5^3+3.5^2y+3.5.y^2+y^3$ where $y=x+x^2+x^3+x^4+x^5$.

There is no term involving x^{10} in 5^3 or in 3.5^2y; the coefficient of x^{10} in $3.5.y^2$ is 15; the coefficient of x^{10} in y^3 will be found to be 18. Thus the whole coefficient is 33.

42. The probability is now the same as when 3 tickets are drawn *simultaneously* from the set of 10. The whole number of cases is the same as the number of the combinations of 10 things taken 3 at a time, that is $\frac{10.9.8}{\underline{|3}}$, that is 120. There are 2 favourable cases, namely that in which the tickets marked 1, 4, 5 are drawn, and that in which the tickets marked 2, 3, 5 are drawn. Thus the probability $= \frac{2}{120}$.

43. The whole number of cases is $\frac{\underline{|n}}{\underline{|p}\,\underline{|n-p}} \times \frac{\underline{|n}}{\underline{|q}\,\underline{|n-q}}$. Fix upon a particular set of r balls; the number of ways in which this particular set can occur in the first drawing is the same as the number of the combinations of the remaining $n-r$ balls taken $p-r$ at a time; that is the number is $\frac{\underline{|n-r}}{\underline{|p-r}\,\underline{|n-p}}$. In the second drawing we want to have the r balls upon which we have fixed together with $q-r$ balls which did not occur in the first drawing; so the $q-r$ balls must be taken from the $n-p$ balls which did not occur in the first drawing: the number of ways in which this can be done is $\frac{\underline{|n-p}}{\underline{|q-r}\,\underline{|n-p-q+r}}$. And there are $\frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$ ways in which we can fix upon a particular set of r balls. Thus the number of favourable cases $=$

$$\frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} \times \frac{\underline{|n-r}}{\underline{|p-r}\,\underline{|n-p}} \times \frac{\underline{|n-p}}{\underline{|q-r}\,\underline{|n-p-q+r}} = \frac{\underline{|n}}{\underline{|r}\,\underline{|p-r}\,\underline{|q-r}\,\underline{|n-p-q+r}}.$$

44. Denote the two assigned persons by A and B. There are three cases in which A and B will play together. I. A may be opposed to B in the four games: the chance is $\frac{1}{7}$. II. A and B may have other opponents in the four games, may each vanquish his opponent, and may then be opposed in the two games: the chance is $\frac{6}{7}.\left(\frac{1}{2}\right)^2.\frac{1}{3}$. III. A and B may have other opponents in the four games, may each vanquish his opponent, may then have other opponents in the two games, and each vanquish his opponent: the chance is $\frac{6}{7} \times \left(\frac{1}{2}\right)^2 \times \frac{2}{3} \times \left(\frac{1}{2}\right)^2$.

45. The chance of a white ball at the first drawing is $\frac{m}{m+n}$; then the chance of a black ball at the second drawing is $\frac{n}{m+n-1}$; then the chance of a white ball at the third drawing is $\frac{m-1}{m+n-2}$: and so on.

In the second part of the Example the whole number of cases is the number of the combinations of $m+n$ things taken m at a time; and there is only one favourable case.

46. The chance that the first ball drawn is of the first colour is $\frac{p_1}{n}$; then the chance that the second ball drawn is of the first colour is $\frac{p_1-1}{n-1}$; and so on.

47. As any ball may be taken out each time, the whole number of cases is n^n. The number of favourable cases is the number of permutations of n things taken all together, that is $\underline{|n}$.

48. A in order to win the set must win 2 games before B wins 3. Thus by Art. 741 we find that the probability for A is $\left(\frac{1}{2}\right)^2\left\{1+2.\frac{1}{2}+\frac{2.3}{1.2}\left(\frac{1}{2}\right)^2\right\}$, that is $\frac{11}{16}$. Therefore the probability for B is $\frac{5}{16}$.

49. The number of different ways is $\underline{|3}$, that is 6. As each way is equally likely to occur there are 6^6 cases; and the number of favourable cases is $\underline{|6}$.

50. Since N is a *given* number, the number of its divisors is known, see Art. 722; call M the number of divisors, then $\frac{M}{N}$ is the chance.

51. Let N denote the number of shot in the bag, which must be supposed known; let M denote the number of positive integers which are less than N and prime to N; see Art. 721: then assuming that the handful is equally likely to contain any number not exceeding N the chance is $\frac{M}{N}$. If $N=3.5.7$ we have $M=N\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)$; therefore $\frac{M}{N}=\frac{16}{35}$.

52. The whole number of cases is n; there are $\frac{n}{a^s}$ numbers not exceeding n which contain a^s as a factor: from these we must exclude those which contain a^{s+1} as a factor, which are $\frac{n}{a^{s+1}}$ in number. Thus the number of favourable cases is $\frac{n}{a^s}-\frac{n}{a^{s+1}}$.

53. Let A's chance of winning a single game be x. Now A may win the match in 2, in 3, or in 4, ... games. Thus A's whole chance is

$$x^2+(1-x)x^2+x(1-x)x^2+(1-x)x(1-x)x^2+\dots.$$

For instance, if A wins in 4 games he must win the first game, lose the second, and win the third and fourth. Thus we see that A's whole chance consists of two infinite geometrical progressions, one having x^2 for its first term, and the other having $(1-x)x^2$ for its first term, and each having $x(1-x)$ for the common ratio. Thus we obtain $\frac{x^2+(1-x)x^2}{1-x(1-x)}$, that is $\frac{x^2(2-x)}{1-x(1-x)}$.

We may obtain B's chance by changing x into $1-x$ in this expression; or we may obtain B's chance by subtracting A's chance from unity: thus B's chance $=\frac{(1-x)^2(1+x)}{1-x(1-x)}$. If $x=\frac{2}{3}$ we get $\frac{16}{21}$ for A's chance.

This problem will be found generalised in Art. 683 of the *History of the Theory of Probability*.

54. The faces of the tetrahedron are supposed to be marked 1, 2, 3, 4 respectively; and those of the octahedron are supposed to be marked 1, 2, 3, 4, 5, 6, 7, 8 respectively. The whole number of cases is 32. There are 3 favourable cases in which the face marked 1 of the tetrahedron is thrown, 4 in which the face marked 2 is thrown, and so on: thus there are in all $3+4+5+6$, that is 18, favourable cases.

55. The probability of the failing of all the events is $(1-p_1)(1-p_2)(1-p_3)$; subtract this from unity, and we have the probability of the happening of one at least of the events.

Two events at least happen if all three happen, or if the first alone fails, or if the second alone fails, or if the third alone fails: thus the probability is $p_1p_2p_3+(1-p_1)p_2p_3+(1-p_2)p_1p_3+(1-p_3)p_1p_2$.

56. The chance of throwing 10 at a single throw is $\frac{27}{6^3}$, that is $\frac{1}{8}$: see Example 36. The chance that A wins at the first throw is therefore $\frac{1}{8}$; if A, B, C fail in succession A has a second throw; thus the chance of his winning at a second throw is $\left(\frac{7}{8}\right)^3\frac{1}{8}$. In this way we find that A's whole chance is an infinite geometrical progression of which the first term is $\frac{1}{8}$, and the common ratio is $\left(\frac{7}{8}\right)^3$. In like manner the chances of B and C are infinite geometrical progressions in which the common ratio is $\left(\frac{7}{8}\right)^3$, and the first terms are respectively $\frac{7}{8}\cdot\frac{1}{8}$ and $\left(\frac{7}{8}\right)^2\frac{1}{8}$.

57. Consider the figures which occur in a particular place, say the seventh place. The upper figure may be 0, or 1, or 2,......or 9; and so may the lower figure: and as any upper figure may occur with any lower figure there are 100 cases in all. We must now find how many cases are favourable. If the upper figure be 0 there is 1 favourable case, namely when the lower figure is also 0; if the upper figure is 1 there are 2 favourable cases, namely when the lower figure is 0 or 1; and so on: thus the number of favourable cases is $1+2+3+\ldots+9+10$, that is 55. Therefore the chance that in a given place the lower figure will not exceed the upper figure is $\frac{55}{100}$; and the chance that this will be the case in all the seven places is $\left(\frac{55}{100}\right)^7$.

58. The chances of throwing 6 and 7 at a single throw with a pair of dice are respectively $\frac{5}{36}$ and $\frac{6}{36}$: see Example 8. The chance that A wins at the first throw is therefore $\frac{5}{36}$. If A and B fail in succession A has a second throw; then the chance of his winning at a second throw is $\frac{31}{36} \cdot \frac{30}{36} \cdot \frac{5}{36}$; and so on. Thus A's whole chance is an infinite geometrical progression of which the first term is $\frac{5}{36}$; and the common ratio is $\frac{31}{36} \cdot \frac{5}{6}$. Similarly B's whole chance is an infinite geometrical progression having the same common ratio of which the first term is $\frac{31}{36} \cdot \frac{1}{6}$.

59. He may draw two sovereigns, or a sovereign and a shilling, or two shillings; the respective chances are $\frac{3}{14}, \frac{4}{7}, \frac{3}{14}$: thus the expectation in shillings is $\frac{3}{14} \times 40 + \frac{4}{7} \times 21 + \frac{3}{14} \times 2$, that is 21. Or thus: as 2 coins are to be drawn out the chance that a particular one is drawn is $\frac{1}{4}$; therefore the expectation is $\frac{1}{4}$ of a sovereign for each of the 4 sovereigns, and $\frac{1}{4}$ of a shilling for each of the 4 shillings; that is the sum of a sovereign and a shilling.

60. The chance that a particular coin is drawn out is $\frac{1}{6}$; therefore the expectation is $\frac{1}{6}$ of a guinea for each of the 6 guineas, $\frac{1}{6}$ of a sovereign for each of the 6 sovereigns, and $\frac{1}{6}$ of a shilling for each of the 6 shillings; that is the sum of a guinea, a sovereign, and a shilling.

61. There are 36 ships; the number of the combinations of them taken two at a time is $\frac{36 \cdot 35}{1 \cdot 2}$, that is $18 \cdot 35$. Hence the chance that one of the first two ships which arrive is a Russian ship and the other a French ship is $\frac{10 \cdot 12}{18 \cdot 35}$. Therefore the expectation in pounds is $\frac{10 \cdot 12}{18 \cdot 35} \times 2100$, that is 400.

62. As in Examples 59 and 60 the expectation is in shillings $\frac{3}{9}(63+40+4)$, that is $\frac{107}{3}$, that is $35\frac{2}{3}$.

63. The expectation in pounds is $\frac{1}{100}\left\{4 \times 100 + 10 \times 50 + 20 \times 5\right\}$, that is $\frac{1}{100} \times 1000$, that is 10.

64. Let each of the coins be worth x shillings. The expectation in shillings is $\frac{2}{9}\left(100+4x\right)$: this is to be equal to 24; so that we obtain $x=2$.

65. Suppose the gold coin to be worth x shillings, and each silver coin to be worth y shillings. The expectation in shillings is $\frac{4}{8}\left\{x+3y+4\right\}$; this is to be equal to 15: therefore $x+3y=26$. The only admissible solution is $x=20$, $y=2$.

66. As in Example 58 we find that the chances of A and B are infinite geometrical progressions of which the common ratio is $\frac{1}{4}$; the first term in A's chance is $\frac{1}{2}$, and the first term in B's chance is $\frac{1}{4}$. Thus A's chance is $\frac{2}{3}$ and B's chance is $\frac{1}{3}$.

Suppose that each stakes £a, and that A gives £x to B for the first throw. Then A's expectation in pounds is $\frac{2}{3}\times 2a-x$; and B's expectation is $\frac{1}{3}\times 2a+x$. Equating these we get $2x=\frac{2a}{3}$; therefore $x=\frac{a}{3}$.

67. As there are m counters marked m the expectation which arises from these counters in shillings is $\frac{m^2}{n}$, where n is the whole number of counters. Thus the whole expectation in shillings is $\frac{1}{n}\left\{1^2+2^2+\dots\dots+r^2\right\}$, that is $\frac{1}{n}\frac{r(r+1)(2r+1)}{6}$; and $n=1+2+\dots\dots+r=\frac{r(r+1)}{2}$.

68. As in the preceding Example the whole expectation in shillings is $\frac{1}{n}\left\{1^3+2^3+\dots\dots+r^3\right\}$, where $n=1^2+2^2+\dots\dots+r^2$. Then see Art. 461.

69. Consider the expectation which arises from the 5; this 5 may be in shillings 5 or 50 or 500 or 5000 or 50000, one case being as likely as another: thus the expectation is $\frac{5(1+10+100+1000+10000)}{5}$, that is $\frac{5\times 11111}{5}$. Similarly we proceed with the 1, the 2, the 3, and the 4. Thus the whole expectation in shillings is $\frac{1+2+3+4+5}{5}\times 11111$.

70. Suppose any particular card is marked with the number m; then the chance of drawing this card is $\frac{b}{a}$, and the expectation in shillings arising from this card is $\frac{bm}{a}$. And thus the expectation from all the cards is $\frac{bn}{a}$.

71. The chance of drawing a white ball from the first urn is $\frac{8}{12}$, the chance of drawing a white ball from the second urn is $\frac{4}{16}$: thus if a white ball has been drawn the chance that it came from the first urn is to the chance it came from the second as $\frac{2}{3}$ is to $\frac{1}{4}$, that is as 8 is to 3. Therefore the chance that it came from the first urn is $\frac{8}{11}$.

72. Before the observed event we suppose that any number of white balls is equally likely. Now the probability of drawing two white balls is 1 on the hypothesis that there are 5 white balls, $\frac{3}{5}$ on the hypothesis that there are 4 white balls, $\frac{3}{10}$ on the hypothesis that there are 3 white balls, and $\frac{1}{10}$ on the hypothesis that there are 2 white balls. Hence after the observed event the probability of the first hypothesis is $1 \div \left\{1 + \frac{3}{5} + \frac{3}{10} + \frac{1}{10}\right\}$, that is $\frac{1}{2}$.

73. Before the observed event we suppose that any number of sovereigns is equally likely. Now the probability of drawing a sovereign is $\frac{1}{n}$ on the hypothesis that there is 1 sovereign, $\frac{2}{n}$ on the hypothesis that there are 2 sovereigns, and so on. Hence after the observed event the probability of the first hypothesis is $\frac{1}{n} \div \left\{\frac{1}{n} + \frac{2}{n} + \ldots + \frac{n}{n}\right\}$, that is $1 \div \frac{n(n+1)}{2}$.

74. There are two hypotheses as to the two balls put into the smaller bag; either both are white, or one is white and one black. The probability of drawing two white balls from the first bag is $\frac{3}{14}$; the probability of drawing one white ball and one black is $\frac{4}{7}$. The probability that two successive drawings from the smaller bag will produce a white ball is 1 on the first hypothesis and $\frac{1}{4}$ on the second. Hence after the observed event the probability of the first hypothesis is $\frac{3}{14} \times 1 \div \left\{\frac{3}{14} \times 1 + \frac{4}{7} \times \frac{1}{4}\right\}$, that is $\frac{3}{5}$.

75. The probability of drawing 4 sovereigns from the first purse is 1; the probability of drawing 4 sovereigns from the second purse is $\frac{10.9.8.7}{25.24.23.22}$, that is $\frac{21}{1265}$. Hence the probability that the 4 sovereigns came from the first purse is $1 \div \left\{1 + \frac{21}{1265}\right\}$, that is $\frac{1265}{1286}$. Also the proba-

bility that the 4 sovereigns came from the second purse is $\frac{21}{1286}$. Put p for $\frac{1265}{1286}$. Then there is the probability p that the next coin will be drawn from a purse containing 21 sovereigns; this gives an expectation of p pounds. There is the probability $1-p$ that the next coin will be drawn from a purse containing 6 sovereigns and 15 shillings; this gives an expectation of $(1-p)\left\{£\frac{6}{21}+\frac{15}{21}s.\right\}$, that is $(1-p)\frac{6\frac{3}{4}}{21}$ pounds. Hence the whole expectation in pounds is $\frac{1265}{1286}+\frac{27}{4\times 1286}$.

76. There are six hypotheses with respect to the notes to be regarded as equally probable before the observed event. I. Three £5. II. Two £5, one £10. III. Two £5, one £20. IV. One £5, two £10. V. One £5, two £20. VI. One £5, one £10, one £20. The probability of the observed event on these hypotheses is respectively

$$1,\ \left(\frac{2}{3}\right)^3,\ \left(\frac{2}{3}\right)^3,\ \left(\frac{1}{3}\right)^3,\ \left(\frac{1}{3}\right)^3,\ \left(\frac{1}{3}\right)^3.$$

Hence after the observed event the probabilities of the hypotheses are respectively $\frac{27}{46}$, $\frac{8}{46}$, $\frac{8}{46}$, $\frac{1}{46}$, $\frac{1}{46}$, $\frac{1}{46}$. The probable value of the contents in pounds is therefore $\frac{1}{46}\left\{27\times 15+8\times 20+8\times 30+25+45+35\right\}$, that is $\frac{910}{46}$.

77. There are two possible hypotheses, that the event took place, and that it did not. The probability of A's assertion, B's assertion, and C's denial is on the first hypothesis $\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{1}{7}$, and on the second $\frac{1}{4}\cdot\frac{1}{5}\cdot\frac{6}{7}$. Hence the probability of the first hypothesis is $\frac{12}{140}\div\left\{\frac{12}{140}+\frac{6}{140}\right\}$, that is $\frac{12}{18}$.

78. This is an example of Art. 754. Here $p=\frac{3}{4}$, $p'=\frac{4}{5}$, $n=9$: thus the odds for the truth are as $\frac{3}{5}$ to $\frac{1}{160}$, that is as 96 to 1.

79. Suppose that there are 13 witnesses, and that p is the probability that each speaks the truth: let q be the *à priori* probability of the event asserted to have happened. Then by Arts. 752 and 753 the probability of the hypothesis that the event happened is to the probability of the hypothesis that it did not happen as $p^{13}q$ is to $(1-q)(1-p)^{13}$. If $p=\frac{10}{11}$ the ratio is that of $10^{13}q$ to $1-q$; and this is the ratio of 10 to 1 if $q=\frac{1}{10^{12}+1}$.

80. As there are 4 suits of cards the chance, before the observed event, that the missing card is a spade is $\frac{1}{4}$; and the chance that it is not a spade

is $\frac{3}{4}$. The chance of drawing two spades is $\frac{12.11}{51.50}$ if the missing card is a spade, and $\frac{13.12}{51.50}$ if it is not. Hence after the observed event the chance that the missing card is a spade is

$$\frac{1}{4}\cdot\frac{12.11}{51.50} \div \left\{\frac{1}{4}\cdot\frac{12.11}{51.50}+\frac{3}{4}\cdot\frac{13.12}{51.50}\right\}, \text{ that is } \frac{11}{50}.$$

81. Let us first find the chance that the persons do not meet. Let A denote the first person, and B the second. If they do not meet either A must reach B's end before B starts, or B must reach A's end before A starts: we shall shew that the chance of each event is $\frac{c^2}{2(a+c)(b+c)}$. A must arrive at B's end at some instant during the last $b+c$ minutes; and B must leave his end at some instant during the first $a+c$ minutes. Hence $\frac{c^2}{(a+c)(b+c)}$ is the chance that A's arrival and B's departure both occur during the c minutes which are *common* to the two intervals; and since one event is as likely to occur before as after the other $\frac{c^2}{2(a+c)(b+c)}$ is the chance that A's arrival occurs before B's departure. Similarly we have the same chance that B's arrival at A's end occurs before A's departure. Hence finally the chance that they *do* meet is $1-\frac{c^2}{(a+c)(b+c)}$.

82. We shall first shew that when n odd numbers are multiplied together the chance that the last figure is not 5 is $\left(\frac{4}{5}\right)^n$. An odd number must end with 1, 3, 5, 7, or 9: thus $\frac{4}{5}$ is the chance that it does not end with 5. Now in order that the product of n odd numbers may end with a figure which is not 5 every one of the n factors must end with a figure which is not 5. The chance then is $\left(\frac{4}{5}\right)^n$. Hence the chance that the product does end with 5 is $1-\left(\frac{4}{5}\right)^n$. Therefore we require that $1-\left(\frac{4}{5}\right)^n$ should not be less than $\frac{1}{2}$; thus $\left(\frac{4}{5}\right)^n$ must not be greater than $\frac{1}{2}$.

Put $\left(\frac{4}{5}\right)^x=\frac{1}{2}$, that is $\left(\frac{8}{10}\right)^x=\frac{1}{2}$: then $x\log\frac{8}{10}=\log\frac{1}{2}$; therefore $x(\log 10-\log 8)=\log 2$, that is $x=\frac{\log 2}{1-3\log 2}$. From the given value of $\log 2$ we find that x lies between 3 and 4: hence 4 is the least positive integral value of n which makes $\left(\frac{4}{5}\right)^n$ not greater than $\frac{1}{2}$.

LIV.

1. Square; $2-2\sqrt{(1-x^4)}=1-x^4$: therefore $\{\sqrt{(1-x^4)}+1\}^2=3$; therefore $\sqrt{(1-x^4)}=\pm\sqrt{3}-1$. Square again, and we find the value of x^4.

2. From the first equation $x^2=\frac{ay(y-n)}{b-y}$; substitute in the second equation; thus $y^2(a-x)=\frac{bay(y-n)}{b-y}-bnx$; therefore $ay^2-\frac{bay(y-n)}{b-y}=(y^2-bn)x$, that is $-\frac{ay(y^2-bn)}{b-y}=(y^2-bn)x$. Thus either $y^2-bn=0$ or $-\frac{ay}{b-y}=x$. If we take $y^2=bn$ we find from the second of the given equations $x^2=an$. If we take $x=\frac{ay}{y-b}$ and substitute in either of the given equations we find that y is 0 or is a root of the quadratic $y^2+(a-b-n)y+nb=0$.

3. Subtract the second equation from the first; thus $(y-z)(x+y+z)=c^2-b^2$; two similar results can be obtained by subtracting the third equation from the second, and the first from the third. Square and add; thus

$$(x+y+z)^2(x^2+y^2+z^2-yz-zx-xy)=a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2.$$

Substitute in this the value of $x^2+y^2+z^2$ which we obtain by adding together the three given equations, namely

$$x^2+y^2+z^2=\frac{1}{2}(a^2+b^2+c^2-yz-zx-xy);$$

thus $(a^2+b^2+c^2)^2-\{3(yz+zx+xy)\}^2=4(a^4+b^4+c^4)-4(b^2c^2+c^2a^2+a^2b^2)$.

Hence we can obtain the value of $yz+zx+xy$; denote it by p. Then by adding the given equations we have $2(x+y+z)^2-3p=a^2+b^2+c^2$; thus the value of $x+y+z$ can be obtained; denote it by q. Then the three results at the beginning of this solution become

$$q(y-z)=c^2-b^2,\quad q(x-y)=b^2-a^2,\quad q(z-x)=a^2-c^2;$$

so that x, y, and z can be found.

4. Square and multiply up; $(x^2-4x-8)^2=8(x^2+2x+11)$; therefore $(x^2-4x)^2-16(x^2-4x)+64=8(x^2+2x)+88$; therefore $(x^2-4x)^2=24(x-1)^2$. Extract the square root; &c.

5. By Art. 634 the number of solutions cannot differ by *more* than 1 from $\frac{c}{ab}$, that is from $\frac{c}{20}$ in this case: hence c *cannot be greater* than 220. Since 220 is divisible by 2 and by 5 we see by Case IV. of Art. 634 that the number of solutions is $\frac{220}{20}-1$, that is 10: hence c may be as great as 220.

6. Multiply up in the *given* relation, and transpose; thus

$$(x-y)xy+(x^2-y^2)z=(x^2-y^2)xyz+(x-y)xyz^2:$$

divide by $x-y$; thus $xy+yz+zx=xyz(x+y+z)$. And if we take any one of the *proposed* relations and simplify we shall find that it reduces to the result just obtained, and is therefore true.

7. $\left(\dfrac{N}{n}\right)^{\frac{1}{2}} = \left\{\dfrac{(N+n)^2}{4n^2} - \dfrac{(N-n)^2}{4n^2}\right\}^{\frac{1}{2}}$ and $= \left\{\dfrac{(N+n)^2}{4N^2} - \dfrac{(N-n)^2}{4N^2}\right\}^{-\frac{1}{2}}$.

Expand the two binomial expressions by the Binomial Theorem as far as two terms, and add; thus very approximately

$$2\left(\frac{N}{n}\right)^{\frac{1}{2}} = \frac{N+n}{2n} + \frac{2N}{N+n} - \frac{(N-n)^2}{4n(N+n)} + \frac{N(N-n)^2}{(N+n)^3}.$$

The last two terms are equal to $\dfrac{(N-n)^2}{(N+n)^2}\left\{\dfrac{N}{N+n} - \dfrac{N+n}{4n}\right\}$, that is to $-\dfrac{(N-n)^4}{4n(N+n)^3}$; therefore

$$\left(\frac{N}{n}\right)^{\frac{1}{2}} = \frac{N}{N+n} + \frac{N+n}{4n} - \frac{(N-n)^4}{8n(N+n)^3} \text{ very nearly.}$$

8. Let u_n denote the capital in pounds at the beginning of the n^{th} year. During the n^{th} year the income is $\dfrac{3u_n}{100}+200$; the expenditure is $\dfrac{5}{4}\left(\dfrac{3u_n}{100}+200\right)-95$; thus the saving is $95-\dfrac{1}{4}\left(\dfrac{3u_n}{100}+200\right)$, that is $45-\dfrac{3u_n}{400}$. Therefore $u_{n+1}=u_n+45-\dfrac{3u_n}{400}$; this may be written $u_{n+1}-6000=\dfrac{397}{400}(u_n-6000)$. Thus if 6000 be subtracted from the numbers which express the capital in pounds at the ends of successive years the remainders form a series in Geometrical Progression having the common ratio $\dfrac{397}{400}$. Therefore $u_{n+1}-6000=(u_1-6000)\left(\dfrac{397}{400}\right)^n$. Now whatever be the value of u_1-6000 we can take n so large that the product of this into $\left(\dfrac{397}{400}\right)^n$ shall be as small as we please; and so $u_{n+1}-6000$ will not differ sensibly from zero.

If $u_1=1000$ we have $u_{n+1}=6000-5000\left(\dfrac{397}{400}\right)^n$. Put this equal to 2000; therefore $\left(\dfrac{397}{400}\right)^n=\dfrac{4}{5}=\dfrac{8}{10}$. Therefore $n\log\dfrac{397}{400}=\log\dfrac{8}{10}$; therefore $n=\dfrac{\log 10-\log 8}{\log 400-\log 397}=\dfrac{1-3\log 2}{2+2\log 2-\log 397}$: this will be found to be a little less than 30, so that at the beginning of the 31st year the capital is a little more than £2000.

9. $$\frac{1}{1-x+cx^2}=\frac{1}{1-x}\left(1+\frac{cx^2}{1-x}\right)^{-1}=\frac{1}{1-x}-\frac{cx^2}{(1-x)^2}+\frac{c^2x^4}{(1-x)^3}-\dots.$$

Expand each term, and pick out the coefficient of x^n, as in Example L. 13. Again, putting $\dfrac{m}{(m+1)^2}$ for c we find that $1-x+cx^2=c\left(\dfrac{m+1}{m}-x\right)(m+1-x)$;

therefore $\dfrac{1}{1-x+cx^2}=\dfrac{m+1}{m-1}\left(\dfrac{1}{\dfrac{m+1}{m}-x}-\dfrac{1}{m+1-x}\right)$; and by expanding we see that the coefficient of x^n is $\dfrac{m}{m-1}\left(\dfrac{m}{m+1}\right)^n-\dfrac{1}{m-1}\left(\dfrac{1}{m+1}\right)^n$.

10. Let I denote the integral part of $(1+\sqrt{2})^x$, and F the fractional part. By the Binomial Theorem

$$I+F=1+x2^{\frac{1}{2}}+\frac{x(x-1)}{\underline{|2}}2+\frac{x(x-1)(x-2)}{\underline{|3}}2^{\frac{3}{2}}+\dots$$

Now $1-\sqrt{2}$ is a negative proper fraction, therefore $(1-\sqrt{2})^x$ is a negative proper fraction; denote it by $-F'$: thus

$$-F'=1-x2^{\frac{1}{2}}+\frac{x(x-1)}{\underline{|2}}2-\frac{x(x-1)(x-2)}{\underline{|3}}2^{\frac{3}{2}}+\dots$$

By addition $I+F-F'=2+2\dfrac{x(x-1)}{2}2+\dots$. Hence $F-F'$ must be zero; and $I-2$ is equal to a series of terms every one of which is divisible by $2x\times 2$, that is by $4x$.

11. Suppose that n integers are multiplied together. In each integer the last figure may be 0, or 1, or 2, ... or 9; so that there are 10^n cases in all. The favourable cases are those in which all the last figures are odd numbers, one or more being 5. Now there are 5^n cases in which the last figure is odd, and there are 4^n cases among them in which no last figure is 5; thus there are 5^n-4^n cases in which all the last figures are odd numbers, one or more of them being 5. Therefore the chance is $\dfrac{5^n-4^n}{10^n}$. Therefore when $n+1$ integers are multiplied together the chance is $\dfrac{5^{n+1}-4^{n+1}}{10^{n+1}}$; this will be found to be less than $\dfrac{5^n-4^n}{10^n}$, by bringing the fractions to a common denominator.

12. Suppose that the first purse contains a sovereigns and m shillings, and that the second purse contains b sovereigns and n shillings. If a purse is taken at random the chance of drawing out a sovereign is $\dfrac{1}{2}\dfrac{a}{a+m}+\dfrac{1}{2}\dfrac{b}{b+n}$; if all the coins are put into one purse the chance of drawing out a sovereign is $\dfrac{a+b}{a+b+m+n}$: these chances are equal if

$$(a+b+m+n)\{a(b+n)+b(a+m)\}=2(a+b)(a+m)(b+n).$$

Bring all the terms to one side and simplify, and this reduces to $(bm-an)(a+m-b-n)=0$; and thus the first part of the Example is established.

Again, suppose that $a+m$ is greater than $b+n$; the chance is in favour of the purse taken at random if $(bm-an)(a+m-b-n)$ is positive: this requires that an should be less than bm, and therefore $\frac{a}{m}$ less than $\frac{b}{n}$; and thus the second part of the Example is established.

LV.

2. See Arts 307, 308.

3. Let x denote the radix; then $16640=4x^4+4x^2$.

4. $\left(1-\frac{1}{2}\right)^{-2}=1+2.\frac{1}{2}+3.\left(\frac{1}{2}\right)^2+4.\left(\frac{1}{2}\right)^3+\ldots\ldots$ Subtract 2 from both sides; thus $2=3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots\ldots$

5. Let x denote the number of persons to be elected; then $2x+1$ denotes the number of candidates: put n for $2x+1$. There are n ways in which an elector may vote for one person, $\frac{n(n-1)}{2}$ ways in which he may vote for two persons, $\frac{n(n-1)(n-2)}{\lfloor 3}$ ways in which he may vote for three persons, and so on. Thus $15=n+\frac{n(n-1)}{2}+\frac{n(n-1)(n-2)}{\lfloor 3}+\ldots$ to x terms. If from the expansion of $(1+1)^{2x+1}$ we subtract the first and the last terms the remainder is twice the above series of x terms; thus $15=\frac{1}{2}(2^{2x+1}-2)$; therefore $2^{2x+1}=32=2^5$. Thus $2x+1=5$.

6. Let x represent the number of shillings, and y the number of francs; then $x+\frac{21y}{26}=495$; therefore $26x+21y=26\times 495$. One solution is $x=495$, $y=0$; the general solution is $x=495-21t$, $y=26t$. Thus including the zero solution there are 24 solutions, and excluding it there are 23.

7. Add together the first two terms; thus we get $\frac{1+z+z^2}{(1+z)(1+z^2)}$, that is $1-\frac{z^3}{(1+z)(1+z^2)}$. Now annex the third term to this; thus we get $1-\frac{z^3}{(1+z)(1+z^2)}\left(1-\frac{1}{1+z^4}\right)$, that is $1-\frac{z^7}{(1+z)(1+z^2)(1+z^4)}$. Proceed in this way, and we obtain for the sum of n terms $1-\frac{z^{p-1}}{(1+z)(1+z^2)\ldots(1+z^q)}$, where $p=2^n$, and $q=2^{n-1}$. If we multiply both numerator and denominator of the fraction by $1-z$ we get for the sum $1-\frac{(1-z)z^{p-1}}{1-z^p}$.

8. $1+2x^4-(x^2+2x^3)=1-x^2-2x^3(1-x)=(1-x)(1+x-2x^3)$
$=(1-x)(1-x^3+x-x^3)=(1-x)^2(1+x+x^2+x+x^2)=(1-x)^2\{(1+x)^2+x^2\}$;
this is never negative.

9. Let a and b denote the two quantities; suppose n means are inserted between them. The m^{th} term of the A.P. is $a+\dfrac{m-1}{n+1}(b-a)$. The m^{th} term of the G.P. is $a\left(\dfrac{b}{a}\right)^{\frac{m-1}{n+1}}$. The former $=\dfrac{(n-m+2)\,a+(m-1)\,b}{n+1}$; the latter $=(a^{n-m+2}b^{m-1})^{\frac{1}{n+1}}$: these may be regarded respectively as the arithmetical mean and the geometrical mean of $n+1$ quantities of which $n-m+2$ are equal to a, and $m-1$ are equal to b; and therefore the former is the greater by Art. 681.

10. Let I denote the integral part of $(2+\sqrt{3})^x$, and F the fractional part. By the Binomial Theorem

$$I+F=2^x+x2^{x-1}\sqrt{3}+\frac{x(x-1)}{2}2^{x-2}.\,3+\ldots$$

Now $(2-\sqrt{3})^x$ is a proper fraction; denote it by F': thus

$$F'=2^x-x2^{x-1}\sqrt{3}+\frac{x(x-1)}{2}2^{x-2}.\,3-\ldots$$

By addition $I+F+F'=2^{x+1}+2\dfrac{x(x-1)}{2}2^{x-2}.\,3+\ldots$ Hence $F+F'$ must be equal to unity; and $I+1-2^{x+1}$ is equal to a series of terms every one of which is divisible by $2x\times 2\times 3$, that is by $12x$.

11. Let a denote any prime number. By Art. 709 the highest power of a which is contained in $\underline{|n}$ is $I\left(\dfrac{n}{a}\right)+I\left(\dfrac{n}{a^2}\right)+\ldots$; and the highest power of a which is contained in $(\underline{|p})^q\,\underline{|q}$ is $q\left\{I\left(\dfrac{p}{a}\right)+I\left(\dfrac{p}{a^2}\right)+\ldots\right\}+I\left(\dfrac{q}{a}\right)+I\left(\dfrac{q}{a^2}\right)+\ldots$
We have to shew that the former power of a is at least as high as the latter. Suppose that a^r is the highest power of a which is not greater than p. Then there are r terms in $I\left(\dfrac{p}{a}\right)+I\left(\dfrac{p}{a^2}\right)+\ldots$; and the product of q into the sum of these r terms cannot be greater than the sum of the first r terms of $I\left(\dfrac{n}{a}\right)+I\left(\dfrac{n}{a^2}\right)+\ldots$ The $(r+1)^{\text{th}}$ term of the last series is $I\left(\dfrac{n}{a^{r+1}}\right)$, and this cannot be less than $I\left(\dfrac{q}{a}\right)$; for $I\left(\dfrac{q}{a}\right)=I\left(\dfrac{qa^r}{a^{r+1}}\right)$, and qa^r is not greater than n. Similarly $I\left(\dfrac{n}{a^{r+2}}\right)$ is not less than $I\left(\dfrac{q}{a^2}\right)$. And so on. Thus the required result is established.

13. The observed event is the result of the experiment. The probability of this event on the hypothesis that the theory is true is p, and on the hypothesis that the theory is false is $(1-p)\,q$. Therefore after the observed event the probability of the first hypothesis is $p \div \{p+(1-p)\,q\}$.

14. Denote the bag with two sovereigns and a shilling by A, and the bag with a sovereign and a shilling by B. The chance of drawing a sovereign from A is $\frac{2}{3}$ and from B is $\frac{1}{2}$: hence, when a sovereign has been drawn the chance that it came from A is $\frac{2}{3} \div \left\{\frac{2}{3}+\frac{1}{2}\right\}$, that is $\frac{4}{7}$; and the chance that it came from B is $\frac{1}{2} \div \left\{\frac{2}{3}+\frac{1}{2}\right\}$, that is $\frac{3}{7}$. If the second drawing be made from the *same* bag as before the chance is $\frac{4}{7}$ that it now contains a sovereign and a shilling, and $\frac{3}{7}$ that it now contains a shilling; therefore the chance of drawing a sovereign is $\frac{4}{7} \times \frac{1}{2}$, that is $\frac{2}{7}$. If the second drawing be made from the *other* bag the chance is $\frac{4}{7}$ that it contains a sovereign and a shilling, and $\frac{3}{7}$ that it contains two sovereigns and a shilling; therefore the chance of drawing a sovereign is $\frac{4}{7} \times \frac{1}{2} + \frac{3}{7} \times \frac{2}{3}$, that is $\frac{4}{7}$. For the expectation see the *Algebra*.

15. There are two hypotheses; that the white bag contains the sovereign and the four shilling pieces, or that it contains the two sovereigns and the three shilling pieces. The observed event is the drawing of a sovereign from the white bag, and a shilling from the red bag. The probability of this event on the first hypothesis is $\frac{1}{5} \times \frac{3}{5}$, and on the second hypothesis is $\frac{2}{5} \times \frac{4}{5}$. Hence after the observed event the probability of the hypotheses are $\frac{3}{11}$ and $\frac{8}{11}$ respectively. The coins drawn being now put back we have four cases which will be sufficiently described by referring as before to the *white* bag; the following are the probabilities: $\frac{3}{22}$ that it has now a sovereign and four shilling pieces, $\frac{3}{22}$ that it has now five shilling pieces, $\frac{8}{22}$ that it has now two sovereigns and three shilling pieces, and $\frac{8}{22}$ that it has now a sovereign and four shilling pieces. The probability of drawing a sovereign from the white bag is therefore $\frac{3}{22} \times \frac{1}{5} + \frac{3}{22} \times 0 + \frac{8}{22} \times \frac{1}{5} + \frac{8}{22} \times \frac{2}{5}$, that is $\frac{27}{110}$

Similarly the probability of drawing a sovereign from the red bag is $\frac{3}{22} \times \frac{2}{5} + \frac{3}{22} \times \frac{3}{5} + \frac{8}{22} \times \frac{2}{5} + \frac{8}{22} \times \frac{1}{5}$, that is $\frac{39}{110}$.

16. Out of n persons of the individual's age one will die every year: thus $\frac{n-1}{n}$, $\frac{n-2}{n}$, $\frac{n-3}{n}$,... are the respective probabilities of his living 1, 2, 3, ... years. Therefore $\frac{n-1}{nR}$, $\frac{n-2}{nR^2}$, $\frac{n-3}{nR^3}$, ... are the present values of the payments to be made respectively at the end of 1, 2, 3, ... years during the life of the individual. The sum of this series is found from Art. 473 by putting $a = \frac{n-1}{nR}$, $b = -\frac{1}{nR}$, $r = \frac{1}{R}$.

LVI.

1. Take the logarithm of the product; thus we have an infinite series of which the n^{th} term is $\log u_n$; and we must examine whether this series is convergent: if it is convergent the logarithm of the product is finite, and so the product is finite. If the series of which the n^{th} term is $\log u_n$ is divergent and *negative* the logarithm of the product is *negative* and numerically indefinitely great; in this case the product is indefinitely small.

2. Denote the given expression by P; then

$$\log P = x \log n + \log \frac{1}{x+1} + \log \frac{2}{x+2} + \log \frac{3}{x+3} + \dots + \log \frac{n}{x+n}$$

$$= x \log n - \log (1+x) - \log \left(1 + \frac{x}{2}\right) - \log \left(1 + \frac{x}{3}\right) - \dots - \log \left(1 + \frac{x}{n}\right).$$

Now $\log n = \log \left(\frac{1}{1} \cdot \frac{2}{1} \cdot \frac{3}{2} \cdots \frac{n}{n-1}\right) = \log \frac{1}{1} + \log \frac{2}{1} + \log \frac{3}{2} + \dots + \log \frac{n}{n-1}$;

and thus $\log P$ may be considered as a sum of n terms, the first term being $-\log(1+x)$, and for all values of r greater than 1 the r^{th} term being $x \log \frac{r}{r-1} - \log \left(1 + \frac{x}{r}\right)$, that is $-x \log \left(1 - \frac{1}{r}\right) - \log \left(1 + \frac{x}{r}\right)$. Thus if x is a negative integer we have in $\log P$ the term $-\log(0)$, so that $\log P$ is infinite. But if x is not a negative integer every term in $\log P$ is finite, and for all values of r which are numerically greater than x we can expand the r^{th} term, so that it becomes $x\left(\frac{1}{r} + \frac{1}{2r^2} + \frac{1}{3r^3} + \dots\right) - \left(\frac{x}{r} - \frac{x^2}{2r^2} + \frac{x^3}{3r^3} - \dots\right)$, that is $\frac{x+x^2}{2r^2}$ + terms involving $\frac{1}{r^3}$, $\frac{1}{r^4}$,.... Thus the r^{th} term bears a finite ratio to $\frac{1}{r^2}$, and so the series which forms $\log P$ is convergent: see Arts. 771 and 562.

3. We may consider u_n as the product of n factors, the $(r+1)^{\text{th}}$ factor being $\dfrac{(\alpha+r)(\beta+r)}{(\gamma+r)(1+r)}$, that is $\dfrac{\left(1+\frac{\alpha}{r}\right)\left(1+\frac{\beta}{r}\right)}{\left(1+\frac{\gamma}{r}\right)\left(1+\frac{1}{r}\right)}$: denote this by v_{r+1}. Then $\log v_{r+1}=\log\left(1+\frac{\alpha}{r}\right)+\log\left(1+\frac{\beta}{r}\right)-\log\left(1+\frac{\gamma}{r}\right)-\log\left(1+\frac{1}{r}\right)$. Suppose r so large that $\frac{\alpha}{r}$, $\frac{\beta}{r}$, and $\frac{\gamma}{r}$ are all proper fractions; then by expanding we have $\log v_{r+1}=\dfrac{\alpha+\beta-\gamma-1}{r}-\dfrac{\alpha^2+\beta^2-\gamma^2-1}{2r^2}+\ldots$ If $\alpha+\beta-\gamma-1$ is positive $\log v_{r+1}$ bears a finite ratio to $\frac{1}{r}$; and thus the series which forms $\log u_n$ is divergent and positive: therefore u_n increases indefinitely with n.

4. Proceed as in Example 3. If $\alpha+\beta-\gamma-1=0$ we see that $\log v_{r+1}$ bears a finite ratio to $\frac{1}{r^2}$; therefore $\log u_n$ is finite when n increases indefinitely: therefore u_n is finite when n increases indefinitely.

5. Proceed as in Example 3. If $\alpha+\beta-\gamma-1$ is negative the series which forms $\log u_n$ is divergent and negative; therefore $\log u_n$ is negative and numerically increases indefinitely with n: therefore u_n is indefinitely small when n increases indefinitely.

6. Here $\dfrac{u_n}{u_{n+1}}=\dfrac{n+1}{(a+n)x}$; thus by Art. 762 the series is convergent if x is less than unity, and divergent if x is greater than unity; if x is unity Art. 762 will not always decide. Put $x=1$, then

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=\frac{n(1-a)}{a+n}=\frac{1-a}{1+\frac{a}{n}};$$

if a is negative the series is convergent by Art. 766, and if a is positive the series is divergent by Art. 767.

7. $u_n n\lambda(n)=\dfrac{n\log n}{n^{\frac{n+1}{n}}}=\dfrac{\log n}{n^{\frac{1}{n}}}$; this is indefinitely great when n is; for $n^{\frac{1}{n}}$ becomes unity since its logarithm, which is $\frac{1}{n}\log n$, vanishes by Art. 769. Hence the series of which u_n is the n^{th} term is divergent by Art. 771.

8. Here $\dfrac{u_n}{u_{n+1}}=\dfrac{2n-1}{2n-3}\cdot\dfrac{2n-2}{2n-3}\cdot\dfrac{1}{x}$; thus by Art. 762 the series is convergent if x is less than unity, and divergent if x is greater than unity. If $x=1$ we have $n\left(\dfrac{u_n}{u_{n+1}}-1\right)=n\left\{\dfrac{(2n-1)(2n-2)}{(2n-3)^2}-1\right\}=\dfrac{6n^2-7n}{(2n-3)^2}$; and the series is convergent by Art. 766.

9. Here $\frac{u_n}{u_{n+1}} = \frac{n^2}{(n-1+\beta)(n-\beta)} = \frac{n^2}{n^2-n+\beta(1-\beta)}$; with the notation of Art. 776 we have $a-A-1=0$, and so the series is divergent.

10. Here $\frac{u_n}{u_{n+1}} = \frac{n^p}{(n-1)^q} \times \frac{n^q}{(n+1)^p} = \frac{n^{p+q}}{n^{p+q}+(p-q)n^{p+q-1}+\dots}$: with the notation of Art. 776 we have $a-A-1=q-p-1$; the series is convergent if $q-p-1$ is positive, and divergent if $q-p-1$ is negative or zero.

11. Suppose that from and after some fixed value of n the value of $n\log\frac{u_n}{u_{n+1}}$ is always greater than γ, where γ is positive and greater than unity. Then $\log\frac{u_n}{u_{n+1}}$ is greater than $\frac{\gamma}{n}$, and therefore when n is large enough $\log\frac{u_n}{u_{n+1}}$ is greater than $\log\left(1+\frac{\gamma}{n}\right)$; see Art. 687. Therefore when n is large enough $\frac{u_n}{u_{n+1}}$ is greater than $1+\frac{\gamma}{n}$; and therefore, as in Art. 766, the series of which the n^{th} term is u_n is convergent.

12. Here from and after some fixed value of n the value of $\log\frac{u_n}{u_{n+1}}$ is positive and not greater than $\frac{1}{n}$ or is negative. In the former case $\log\frac{u_n}{u_{n+1}}$ is less than $\log\frac{n}{n-1}$, by Art. 688; and therefore $\frac{u_n}{u_{n+1}}$ is less than $\frac{n}{n-1}$: in the latter case $\frac{u_n}{u_{n+1}}$ is less than unity. Thus in both cases $\frac{u_{n+1}}{u_{n+2}}$ is less than $\frac{n+1}{n}$. Hence by Arts. 765 and 562 the series of which the n^{th} term is u_n is divergent.

13. Here $\frac{u_n}{u_{n+1}} = \frac{(n+1)(a+nx)^n}{(a+nx+x)^{n+1}} = \frac{1}{x\left(1+\frac{a}{nx+x}\right)}\left(1+\frac{x}{nx+a}\right)^{-n}$;

therefore $$\log\frac{u_n}{u_{n+1}} = -\log x - \log\left(1+\frac{a}{nx+x}\right) - n\log\left(1+\frac{x}{nx+a}\right)$$
$$= -\log x - \frac{a}{nx+x} + \frac{a^2}{2(nx+x)^2} - \dots - \frac{nx}{nx+a} + \frac{nx^2}{2(nx+a)^2} - \dots$$

But $\frac{nx}{nx+a} = 1 - \frac{a}{nx+a}$; thus we have

$$n\log\frac{u_n}{u_{n+1}} = n\left\{-\log x - 1 + \frac{a}{nx+a} - \frac{a}{nx+x} + \frac{nx^2}{2(nx+a)^2} + \dots\right\}.$$

If x is greater than e^{-1} then $-\log x-1$ is negative, and the series is divergent by Example 12. If x is less than e^{-1} then $-\log x-1$ is positive, and the series is convergent by Example 11. If $x=e^{-1}$ then $-\log x-1=0$; in this case we have $n\log\dfrac{u_n}{u_{n+1}}=\dfrac{n^2x^2}{2(nx+a)^2}+\dfrac{na(x-a)}{(nx+a)(nx+x)}+\ldots$; and by taking n large enough this approximates as closely as we please to the value $\frac{1}{2}$: therefore the series is divergent by Example 12.

14. We have only to examine the case in which $x=1$; then

$$\frac{u_n}{u_{n+1}}=\frac{(n+1)(n+\gamma)}{(n+\alpha)(n+\beta)}=\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}.$$

I. Suppose that $\gamma-\alpha-\beta$ is positive.

Then a positive quantity h can be found greater than unity, such that when n is large enough $\dfrac{u_n}{u_{n+1}}$ shall be greater than $1+\dfrac{h}{n}$. For this will be secured if $1+\dfrac{\gamma+1}{n}+\dfrac{\gamma}{n^2}$ is greater than $\left(1+\dfrac{h}{n}\right)\left(1+\dfrac{\alpha+\beta}{n}+\dfrac{\alpha\beta}{n^2}\right)$, that is if $\gamma+1-\alpha-\beta-h$ is greater than $\dfrac{h(\alpha+\beta)+\alpha\beta-\gamma}{n}+\dfrac{h\alpha\beta}{n^2}$; and this condition can obviously be satisfied by taking n large enough. Now, by Art. 686, a positive quantity p greater than unity can be found such that when n is large enough $\left(\dfrac{n+1}{n}\right)^p$ is less than $1+\dfrac{h}{n}$. Hence when n is large enough $\dfrac{u_n}{u_{n+1}}$ is greater than $\left(\dfrac{n+1}{n}\right)^p$. But, by Art. 562, the series of which the n^{th} term is $\dfrac{1}{n^p}$ is convergent when p is positive and greater than unity; hence, by Art. 764, the series of which the n^{th} term is u_n is convergent.

II. Suppose that $\gamma-\alpha-\beta$ is zero.

Then $\dfrac{u_n}{u_{n+1}}$ will be less than $\dfrac{n+1}{n}$ provided $1+\dfrac{\alpha+\beta+1}{n}+\dfrac{\alpha+\beta}{n^2}$ is less than $\left(1+\dfrac{1}{n}\right)\left(1+\dfrac{\alpha+\beta}{n}+\dfrac{\alpha\beta}{n^2}\right)$, that is provided $\dfrac{\alpha\beta}{n^2}+\dfrac{\alpha\beta}{n^3}$ is positive. Thus if α and β are of the same sign $\dfrac{u_n}{u_{n+1}}$ is less than $\dfrac{n+1}{n}$. But, by Art. 562, the series of which the n^{th} term is $\dfrac{1}{n}$ is divergent; hence by Art. 765 the series of which the n^{th} term is u_n is divergent. If α and β are not of the same sign, take m a fixed positive integer so large that $m+\alpha\beta$ is positive. Then

$$\frac{u_{n+m}}{u_{n+m+1}}=\frac{(n+m+1)(n+m+\gamma)}{(n+m+\alpha)(n+m+\beta)}=\frac{\left(1+\frac{m+1}{n}\right)\left(1+\frac{m+\gamma}{n}\right)}{\left(1+\frac{m+\alpha}{n}\right)\left(1+\frac{m+\beta}{n}\right)};$$

this by multiplying out will be seen to be less than $\dfrac{n+1}{n}$, and so the series of which the n^{th} term is u_{n+m} is divergent. Hence the proposed series is divergent.

III. Suppose that $\gamma - \alpha - \beta$ is negative.

Here $\frac{u_n}{u_{n+1}}$ is always less than it is for a series with the same values of α and β in which $\gamma - \alpha - \beta$ is zero; that is $\frac{u_n}{u_{n+1}}$ is always less than the corresponding ratio for a series which has been shewn to be divergent in II.: therefore, by Art. 765, the series of which the nth term is u_n is divergent.

15. I. Suppose that $a - A - 1$ is positive.

Then a positive quantity h can be found greater than unity, such that when n is large enough $\frac{u_n}{u_{n+1}}$ shall be greater than $\frac{n+h}{n}$. For this will be secured if $1+\frac{a}{n}+\frac{b}{n^2}+\frac{c}{n^3}+\dots$ is greater than $\left(1+\frac{h}{n}\right)\left(1+\frac{A}{n}+\frac{B}{n^2}+\frac{C}{n^3}+\dots\right)$, that is if $a - A - h$ is greater than $\frac{Ah+B-b}{n}+\frac{Bh+C-c}{n^2}+\dots$; and this condition can obviously be satisfied by taking n large enough. Then continuing the investigation as in I. of the preceding solution, we see that the series of which the nth term is u_n is convergent.

II. Suppose that $a - A - 1$ is zero.

Then $\frac{u_n}{u_{n+1}}$ will be less than $\frac{n+1}{n}$ provided $1+\frac{a}{n}+\frac{b}{n^2}+\frac{c}{n^3}+\dots$ is less than $\left(1+\frac{1}{n}\right)\left(1+\frac{A}{n}+\frac{B}{n^2}+\frac{C}{n^3}+\dots\right)$, that is provided $A+B-b+\frac{B+C-c}{n}+\dots$ is positive. Thus if $A+B-b$ is positive, the condition will obviously be satisfied when n is large enough. If $A+B-b$ is not positive, take m a fixed positive integer so large that $A+B-b+m$ is positive. Then $\frac{u_{n+m}}{u_{n+m+1}}$ will be less than $\frac{n+1}{n}$ provided $1+\frac{a}{n+m}+\frac{b}{(n+m)^2}+\dots$ is less than $\left(1+\frac{1}{n}\right)\left\{1+\frac{A}{n+m}+\frac{B}{(n+m)^2}+\dots\right\}$, that is if $1+\frac{a}{n}+\frac{b-ma}{n^2}+\frac{c-2mb+m^2a}{n^3}+\dots$ is less than $\left(1+\frac{1}{n}\right)\left\{1+\frac{A}{n}+\frac{B-mA}{n^2}+\frac{C-2mB+m^2A}{n^3}+\dots\right\}$, that is if $A+B-mA-(b-ma)$ is positive, that is if $A+B-b+m$ is positive: and this by supposition is the case. Then, continuing the investigation as in II. of the preceding solution, we see that the proposed series is divergent.

III. Suppose that $a - A - 1$ is negative.

Here $\frac{u_n}{u_{n+1}}$ is always less than the corresponding ratio for a series with the same values of $b, c, \dots A, B, C, \dots$ in which $a - A - 1$ is zero; that is $\frac{u_n}{u_{n+1}}$ is always less than the corresponding ratio for a series which has been shewn to be divergent in II.: therefore by Art. 765 the series of which the nth term is u_n is divergent.

16. Here $P_0 = n\dfrac{(a-A)n^\beta + (b-B)n^\gamma + \dots}{n^a + An^\beta + Bn^\gamma + \dots}$.

I. Suppose $a-A$ positive. If $\beta+1$ is greater than a the series is convergent, by Art. 766. If $\beta+1$ is less than a the series is divergent, by Art. 767. If $\beta+1=a$ the series is convergent if $a-A$ is greater than 1, and divergent if $a-A$ is less than 1; but if $a-A=1$ we use Art. 773; in this case we have

$$\lambda(n)(P_0-1) = \frac{\lambda(n)\{(b-B)n^{\gamma+1} - An^{a-1} + \dots\}}{n^a + An^{a-1} + Bn^\gamma + \dots}:$$

and since γ is less than β we have $\gamma+1$ less than a, and therefore $\lambda(n)(P_0-1)$ can be made as small as we please by taking n large enough, and the series is divergent.

II. Suppose $a-A$ negative. The series is divergent by Art. 767.

III. Suppose $a-A=0$. Then P_0 becomes $n\dfrac{(b-B)n^\gamma + \dots}{n^a + An^\beta + Bn^\gamma + \dots}$ and we must continue the process as in Case I.

17. The second series is less than $\dfrac{1}{u_0}$ of the first series, and therefore if the first series is convergent so also is the second.

If x be positive $\log(1+x)$ is less than x; see Art. 687.

Thus
$$\log\frac{u_0+u_1}{u_0} \text{ is less than } \frac{u_1}{u_0},$$
$$\log\frac{u_0+u_1+u_2}{u_0+u_1} \text{ is less than } \frac{u_2}{u_0+u_1},$$
$$\log\frac{u_0+u_1+u_2+u_3}{u_0+u_1+u_2} \text{ is less than } \frac{u_3}{u_0+u_1+u_2},$$
and so on; hence by addition $\log(u_0+u_1+u_2+\dots+u_n) - \log u_0$ is less than

$$\frac{u_1}{u_0} + \frac{u_2}{u_0+u_1} + \frac{u_3}{u_0+u_1+u_2} + \dots + \frac{u_n}{u_0+u_1+\dots+u_{n-1}};$$

this shews that if the first series is divergent so also is the second.

18. I. Suppose that from and after some fixed value of n the value of $\lambda^2(n)(P_1-1)$ is always positive and greater than γ, where γ is positive and greater than unity. Then P_1 is greater than $1+\dfrac{\gamma}{\lambda^2(n)}$; therefore P_0 is greater than $1+\dfrac{1}{\lambda(n)} + \dfrac{\gamma}{\lambda(n)\lambda^2(n)}$; and therefore $\dfrac{u_n}{u_{n+1}}$ is greater than $1+\dfrac{1}{n} + \dfrac{1}{n\lambda(n)} + \dfrac{\gamma}{n\lambda(n)\lambda^2(n)}$.

Let $v_n=\dfrac{1}{n\lambda(n)\{\lambda^2(n)\}^p}$; then $\dfrac{v_n}{v_{n+1}}=\dfrac{n+1}{n}\dfrac{\lambda(n+1)}{\lambda(n)}\left\{\dfrac{\lambda^2(n+1)}{\lambda^2(n)}\right\}^p$. Now $\lambda(n+1)$ is less than $\lambda(n)\left\{1+\dfrac{1}{n\lambda(n)}\right\}$; therefore $\lambda^2(n+1)$ is less than $\lambda^2(n)+\lambda\left\{1+\dfrac{1}{n\lambda(n)}\right\}$, and is therefore less than $\lambda^2(n)+\dfrac{1}{n\lambda(n)}$. Thus $\dfrac{v_n}{v_{n+1}}$ is less than $\left(1+\dfrac{1}{n}\right)\left\{1+\dfrac{1}{n\lambda(n)}\right\}\left\{1+\dfrac{1}{n\lambda(n)\lambda^2(n)}\right\}^p$; and therefore when n is large enough $\dfrac{v_n}{v_{n+1}}$ is less than $\left(1+\dfrac{1}{n}\right)\left\{1+\dfrac{1}{n\lambda(n)}\right\}\left\{1+\dfrac{q}{n\lambda(n)\lambda^2(n)}\right\}$; provided q be greater than p. Hence when n is large enough $\dfrac{v_n}{v_{n+1}}$ is less than $1+\dfrac{1}{n}+\dfrac{1}{n\lambda(n)}+\dfrac{r}{n\lambda(n)\lambda^2(n)}$, provided r be greater than q.

Since γ is greater than unity we may suppose that γ is greater than r, and yet have p positive and greater than unity. Since γ is greater than r we have $\dfrac{u_n}{u_{n+1}}$ greater than $\dfrac{v_n}{v_{n+1}}$. But, by Art. 770, the series of which the nth term is v_n is convergent when p is positive and greater than unity; hence, by Art. 764, the series of which the nth term is u_n is convergent.

II. Suppose that from and after some fixed value of n the value of $\lambda^2(n)(P_1-1)$ is never positive and greater than unity. Then P_1-1 is positive and not greater than $\dfrac{1}{\lambda^2(n)}$ or is negative. In both cases $\dfrac{u_{n+1}}{u_{n+2}}$ is less than $1+\dfrac{1}{n}+\dfrac{1}{n\lambda(n)}+\dfrac{1}{n\lambda(n)\lambda^2(n)}$.

Let $v_n=\dfrac{1}{n\lambda(n)\lambda^2(n)}$; then $\dfrac{v_n}{v_{n+1}}=\dfrac{n+1}{n}\dfrac{\lambda(n+1)}{\lambda(n)}\dfrac{\lambda^2(n+1)}{\lambda^2(n)}$. Now $\lambda(n+1)$ is greater than $\lambda(n)\left\{1+\dfrac{1}{n\lambda(n)}-\dfrac{1}{2n^2\lambda(n)}\right\}$ by Art. 688; hence $\lambda^2(n+1)$ is greater than $\lambda^2(n)+\lambda\left\{1+\dfrac{1}{n\lambda(n)}-\dfrac{1}{2n^2\lambda(n)}\right\}$, and therefore greater than $\lambda^2(n)+\dfrac{1}{n\lambda(n)}-\dfrac{1}{n^2\lambda(n)}$, when n is large enough. Thus when n is large enough $\dfrac{v_n}{v_{n+1}}$ is greater than $\left(1+\dfrac{1}{n}\right)\left\{1+\dfrac{1}{n\lambda(n)}-\dfrac{1}{2n^2\lambda(n)}\right\}\left\{1+\dfrac{1}{n\lambda(n)\lambda^2(n)}-\dfrac{1}{n^2\lambda(n)\lambda^2(n)}\right\}$, and therefore greater than $1+\dfrac{1}{n}+\dfrac{1}{n\lambda(n)}+\dfrac{1}{n\lambda(n)\lambda^2(n)}$.

Hence when n is large enough $\dfrac{u_{n+1}}{u_{n+2}}$ is less than $\dfrac{v_n}{v_{n+1}}$. But, by Art. 770, the series of which the n^{th} term is v_n is divergent; therefore, by Art. 765, the series of which the nth term is u_{n+1} is divergent.

LVII.

1. In Art. 795 put $a=5$, $b=1$; thus we find that the value of the expression is $\sqrt{24}$.

2. By Arts. 794 and 795 the expression $=(\sqrt{n^2+1})^2-(\sqrt{n^2-1})^2$, that is $n^2+1-(n^2-1)$, that is 2.

3. This may be shewn by induction. Assume that $p_n=bq_{n-1}$, and $p_{n-1}=bq_{n-2}$; we have

$$p_{n+1}=ap_n+bp_{n-1}=abq_{n-1}+b^2q_{n-2}=b\,(aq_{n-1}+bq_{n-2})=bq_n.$$

Thus if the relation is true up to a certain value of n it is true for the next value; and it can be shewn to be true when $n=1$, and when $n=2$: therefore it is always true.

4. Here p_n is the coefficient of x^{n-1} in the expansion of $\dfrac{b}{1-ax}\left(1-\dfrac{bx^2}{1-ax}\right)^{-1}$, by Art. 796, that is in the expansion of $\dfrac{b}{1-ax}+\dfrac{b^2x^2}{(1-ax)^2}+\dfrac{b^3x^4}{(1-ax)^3}+\dots$; then expand each of these terms by the Binomial Theorem, and pick out the coefficient of x^{n-1} in each. Again q_n is the coefficient of x^{n-1} in the expansion of $\dfrac{a+bx}{1-ax-bx^2}$, that is in the expansion of $\dfrac{1}{x(1-ax-bx^2)}-\dfrac{1}{x}$; so that q_n is the coefficient of x^n in the expansion of $\dfrac{1}{1-ax-bx^2}$, that is in the expansion of $\dfrac{1}{1-ax}+\dfrac{bx^2}{(1-ax)^2}+\dfrac{b^2x^4}{(1-ax)^3}+\dots$. We may observe that these formulæ for p_n and q_n will furnish a direct proof of the statement of Example 3.

5. This may be shewn by induction. We have $p_n=n\,(p_{n-1}+p_{n-2})$, $q_n=n\,(q_{n-1}+q_{n-2})$; therefore $p_n+q_n=n\,(p_{n-1}+q_{n-1}+p_{n-2}+q_{n-2})$. Assume that $p_{n-2}+q_{n-2}=\lfloor n-1$, and $p_{n-1}+q_{n-1}=\lfloor n$; then $p_n+q_n=n\lfloor n+\lfloor n=\lfloor n+1$. Thus if the relation is true up to a certain value of n it is true for the next value; and it can be shewn to be true when $n=1$, and when $n=2$: therefore it is always true.

6. We have $p_n=a_np_{n-1}+b_np_{n-2}=(b_{n+1}-1)\,p_{n-1}+b_np_{n-2}$; therefore $p_n-b_{n+1}p_{n-1}=-(p_{n-1}-b_np_{n-2})$. Denote $p_n-b_{n+1}p_{n-1}$ by u_n, thus $u_n=-u_{n-1}$. Hence we see that $u_2, u_3, u_4, \dots u_n$ may be considered to form a Geometrical Progression in which the common ratio is -1; therefore $u_n=u_2(-1)^{n-2}=u_2(-1)^n$. Similarly we can shew that $q_n-b_{n+1}q_{n-1}=(q_2-b_3q_1)\,(-1)^n$.

7. We have $p_n=a_np_{n-1}+b_np_{n-2}=n^2p_{n-1}+\{(n-1)^2+1\}\,p_{n-2}$;

therefore $$p_n-(n^2+1)\,p_{n-1}=-\{p_{n-1}-[(n-1)^2+1]\,p_{n-2}\}.$$

Denote $p_n-(n^2+1)\,p_{n-1}$ by u_n, thus $u_n=-u_{n-1}$. Hence we see that $u_2, u_3, u_4, \dots u_n$ may be considered to form a Geometrical Progression in which the common ratio is -1; therefore $u_n=u_2\,(-1)^{n-2}=u_2\,(-1)^n$.

8. In Art. 799 suppose $v_0=1$, $v_1=1$, $v_2=2$, $v_3=3, \dots$ and let n increase indefinitely. Then the series becomes equal to e^{-x}, and the continued fraction has the form stated.

9. In Art. 797 put $x=-1$, $u_0=1$, $u_1=2$, $u_2=3, \ldots$ and let n increase indefinitely. Then the series becomes equal to $\log(1+1)$, that is to $\log 2$, and the continued fraction has the form stated.

10. In Art. 801 put $\alpha=1$, $\beta=0$, $\gamma=1$, and change the sign of x; thus we deduce an infinite continued fraction of the first class for $\frac{1}{x}\log(1+x)$; every component has unity for denominator, the numerator of the first component is 1, of the second is $\frac{1}{2}x$, and generally of the $(2r)^{\text{th}}$ is $\frac{r^2x}{(2r-1)\,2r}$, and of the $(2r+1)^{\text{th}}$ is $\frac{r^2x}{2r(2r+1)}$.

LVIII.

1. From the first and second equations by subtraction we get $x-y=by-ax$; therefore $x(1+a)=y(1+b)$. Similarly from the second and third equations $y(1+b)=z(1+c)$; and from the third and fourth equations $z(1+c)=u(1+d)$. Thus $x(1+a)=y(1+b)=z(1+c)=u(1+d)=k$ say. Substitute in the first equation; thus we get $\frac{1}{1+a}=\frac{b}{1+b}+\frac{c}{1+c}+\frac{d}{1+d}$, and as $\frac{1}{1+a}=1-\frac{a}{1+a}$ this is equivalent to the required result.

2. $\frac{x}{a}=y+z$, $\frac{y}{b}=z+x$, $\frac{z}{c}=x+y$. Subtract the second equation from the first; thus $\frac{x}{a}-\frac{y}{b}=y-x$; therefore $\frac{x(1+a)}{a}=\frac{y(1+b)}{b}$. Similarly by subtracting the third equation from the second we get $\frac{y(1+b)}{b}=\frac{z(1+c)}{c}$. Thus $\frac{x(1+a)}{a}=\frac{y(1+b)}{b}=\frac{z(1+c)}{c}=k$ say. Substitute in the first equation; thus we get $\frac{1}{1+a}=\frac{b}{1+b}+\frac{c}{1+c}$. Clear of fractions and reduce; thus $ab+bc+ca+2abc=1$.

Again $\frac{x^2}{a(1-bc)}=\frac{k^2a}{(1+a)^2(1-bc)}=\frac{k^2a}{1+2a+a^2-bc-2abc-a^2bc}$; but $1-bc-2abc=ab+ca$, so that $\frac{x^2}{a(1-bc)}$ reduces to $\frac{k^2a}{2a+a^2+ab+ca-a^2bc}$, that is to $\frac{k^2}{2+a+b+c-abc}$. Similarly $\frac{y^2}{b(1-ca)}$ and $\frac{z^2}{c(1-ab)}$ reduce to the same symmetrical expression.

3. $\frac{x}{a}+\frac{a}{x}=\frac{y}{b}+\frac{b}{y}$; this may be regarded as a quadratic for finding y in terms of x: solve it in the ordinary way, or use Art. 336, and we shall obtain $\frac{y}{b}=\frac{x}{a}$ or $\frac{a}{x}$. Similarly we get $\frac{z}{c}=\frac{x}{a}$ or $\frac{a}{x}$. If we take $\frac{y}{b}=\frac{a}{x}$ and $\frac{z}{c}=\frac{a}{x}$ we get $\frac{a^2bc}{x}=abc$; therefore $x=a$; and therefore $y=b$ and $z=c$. If we take $\frac{y}{b}=\frac{x}{a}$ and $\frac{z}{c}=\frac{a}{x}$ we also get $x=a$, $y=b$, $z=c$: likewise if we take

$\frac{y}{b}=\frac{a}{x}$ and $\frac{z}{c}=\frac{x}{a}$ we get $x=a$, $y=b$, $z=c$. If we take $\frac{y}{b}=\frac{x}{a}$ and $\frac{z}{c}=\frac{x}{a}$ we get $x^3=a^3$; therefore $x=ka$, $y=kb$, $z=kc$, where k is one of the cube roots of unity; see Art. 360: the last result includes all the others since unity is itself one of the cube roots of unity. Substitute in the last given equation; thus $k^2(a^2+b^2+c^2)=-2(ab+bc+ca)$; this gives the required relation; if we cube both sides it takes the form $(a^2+b^2+c^2)^3=-8(ab+bc+ca)^3$.

4. Square and add the given equations; thus

$$\frac{y^2}{z^2}+\frac{z^2}{y^2}+\frac{z^2}{x^2}+\frac{x^2}{z^2}+\frac{x^2}{y^2}+\frac{y^2}{x^2}+6=a^2+b^2+c^2.$$

Multiply the given equations together; thus

$$\frac{y^2}{z^2}+\frac{z^2}{y^2}+\frac{z^2}{x^2}+\frac{x^2}{z^2}+\frac{x^2}{y^2}+\frac{y^2}{x^2}+2=abc.$$

Hence, by subtraction, $4=a^2+b^2+c^2-abc$.

5. Multiply together; $x^2y^2z^2(y+z)(z+x)(x+y)=a^3b^3c^3$,

that is $$a^2b^2c^2\{x^2(y+z)+y^2(z+x)+z^2(x+y)+2xyz\}=a^3b^3c^3,$$

that is $$a^2b^2c^2\{a^3+b^3+c^3+2abc\}=a^3b^3c^3.$$

6. $\dfrac{a^3-x^3+b^3-y^3}{a^3-x^3-b^3+y^3}=\dfrac{5(y+x)}{y-x}$; see Art. 396; therefore

$$a^3+b^3-x^3-y^3=\frac{5(y+x)}{y-x}\left\{(x-y)^3-x^3+y^3\right\}=15xy(x+y);$$

therefore $a^3+b^3=x^3+y^3+15xy(x+y)$; also $a^3-b^3=(x-y)^3$; therefore by addition $a^3=x^3+6x^2y+9xy^2=x(x+3y)^2$, similarly by subtraction $b^3=y(y+3x)^2$. Hence $a^{\frac{3}{2}}+b^{\frac{3}{2}}=x^{\frac{1}{2}}(x+3y)+y^{\frac{1}{2}}(y+3x)=(x^{\frac{1}{2}}+y^{\frac{1}{2}})^3$. Thus $(x^{\frac{1}{2}}+y^{\frac{1}{2}})^3=z^{\frac{3}{2}}$.

7. $$a^5-c^5=(x+y)^5-x^5-y^5=5xy(x^3+y^3)+10x^2y^2(x+y);$$

also $$a^3-b^3=(x+y)^3-x^3-y^3=3xy(x+y)=3xya.$$

Thus $$a^5-c^5=\frac{a^3-b^3}{3a}\times 5b^3+10\left(\frac{a^3-b^3}{3a}\right)^2a=\frac{a^3-b^3}{9a}(10a^3+5b^3),$$

therefore $$9a(a^5-c^5)=5(a^3-b^3)(2a^3+b^3).$$

8. By addition and subtraction

$$32\left(\frac{c}{a}+\frac{a}{c}\right)=\left(\frac{x}{a}+\frac{a}{x}\right)^5,\text{ and } 32\left(\frac{c}{a}-\frac{a}{c}\right)=\left(\frac{x}{a}-\frac{a}{x}\right)^5;$$

therefore $$2\left(\frac{c^2+a^2}{ac}\right)^{\frac{1}{5}}=\frac{x}{a}+\frac{a}{x},\text{ and } 2\left(\frac{c^2-a^2}{ac}\right)^{\frac{1}{5}}=\frac{x}{a}-\frac{a}{x}.$$

Square and subtract; thus $4\left(\dfrac{c^2+a^2}{ac}\right)^{\frac{2}{5}}-4\left(\dfrac{c^2-a^2}{ac}\right)^{\frac{2}{5}}=4$.

9. We shall find that $\alpha\beta=\dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}+\dfrac{y^2}{z^2}+\dfrac{z^2}{y^2}+\dfrac{z^2}{x^2}+\dfrac{x^2}{z^2}+3$, and this is equal to $\gamma+1$.

10. From the first two equations we find $x=-\dfrac{a-b}{a}$, and $y=\dfrac{a-b}{b}$; substitute in the third equation.

11. Multiply the first equation by y, and the second by x, and add; thus $a(x^2+y^2)=y^3+3x^2y$, that is $a=y^3+3x^2y$. Similarly $\beta=x^3+3xy^2$; therefore $a+\beta=(x+y)^3$, and $(a+\beta)^{\frac{2}{3}}=(x+y)^2$. Similarly $(a-\beta)^{\frac{2}{3}}=(x-y)^2$. Add; thus $(a+\beta)^{\frac{2}{3}}+(a-\beta)^{\frac{2}{3}}=2(x^2+y^2)=2$.

12. Multiply the three equations together, and extract the square root: thus $(x+y)(y+z)(z+x)=\pm 8abcxyz$; therefore $\dfrac{(x+y)(y+z)(z+x)}{xyz}=\pm 8abc$; that is $\dfrac{x}{y}+\dfrac{y}{x}+\dfrac{y}{z}+\dfrac{z}{y}+\dfrac{z}{x}+\dfrac{x}{z}+2=\pm 8abc$; therefore

$$4c^2-2+4a^2-2+4b^2-2+2=\pm 8abc: \text{ then divide by 4.}$$

13. Suppose each of the given fractions equal to k; thus

$$x=k(a^2+x^2),\quad 2y=k(a^2+y^2),\quad 4z=k(a^2+z^2);$$

multiply the first of these equations by y^2-z^2, the second by z^2-x^2, and the third by x^2-y^2, and add; it will be found that the sum on the right-hand side vanishes; thus $x(y^2-z^2)+2y(z^2-x^2)+4z(x^2-y^2)=0$.

14. By addition and subtraction and division by 2 we get $3x^2+y^2=ax$, $3y^2+x^2=by$; multiply the former by y, and the latter by x; then by addition and subtraction $(x+y)^3=(a+b)xy$, $(x-y)^3=(b-a)xy$; therefore $(x+y)^2=\{(a+b)c^2\}^{\frac{2}{3}}$, $(x-y)^2=\{(a-b)c^2\}^{\frac{2}{3}}$; subtract; thus

$$4c^2=c^{\frac{4}{3}}\{(a+b)^{\frac{2}{3}}-(a-b)^{\frac{2}{3}}\}.$$

15. $x'=\dfrac{a}{x}$, $y'=\dfrac{b}{y}$, $z'=\dfrac{c}{z}$; substituting we get

$$2a'=\frac{cy}{z}+\frac{bz}{y},\quad 2b'=\frac{az}{x}+\frac{cx}{z},\quad 2c'=\frac{bx}{y}+\frac{ay}{x};$$

square, multiply respectively by a, b, c and add; thus

$$4(aa'^2+bb'^2+cc'^2)=\frac{ac^2y^2}{z^2}+\frac{ab^2z^2}{y^2}+\frac{ba^2z^2}{x^2}+\frac{bc^2x^2}{z^2}+\frac{cb^2x^2}{y^2}+\frac{ca^2y^2}{x^2}+6abc;$$

and by multiplication,

$$8a'b'c'=\frac{ac^2y^2}{z^2}+\frac{ab^2z^2}{y^2}+\frac{ba^2z^2}{x^2}+\frac{bc^2x^2}{z^2}+\frac{cb^2x^2}{y^2}+\frac{ca^2y^2}{x^2}+2abc:$$

therefore $\quad 4(aa'^2+bb'^2+cc'^2)-6abc=8a'b'c'-2abc$;

therefore $\quad aa'^2+bb'^2+cc'^2=abc+2a'b'c'$.

16. There are 11 letters, among which a, i, and n each occur twice. Thus there are 8 distinct letters giving $8.7.6$ permutations, that is, 336 permutations. There are 7×3 permutations in which the letter a occurs

twice; for any one of the other 7 letters may occupy the first, or the second, or the third place in the permutation. Similarly there are 7×3 permutations in which the letter i occurs twice, and as many in which the letter n occurs twice. Thus the whole number of permutations $= 336 + 63 = 399$.

18. Let us suppose for clearness that the $m-1$ walks are parallel to the *length* of the garden, and the $n-1$ walks parallel to the *breadth* of the garden. The person in walking must pass between n pairs of beds in the direction of the length of the garden, and between m pairs of beds in the direction of the breadth; and the $m+n$ portions of which his path may thus be supposed to consist may occur in any order. Thus the number of ways is equal to the number of permutations of $m+n$ things taken all together, in which m are of one kind, and n are of another kind.

21.
$$p^6 = (x^2 + y^2 - 2xy)(y^2 + z^2 - 2yz)(z^2 + x^2 - 2zx),$$
$$q^6 = (x^2 + y^2 + 2xy)(y^2 + z^2 + 2yz)(z^2 + x^2 + 2zx);$$

hence we see that $p^6 + q^6 - 2r^6 =$

$$8xyz\{x(y^2+z^2) + y(z^2+x^2) + z(x^2+y^2)\} = 8xyz(q^3 - 2xyz);$$

thus
$$16x^2y^2z^2 - 8q^3xyz + p^6 + q^6 - 2r^6 = 0.$$

In the same manner we find that
$$16x^4y^4z^4 - 8r^6x^2y^2z^2 + p^6q^6 + r^{12} - 2s^{12} = 0.$$

From the quadratic in xyz we get $xyz = \frac{1}{4}(q^3 \pm \sqrt{2r^6 - p^6})$;

from the quadratic in $x^2y^2z^2$ we get $x^2y^2z^2 = \frac{1}{4}(r^6 \pm \sqrt{2s^{12} - p^6q^6})$;

therefore
$$4(r^6 \pm \sqrt{2s^{12} - p^6q^6}) = (q^3 \pm \sqrt{2r^6 - p^6})^2.$$

22. Since $X = ax + a_1x_1 + a_2$, $Y = bx + b_1x_1 + b_2$, $Z = cx + c_1x_1 + c_2$,

we have
$$X(bc_1 - b_1c) + Y(ca_1 - c_1a) + Z(ab_1 - a_1b)$$
$$= a_2(bc_1 - b_1c) + b_2(ca_1 - c_1a) + c_2(ab_1 - a_1b); \text{ see Art. 207.}$$

And since $aX + bY + cZ = 0$, $a_1X + b_1Y + c_1Z = 0$, we have

$$\frac{X}{bc_1 - b_1c} = \frac{Y}{ca_1 - c_1a} = \frac{Z}{ab_1 - a_1b}; \text{ see Art. 385.}$$

Substituting for Y and Z in the former result, we get

$$X = \frac{(bc_1 - b_1c)\{a_2(bc_1 - b_1c) + b_2(ca_1 - c_1a) + c_2(ab_1 - a_1b)\}}{(bc_1 - b_1c)^2 + (ca_1 - c_1a)^2 + (ab_1 - a_1b)^2}.$$

Similarly the values of Y and Z are known; then by squaring and adding we obtain the required result.

23. Suppose $n=2$; then we have to shew that $\frac{a_1}{b_1} + \frac{a_2}{b_2}$ is less than $\frac{a_1}{b_2} + \frac{a_2}{b_1}$, or that $a_2\left(\frac{1}{b_2} - \frac{1}{b_1}\right)$ is less than $a_1\left(\frac{1}{b_2} - \frac{1}{b_1}\right)$: and this is obvious since a_2 is less than a_1. Now let n have any value; then when the fractions are

arranged so as to give the *least* sum, in *any pair* of fractions the greater denominator must come under the greater numerator by what has just been shewn: hence the arrangement which produces the least sum must be that which is given in the former part of the Example.

In the same way the arrangement which produces the *greatest* sum must be that which is given in the latter part of the Example.

24. $\left(\frac{a}{b}\right)^{a+b} = \left(1 - \frac{b-a}{b}\right)^{a+b}$; the logarithm of this

$$= -(a-b+2b)\left\{\frac{b-a}{b} + \frac{1}{2}\left(\frac{b-a}{b}\right)^2 + \ldots + \frac{1}{n}\left(\frac{b-a}{b}\right)^n + \ldots\right\};$$

thus we may put the logarithm in the form

$$-2(b-a) - \left(\frac{2}{3} - \frac{1}{2}\right)\frac{(b-a)^3}{b^2} - \ldots - \left(\frac{2}{n} - \frac{1}{n-1}\right)\frac{(b-a)^n}{b^{n-1}} - \ldots$$

25. In the preceding Example suppose that the same quantity is added to a and to b; then the logarithm is numerically *diminished*, and as the logarithm is negative, the fraction $\left(\frac{a}{b}\right)^{a+b}$ is increased.

MISCELLANEOUS EXAMPLES.

1.
$$\begin{aligned} &x - [2y + \{3z - 3x - (x+y)\}] + 2x - (y+3z) \\ &= x - [2y + \{3z - 3x - x - y\}] + 2x - y - 3z \\ &= x - [2y + 3z - 4x - y] + 2x - y - 3z \\ &= x - 2y - 3z + 4x + y + 2x - y - 3z = 7x - 2y - 6z. \end{aligned}$$

2.
$$\begin{array}{l|llll|l} ax^4 - bx^2 + c & a^2x^8 & + (2ac - b^2)x^4 & & + c^2 & ax^4 + bx^2 + c \\ & a^2x^8 - abx^6 + acx^4 & & & & \\ \hline & abx^6 + (ac - b^2)x^4 & & & + c^2 & \\ & abx^6 & - b^2x^4 & + bcx^2 & & \\ \hline & & acx^4 & - bcx^2 & + c^2 & \\ & & acx^4 & - bcx^2 & + c^2 & \\ \hline \end{array}$$

3. Multiply the numerator by 7;

$$\begin{array}{l|l|l} 7x^3 - 4x^2 - 21x + 12 & 35x^3 + 14x^2 - 105x - 42 & 5 \\ & 35x^3 - 20x^2 - 105x + 60 & \\ \hline & 34x^2 \qquad\qquad - 102 & \end{array}$$

Divide by 34;
$$\begin{array}{l|l|l} x^2 - 3 & 7x^3 - 4x^2 - 21x + 12 & 7x - 4 \\ & 7x^3 \qquad - 21x & \\ \hline & -4x^2 \qquad + 12 & \\ & -4x^2 \qquad + 12 & \\ \hline \end{array}$$

Thus $x^2 - 3$ is the G.C.M.

4. $$\frac{3x-a}{5x+3a}+\frac{x+3a}{7x+9a}=\frac{(3x-a)(7x+9a)+(x+3a)(5x+3a)}{(5x+3a)(7x+9a)}=\&c.$$

$$\frac{2a+x}{a^2-x^2}-\frac{a-x}{2a^2+3ax+x^2}=\frac{(2a+x)^2-(a-x)^2}{(2a+x)(a^2-x^2)}=\&c.$$

5. Multiply by 15; $4x+1-5(5x-1)=15(x-2)$; &c.

6. Multiply the second equation by 2 and add to the first; &c.

7. Suppose that B overtakes A in x hours; then B goes $x\times\frac{9}{2}$ miles; and A goes $(x+2\frac{1}{2})\times\frac{13}{4}$ miles: therefore $\frac{9x}{2}=\frac{13}{4}(x+2\frac{1}{2})$.

8. Suppose x the number of guineas; then $x+48$ is the number of half-crowns; therefore $21x+\frac{5}{2}(x+48)=2000$.

9.
$$\begin{array}{rl}
 & a^4+2a^3\quad -a+\frac{1}{4}\ \Big(\ a^2+a-\frac{1}{2} \\
 & a^4 \\ \hline
2a^2+a\ \Big) & 2a^3\quad -a+\frac{1}{4} \\
 & 2a^3+a^2 \\ \hline
2a^2+2a-\frac{1}{2}\ \Big) & -a^2-a+\frac{1}{4} \\
 & -a^2-a+\frac{1}{4} \\ \hline
\end{array}$$

10. $(3-x)(3x-1)=\frac{5x}{2}$; $3x^2-\frac{15x}{2}+3=0$; &c.

11.
$$\begin{aligned}
&a-\{2a-3b-[4a-5b-6c-(7a-8b-9c-10d)]\}\\
&=a-\{2a-3b-[4a-5b-6c-7a+8b+9c+10d]\}\\
&=a-\{2a-3b-[-3a+3b+3c+10d]\}=a-\{2a-3b+3a-3b-3c-10d\}\\
&=a-\{5a-6b-3c-10d\}=a-5a+6b+3c+10d=-4a+6b+3c+10d\\
&=-4+3+9+2=10.
\end{aligned}$$

13.
$$\begin{array}{rrl}
x^2+5x+6\ \Big) & x^2+6x+8 & \Big(\ 1 \\
 & x^2+5x+6 & \\ \hline
 & x+2\ \Big) & x^2+5x+6\ \Big(\ x+3 \\
 & & x^2+2x \\ \hline
 & & 3x+6 \\
 & & 3x+6 \\ \hline
\end{array}$$

Thus $x+2$ is the G.C.M.

14. $12x^2+31ax+20a^2=(3x+4a)(4x+5a)$: thus $(3x+4a)(4x+5a)$ may be taken for the common denominator.

15. Clear of fractions; $(x-2)(x-3)+2(x-1)(x-3)=8(x-1)(x-2)$; &c.

16. Add the equations and divide by 16; thus $x-y=5$. Subtract the first equation from the second and divide by 2; thus $x+y=17$. Then add the two results; &c.

17. Let x denote the number of minutes after 9 o'clock. In x minutes the long hand will move over x divisions of the watch face; and as the long hand moves twelve times as fast as the short hand, the short hand will move over $\frac{x}{12}$ divisions in x minutes. At 9 o'clock the short hand is 45 divisions in advance of the long hand; so that in the x minutes the long hand must pass over 45 more divisions than the short hand: therefore $x=\frac{x}{12}+45$; therefore $12x=x+540$.

18. Suppose that A alone could do the work in x days, and B alone in y days; then A can do $\frac{30}{x}$ of the work in 30 days; therefore $\frac{30}{x}=\frac{3}{5}$. Similarly $\frac{10}{x}+\frac{10}{y}=\frac{2}{5}$.

19.
$$\begin{array}{rl} & 4x^2 - x(12y-4z) + 9y^2 - 6yz + z^2 \;(\; 2x-(3y-z) \\ & 4x^2 \\ \hline 4x-(3y-z)\;) & -x(12y-4z)+9y^2-6yz+z^2 \\ & -x(12y-4z)+(3y-z)^2 \\ \hline \end{array}$$

20. $3(x-3)-(x-1)(x-4)=(x-1)(x-3)$; &c.

21. $a-\{3a-5b-[7a-9b-11c-(13a-15b-17c-19d)]\}$
$=a-\{3a-5b-[7a-9b-11c-13a+15b+17c+19d]\}$
$=a-\{3a-5b-[-6a+6b+6c+19d]=a-\{3a-5b+6a-6b-6c-19d\}$
$=a-\{9a-11b-6c-19d\}=a-9a+11b+6c+19d=-8a+11b+6c+19d$
$=-8+22+1+2=17$.

23. We shall find that $x+2$ is the G.C.M. of x^2-4 and $x^2+10x+16$; and $x+2$ divides $x^2-7x-18$; so it is the G.C.M. of the three expressions.

24. $\frac{2x^2-x+2}{4x^3+3x+2}=\frac{1}{2x+1}$, and $\frac{4x^2-1}{2x-1}=2x+1$.

25. Multiply by 60; $20(x-1)+3(11x-3)-6(3x-9)=130$; &c.

26. Multiply the first equation by 12; $7x+y=11$; &c.

27. Suppose that they meet x miles from Ely. The first person takes $\frac{x}{4\frac{4}{9}}$ hours; the second person walks in an hour $\frac{60}{18}$ miles, that is $3\frac{1}{3}$ miles: he therefore takes $\frac{16-x}{3\frac{1}{3}}$ hours to walk $16-x$ miles. Thus $\frac{x}{4\frac{4}{9}}=\frac{16-x}{3\frac{1}{3}}$, that is $\frac{9x}{40}=\frac{3(16-x)}{10}$.

28. Suppose that there are x benches, and that y persons sit on each bench: then the whole number of persons is xy; therefore $xy=(x+10)(y-1)$, and $xy=(x-15)(y+2)$. Thus $10y-x-10=0$, and $-15y+2x-30=0$.

29.
$$\begin{array}{rl}
 & x^6-4x^5+6x^4-8x^3+9x^2-4x+4 \;\big(\; x^3-2x^2+x-2 \\
 & x^6 \\ \hline
2x^3-2x^2 \;\big) & -4x^5+6x^4-8x^3+9x^2-4x+4 \\
 & -4x^5+4x^4 \\ \hline
2x^3-4x^2+x \;\big) & 2x^4-8x^3+9x^2-4x+4 \\
 & 2x^4-4x^3+\ \ x^2 \\ \hline
2x^3-4x^2+2x-2 \;\big) & -4x^3+8x^2-4x+4 \\
 & -4x^3+8x^2-4x+4 \\ \hline
\end{array}$$

30. $11x^2-9x=\dfrac{45}{4}$; &c.

31. $\dfrac{1\times2+3\times4}{2\times3-1\times4}=\dfrac{2+12}{6-4}=\dfrac{14}{2}=7,\qquad 3^2-2^3=9-8=1,$

$$\sqrt[3]{(16+9+2)}=\sqrt[3]{27}=3.$$

32. $\left(\dfrac{3}{10}+x^2\right)\left(\dfrac{2}{10}-x\right)=\dfrac{6}{100}-\dfrac{3x}{10}+\dfrac{2x^2}{10}-x^3.$

33.
$$\begin{array}{rl}
x^4-115x+24 \;\big) & 24x^4-115x^3 \qquad\qquad +\ 1 \;\big(\; 24 \\
 & 24x^4 \qquad\qquad -2760x+576 \\ \hline
 & -115x^3+2760x-575
\end{array}$$

Divide by -115;
$$\begin{array}{rl}
x^3-24x+5 \;\big) & x^4 \qquad\quad -115x+24 \;\big(\; x \\
 & x^4-24x^2+\ \ 5x \\ \hline
 & 24x^2-120x+24
\end{array}$$

Divide by 24;
$$\begin{array}{rl}
x^2-5x+1 \;\big) & x^3 \qquad\quad -24x+5 \;\big(\; x+5 \\
 & x^3-5x^2+\ \ x \\ \hline
 & 5x^2-25x+5 \\
 & 5x^2-25x+5 \\ \hline
\end{array}$$

Thus x^2-5x+1 is the G. C. M.

34. $\dfrac{(1+x)(1+x)+(1-x)(1-x)+1+x^2}{1-x^2}=\dfrac{3(1+x^2)}{1-x^2}.$

35. $x^3-9x^2+27x-27-3(x^3-6x^2+12x-8)+3(x^3-3x^2+3x-1)-x^3=9-x$; that is $-6=9-x$.

36. Multiply the first equation by 5, and the second by 3, and subtract; thus $y=7$; &c.

37. Suppose that he bought x sheep of each kind; then he spent $3x+4x$ pounds, that is $7x$ pounds. Half his money is $\dfrac{7x}{2}$ pounds; thus he can get $\dfrac{7x}{6}$ sheep at £3 each, and $\dfrac{7x}{8}$ sheep at £4 each. Therefore $\dfrac{7x}{6}+\dfrac{7x}{8}=2x+2$.

38. As 20 women are to receive £60, each woman receives £3. Suppose that a man receives £x, and a child £y. Then $15x+30y=177-60$, $x+y=6$.

39.
$$\frac{x^2}{y^2}-\frac{x}{y}-\frac{3}{4}+\frac{y}{2x}+\frac{y^2}{4x^2}\left(\frac{x}{y}-\frac{1}{2}-\frac{y}{2x}\right.$$
$$\frac{x^2}{y^2}$$
$$\frac{2x}{y}-\frac{1}{2}\Big)\ -\frac{x}{y}-\frac{3}{4}+\frac{y}{2x}+\frac{y^2}{4x^2}$$
$$-\frac{x}{y}+\frac{1}{4}$$
$$\frac{2x}{y}-1-\frac{y}{2x}\Big)\ -1+\frac{y}{2x}+\frac{y^2}{4x^2}$$
$$-1+\frac{y}{2x}+\frac{y^2}{4x^2}$$

40. $2x(x+1)+(x-4)(4x-3)=9(x-4)(x+1)$; &c.

41. $\dfrac{5\times7-1\times3}{3\times5-1\times7}=\dfrac{35-3}{15-7}=\dfrac{32}{8}=4$, $\quad 3^3-5^2=27-25=2$,

$\sqrt[3]{(49+15)}=\sqrt[3]{64}=4$.

42. Each expression reduces to $a(a^2-3ax+2x^2)$.

43.
$$x^3-1\Big)\ x^5-x^4+x^3-x^2+x-1\ \Big(x^2-x+1$$
$$x^5 \qquad -x^2$$
$$-x^4+x^3 \qquad +x-1$$
$$-x^4 \qquad +x$$
$$x^3 \qquad -1$$
$$x^3 \qquad -1$$

Thus x^3-1 is the G.C.M.; and therefore $x^5-x^4+x^3-x^2+x-1$ is the L.C.M.

44. $\dfrac{x^2+5x+6}{x^2+5x}\times\dfrac{x^2+6x+5}{x^2+3x}=\dfrac{(x+2)(x+3)}{x(x+5)}\times\dfrac{(x+1)(x+5)}{x(x+3)}=\dfrac{(x+2)(x+1)}{x^2}$.

45. Clear of fractions; thus
$$(2x-1)(3x-1)(6x-1)+4(x+1)(3x-1)(6x-1)+9(x+1)(2x-1)(6x-1)$$
$$=36(x+1)(2x-1)(3x-1);$$
this reduces to $\qquad -84x+12=-144x+36$.

46. Multiply the second equation by 8, and add to the first; thus $58x-29y-29=0$; divide by 29; $2x-y-1=0$. Again, multiply the second equation by 3, and add to the third; &c.

47. Let x denote the number of gallons; then the mixture is sold for $12(x+80)$ shillings: therefore $12(x+80)=15\times80\left(1+\dfrac{10}{100}\right)$.

48. Suppose that A can do the work alone in x days, B in y days, and C in z days; then $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{12}$, $\dfrac{1}{x}+\dfrac{1}{z}=\dfrac{1}{15}$, $\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{20}$. By addition

$2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\frac{1}{5}$; therefore $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{10}$. Thus A, B and C together can do $\frac{1}{10}$ of the work in one day, and therefore can complete the work in 10 days.

49. Assume for the square root $\sqrt{x}+\sqrt{y}$;
then as in Art. 301 we have $x+y=a-c$, $\sqrt{(xy)}=\sqrt{(ab+bc-ac-b^2)}$;
therefore $(x-y)^2=(a-c)^2-4(ab+bc-ac-b^2)=(2b-a-c)^2$; &c.

50. $4-\frac{3}{4-x}=\frac{16-4x-3}{4-x}=\frac{13-4x}{4-x}$; $3\div\frac{13-4x}{4-x}=\frac{12-3x}{13-4x}$;

$$4-\frac{12-3x}{13-4x}=\frac{52-16x-12+3x}{13-4x}=\frac{40-13x}{13-4x};$$

$$3\div\frac{40-13x}{13-4x}=\frac{39-12x}{40-13x}.\quad \text{Thus } x=\frac{39-12x}{40-13x};$$

therefore $13x^2-52x+39=0$; therefore $x^2-4x+3=0$.

51. When the expression is worked out the coefficient of x is $a+b+c+a+b-c-(b+c-a)+c+a-b$, that is $4a$; similarly we can find the coefficient of y, and the coefficient of z.

52. $(s-a+s-b)^3=(s-a)^3+(s-b)^3+3(s-a)(s-b)(s-a+s-b)$;
and $s-a+s-b=2s-a-b=c$.

53.
$$x^4-2x^3y+5x^2y^2-2xy^3+4y^4\ \Big)\ \begin{array}{r} x^4-3x^3y+6x^2y^2-3xy^3+5y^4 \\ \underline{x^4-2x^3y+5x^2y^2-2xy^3+4y^4} \\ -\ x^3y+\ x^2y^2-\ xy^3+\ y^4 \end{array}\ \Big(\ 1$$

Divide by $-y$;
$$x^3-x^2y+xy^2-y^3\ \Big)\ \begin{array}{r} x^4-2x^3y+5x^2y^2-2xy^3+4y^4 \\ \underline{x^4-\ x^3y+\ x^2y^2-\ xy^3\qquad} \\ -\ x^3y+4x^2y^2-\ xy^3+4y^4 \\ \underline{-\ x^3y+\ x^2y^2-\ xy^3+\ y^4} \\ 3x^2y^2\qquad\quad +3y^4 \end{array}\ \Big(\ x-y$$

Divide by $3y^2$;
$$x^2+y^2\ \Big)\ \begin{array}{r} x^3-x^2y+xy^2-y^3 \\ \underline{x^3\qquad\ +xy^2\qquad} \\ -x^2y\qquad\ -y^3 \\ \underline{-x^2y\qquad\ -y^3} \end{array}\ \Big(\ x-y$$

Thus x^2+y^2 is the G.C.M.

54. $$\frac{(x-a)^2+(x-b)^2-(a-b)^2}{(x-a)(x-b)}=\frac{2x^2-2xa-2xb+2ab}{(x-a)(x-b)}=\frac{2(x-a)(x-b)}{(x-a)(x-b)}=2.$$

55. $9x^2-6x+1+16x^2-16x+4=25x^2-30x+9$; &c.

56. Clear of fractions, and simplify; thus we get $x-y=6$, $x+y=-3$.

57. Suppose that A could complete the work alone in x days, B in y days, and C in z days. Since A, B and C together can do a third of the

work in 3 days, $\frac{3}{x}+\frac{3}{y}+\frac{3}{z}=\frac{1}{3}$. Similarly $\frac{4}{y}+\frac{4}{z}=\frac{1}{3}$, and $\frac{5}{z}=\frac{1}{3}$. The last equation finds z, then the second finds y, and then the first finds x.

58. Suppose that in going from A to B the person walks x miles up-hill, y miles on a level, and z miles down-hill; then on returning he walks z miles up-hill, y miles on a level, and x miles down-hill. Thus

$$\frac{x}{3}+\frac{y}{3\frac{1}{4}}+\frac{z}{3\frac{1}{2}}=2\frac{17\frac{1}{2}}{60}=2\tfrac{7}{24}, \quad \frac{x}{3\frac{1}{2}}+\frac{y}{3\frac{1}{4}}+\frac{z}{3}=2\tfrac{1}{3}.$$

Also $x+y+z=7\frac{1}{2}$. Add together the first and second equations; thus $(x+z)\left(\frac{1}{3}+\frac{2}{7}\right)+\frac{8y}{13}=4\frac{5}{8}$. Substitute for $x+z$; thus $\left(\frac{15}{2}-y\right)\frac{13}{21}+\frac{8y}{13}=4\frac{5}{8}$. Clear of fractions; &c.

59.

$$\begin{array}{r} 6x^3-x^2 \\ -2x^2 \\ \hline 6x^3-3x^2-3 \end{array} \qquad \begin{array}{l} 12x^6 \\ \qquad -x^2(6x^3-x^2) \\ \hline 12x^6-6x^5+x^4 \\ \qquad\qquad\quad x^4 \\ \hline 12x^6-12x^5+3x^4 \\ \qquad\qquad -3(6x^3-3x^2-3) \\ \hline 12x^6-12x^5+3x^4-18x^3+9x^2+9 \end{array}$$

$$\begin{array}{r|l} 8x^9-12x^8+6x^7-37x^6+36x^5-9x^4+54x^3-27x^2-27 & 2x^3-x^2-3 \\ 8x^9 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad & \\ \hline -12x^8+6x^7-37x^6+36x^5-9x^4+54x^3-27x^2-27 & \\ -12x^8+6x^7-x^6 \qquad\qquad\qquad\qquad\qquad\quad & \\ \hline -36x^6+36x^5-9x^4+54x^3-27x^2-27 & \\ -36x^6+36x^5-9x^4+54x^3-27x^2-27 & \\ \hline \end{array}$$

60. Clear of fractions; thus

$$(x^2-a^2)\{m(x^2-b^2)+xb(m^2-1)\}=(x^2-b^2)\{m(x^2-a^2)-xa(m^2-1)\};$$

therefore $(x^2-a^2)xb=-(x^2-b^2)xa$; therefore $x=0$ or $b(x^2-a^2)=-a(x^2-b^2)$: the latter gives $x^2(a+b)=(a+b)ab$.

61. $$24\left\{x-\frac{1}{2}(x-1)\right\}\left\{x-\frac{2}{3}(x-2)\right\}\left\{x-\frac{3}{4}(x-1\tfrac{1}{3})\right\}$$

$$=24\,\frac{2x-x+1}{2}\cdot\frac{3x-2x+4}{3}\cdot\frac{4x-3x+4}{4}=(x+1)(x+4)(x+4).$$

And $(x+2)(x+3)(x+4)-(x+1)(x+4)(x+4)$

$$=(x+4)\{(x+2)(x+3)-(x+1)(x+4)\}=(x+4)\{x^2+5x+6-(x^2+5x+4)\}=2(x+4).$$

62. $$\frac{x^2}{a^2}+\frac{a^2}{x^2}-2=\frac{x^4+a^4-2x^2a^2}{x^2a^2}=\frac{(x^2-a^2)^2}{x^2a^2}; \quad \frac{x}{a}-\frac{a}{x}=\frac{x^2-a^2}{xa};$$

$$\left\{\frac{(x^2-a^2)^2}{x^2a^2}\right\}^2\div\frac{x^2-a^2}{xa}=\frac{(x^2-a^2)^4}{x^4a^4}\times\frac{xa}{x^2-a^2}=\frac{(x^2-a^2)^3}{x^3a^3}.$$

63. Multiply the first expression by 7;

$$7x^2-23xy+6y^2 \;)\; 35x^3-126x^2y+77xy^2-42y^3 \;(\; 5x$$
$$35x^3-115x^2y+30xy^2$$
$$-\;11x^2y+47xy^2-42y^3$$

Divide by $-y$, multiply by 7, and continue the division;

$$77x^2-329xy+294y^2 \;(\; 11$$
$$77x^2-253xy+\;66y^2$$
$$-\;76xy+228y^2$$

Divide by $-76y$; $\quad x-3y \;)\; 7x^2-23xy+6y^2 \;(\; 7x-2y$

$$7x^2-21xy$$
$$-\;2xy+6y^2$$
$$-\;2xy+6y^2$$

Thus $x-3y$ is the G.C.M.

64. The fractions may be expressed with the common denominator x^8+x^4+1, and then added together. Or we may proceed thus: add together the first and second fractions; this gives $\dfrac{(x^2-x+1)^2+2x(x-1)^2}{x^4+x^2+1}$, which reduces to $\dfrac{x^4-x^2+1}{x^4+x^2+1}$; then add this to the third fraction, and we obtain in the same way $\dfrac{x^8-x^4+1}{x^8+x^4+1}$.

65. $$\frac{4(x-3)-(x-6)(x-2)}{(x-6)(x-3)}=\frac{x^2-16-2(x-1)(x-5)}{(x-5)(x-4)};\ \text{therefore}$$

$$(x-5)(x-4)(-x^2+12x-24)=(x-6)(x-3)(-x^2+12x-26);$$

that is $(x^2-9x+20)(-x^2+12x-24)=(x^2-9x+18)(-x^2+12x-26)$,

therefore $(x^2-9x)(-x^2+12x)+20(-x^2+12x)-24(x^2-9x)-20\times 24$

$$=(x^2-9x)(-x^2+12x)+18(-x^2+12x)-26(x^2-9x)-18\times 26;$$

therefore $\quad 2(-x^2+12x)+2(x^2-9x)=20\times 24-18\times 26$; &c.

66. Clear of fractions and simplify; then the equations become

$$bx-ay=0,\quad 2bx-ay=ab.$$

67. Let x denote the number of pounds he gave for the house; then the total outlay in pounds was $x+\dfrac{4x}{100}$, that is $\dfrac{26x}{25}$. His loss owing to the house standing empty is reckoned at $\dfrac{26x}{25}\times\dfrac{5}{100}$. Then $1192-\dfrac{26x}{25}-\dfrac{26x}{25}\times\dfrac{5}{100}$ is the gain; and therefore this $=\dfrac{10x}{100}$.

68. Suppose that at the first division x votes were given for the resolution and y against it; then $x-y=\dfrac{1}{8}y$; and $x-10-(y+10)=1$.

69. Transpose and square; $x-2\sqrt{xa}+a=x+a-b-2\sqrt{b(x+a-b)}+b$; therefore $\sqrt{xa}=\sqrt{b(x+a-b)}$; square; $xa=b(x+a-b)$; therefore $x(a-b)=b(a-b)$.

70. This equation can be solved in the ordinary way. Or we may write it thus: $(x-2)(x-3)=\left(2+\frac{1}{77}\right)\left(1+\frac{1}{77}\right)$, and thus it is obvious that one solution is given by $x-2=2+\frac{1}{77}$, that is $x=4\frac{1}{77}$. And as the sum of the roots is 5, the other root is $\frac{76}{77}$. See Art. 336.

71. $\sqrt[3]{\{(2+6-3)^2 5\}}+\sqrt[3]{\{(3+5)(25-24)\}}+\sqrt[3]{\{(6-2)(5-3)\}}$
$$=\sqrt[3]{5^3}+\sqrt[3]{8}+\sqrt[3]{8}=5+2+2=9.$$

72. Each expression $=x(y^2+z^2)+y(z^2+x^2)+z(x^2+y^2)+2xyz$.

73. Multiply the first expression by 4;

$$\begin{array}{r|r|l} 4x^3-15x^2-38x+65 & 20x^3-76x^2+220x-1700 & 5 \\ & \underline{20x^3-75x^2-190x+\ \ 325} & \\ & -x^2+410x-2025 & \end{array}$$

Change the signs of the new divisor;

$$\begin{array}{r|r|l} x^2-410x+2025 & 4x^3-\quad 15x^2-\quad 38x+65 & 4x \\ & \underline{4x^3-1640x^2+8100x\qquad} & \\ & 1625x^2-8138x+65 & \end{array}$$

Divide by 13 and continue the division;

$$\begin{array}{r|l} 125x^2-\quad 626x+\qquad\quad 5 & 125 \\ \underline{125x^2-51250x+253125} & \\ 50624x-253120 & \end{array}$$

Divide by 50624;

$$\begin{array}{r|r|l} x-5 & x^2-410x+2025 & x-405 \\ & \underline{x^2-\quad 5x\qquad\quad} & \\ & -405x+2025 & \\ & \underline{-405x+2025} & \end{array}$$

Thus $x-5$ is the G.C.M.

74. Take for the common denominator $(a-b)(b-c)(c-a)$, then the numerator is $(c-b)bc(x-a)^2+(a-c)ca(x-b)^2+(b-a)ab(x-c)^2$, that is $x^2\{(c-b)bc+(a-c)ca+(b-a)ab\}$, that is $x^2(a-b)(b-c)(c-a)$.

75. Square; $(x-a)^2+2ab+b^2=(x-a)^2+2b(x-a)+b^2$; therefore $2ab=2b(x-a)$; therefore $x=2a$.

76. $ax+cy+bz=h$, $cx+by+az=h$, $bx+ay+cz=h$, where h stands for $a^3+b^3+c^3-3abc$. Subtract the second equation from the first, and the third from the second; thus

$$(a-c)x+(c-b)y+(b-a)z=0, \quad (c-b)x+(b-a)y+(a-c)z=0.$$

Hence $\dfrac{x}{(c-b)(a-c)-(b-a)^2}=\dfrac{y}{(b-a)(c-b)-(a-c)^2}=\dfrac{z}{(a-c)(b-a)-(c-b)^2}$, by Art. 385. It will be found that each of these three denominators reduces to $ab+bc+ca-a^2-b^2-c^2$, so that $x=y=z$. Hence substituting in one of the given equations we have $(a+b+c)x=a^3+b^3+c^3-3abc$; divide by $a+b+c$; therefore $x=a^2+b^2+c^2-bc-ca-ab$.

77. Denote the length of the circuit by a miles; then A walks $3a$ miles in half an hour, that is at the rate of $6a$ miles in an hour; similarly B walks at the rate of $9a$ miles in an hour. And at the end of half an hour we may consider that A is $\dfrac{a}{2}$ miles in advance of B. Suppose that B overtakes A in x hours; then $9a\times x=\dfrac{a}{2}+6a\times x$, thus $3ax=\dfrac{a}{2}$; therefore $x=\dfrac{1}{6}$.

78. Suppose that a mackerel cost x pence on the first day, and y pence on the second day; then $\dfrac{240}{12x}$ is the number of dozens which could be bought for a sovereign on the first day, and $\dfrac{12}{y}$ is the number of mackerel which could be bought for a shilling on the second day. Therefore $\dfrac{12}{y}=2\times\dfrac{240}{12x}$, and $10x+10y=26$.

79. We know that $(p+q)^3=p^3+q^3+3pq(p+q)$. Thus we get

$$x^3=a+\sqrt{a^2+b^3}+a-\sqrt{a^2+b^3}+3x\sqrt[3]{(a+\sqrt{a^2+b^3})}\sqrt[3]{(a-\sqrt{a^2+b^3})},$$

that is $x^3=2a+3x\sqrt[3]{(a^2-a^2-b^3)}$, that is $x^3=2a-3xb$.

80. Transpose and cube; thus

$x^3+8x^2+16x-1=(x+3)^3=x^3+9x^2+27x+27$; therefore $x^2+11x+28=0$.

81. This can be shewn by actual multiplication.

82. This can be shewn by actual development of the expression $X^2+Y^2+Z^2-YZ-ZX-XY$.

83. Multiply the first expression by 8;

$$\begin{array}{r|l|l}
8x^4-13ax^3+5a^2x^2-3a^3x+3a^4 & 56x^4-80ax^3+24a^2x^2-32a^3x+32a^4 & 7 \\
 & 56x^4-91ax^3+35a^2x^2-21a^3x+21a^4 & \\ \hline
 & 11ax^3-11a^2x^2-11a^3x+11a^4 &
\end{array}$$

Divide by $11a$;

$$\begin{array}{r|l|l}
x^3-ax^2-a^2x+a^3 & 8x^4-13ax^3+5a^2x^2-3a^3x+3a^4 & 8x-5a \\
 & 8x^4-8ax^3-8a^2x^2+8a^3x & \\ \hline
 & -5ax^3+13a^2x^2-11a^3x+3a^4 & \\
 & -5ax^3+5a^2x^2+5a^3x-5a^4 & \\ \hline
 & 8a^2x^2-16a^3x+8a^4 &
\end{array}$$

Divide by $8a^2$;

$$\begin{array}{r|l|l}
x^2-2ax+a^2 & x^3-ax^2-a^2x+a^3 & x+a \\
 & x^3-2ax^2+a^2x & \\ \hline
 & ax^2-2a^2x+a^3 & \\
 & ax^2-2a^2x+a^3 & \\ \hline
\end{array}$$

Thus $x^2-2ax+a^2$ is the G.C.M.

84. The common denominator is the product of all the denominators; and the example may be worked in the ordinary way. Or we may proceed thus: $\frac{1}{1-x}-\frac{1}{1+x}=\frac{2x}{1-x^2}$; $\frac{2x}{1-x^2}-\frac{2x}{1+x^2}=\frac{4x^3}{1-x^4}$; $\frac{4x^3}{1-x^4}-\frac{4x^3}{1+x^4}=\frac{8x^7}{1-x^8}$; $\frac{8x^7}{1-x^8}-\frac{8x^7}{1+x^8}=\frac{16x^{15}}{1-x^{16}}$.

85. Clear of fractions. Or we may proceed thus:

$$2x+\frac{2x+1}{2x^2+2x+3}=2x+\frac{1}{x+1}; \text{ therefore } \frac{2x+1}{2x^2+2x+3}=\frac{1}{x+1};$$

therefore $(x+1)(2x+1)=2x^2+2x+3$; &c.

86. Multiply the first equation by $bc+ca+ab$, and the second by $a+b+c$, and subtract; thus $(a^2-bc)x+(b^2-ca)y+(c^2-ab)z=0$. Combine this with the third given equation; thus by Art. 385,

$$\frac{x}{(b+c)h}=\frac{y}{(c+a)h}=\frac{z}{(a+b)h},$$

where h stands for $a^2+b^2+c^2-bc-ca-ab$. Thus we have simply

$$\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a+b}=k \text{ say.}$$

Substitute in the first given equation and we find that $k=\frac{1}{2}$.

87. Suppose the amount of ordinary stock to be x pounds. Thus the dividend paid to the shareholders in pounds is $\frac{5x}{100}+\frac{7\frac{1}{2}}{100}\times 400000$; this therefore must be equal to $\frac{6}{100}(x+400000)$.

88. Suppose that the man walks x miles an hour up-hill, y miles an hour on level ground, and z miles an hour down-hill. Then $\frac{5}{x}+\frac{4}{y}+\frac{6}{z}=3\frac{13}{15}$, $\frac{6}{x}+\frac{4}{y}+\frac{5}{z}=4$. And when he walks half way to B and back again he walks 5 miles up-hill, then $2\frac{1}{2}$ miles on level ground, then $2\frac{1}{2}$ miles again on level ground, and finally 5 miles down-hill: thus $\frac{5}{x}+\frac{5}{y}+\frac{5}{z}=3\frac{11}{12}$. Subtract the first equation from the second; $\frac{1}{x}-\frac{1}{z}=\frac{2}{15}$. Again, multiply the second equation by 5, and the third by 4, and subtract; $\frac{10}{x}+\frac{5}{z}=4\frac{1}{3}$; &c.

89. $$\left\{\frac{1}{161+\sqrt{19360}}\right\}^{\frac{1}{2}}=\left\{\frac{161-\sqrt{19360}}{(161)^2-19360}\right\}^{\frac{1}{2}}=\left\{\frac{161-\sqrt{19360}}{6561}\right\}^{\frac{1}{2}}$$

$$=\frac{1}{81}\left\{161-\sqrt{19360}\right\}^{\frac{1}{2}}=\frac{1}{81}\left\{11-\sqrt{40}\right\}: \text{ see Art. 301.}$$

90. $a(x+b-c)+b(x+a-c)=2(x+a-c)(x+b-c)$; therefore

$$2x^2+x(a+b-4c)=-2c^2+(a+b)c;\ \text{therefore}$$

$$x^2+\frac{x(a+b-4c)}{2}+\left(\frac{a+b-4c}{4}\right)^2=-c^2+\frac{(a+b)c}{2}+\frac{(a+b-4c)^2}{16}=\frac{(a+b)^2}{16};$$

therefore $$x+\frac{a+b-4c}{4}=\pm\frac{a+b}{4};\ \&c.$$

91.
$$\begin{aligned}&3x-[5y-\{2x-(3z-3y)+2z-(x-2y-z)\}]\\ &=3x-[5y-\{2x-3z+3y+2z-x+2y+z\}]\\ &=3x-[5y-\{x+5y\}]=3x-[5y-x-5y]=3x+x\\ &=4x=20.\end{aligned}$$

92. These results may be established by working out the expressions. Or thus: by Example 81 if $p+q+r=0$ we have $p^4+q^4+r^4=2q^2r^2+2r^2p^2+2p^2q^2$; and thus of course each of them is equal to half the sum of the two, that is to $\frac{1}{2}(p^4+q^4+r^4+2q^2r^2+2r^2p^2+2p^2q^2)$, that is to $\frac{1}{2}(p^2+q^2+r^2)^2$. Then if we suppose $p=x-y$, $q=y-z$, $r=z-x$, we have $p+q+r=0$; and $p^2+q^2+r^2=2(x^2+y^2+z^2-yz-zx-xy)$: thus the results are established.

93. $x^3+(m-3)x^2-(2m^2+3m)x+6m^2\ )\ x^3+(5m-3)x^2+(6m^2-15m)x-18m^2\ (\ 1$

$$\begin{array}{l}x^3+(m-3)x^2-(2m^2+3m)x+6m^2\\ \hline 4mx^2+(8m^2-12m)x-24m^2\end{array}$$

Divide by $4m$; $x^2+(2m-3)x-6m\ )\ x^3+(m-3)x^2-(2m^2+3m)x+6m^2\ (\ x-m$

$$\begin{array}{l}x^3+(2m-3)x^2-6mx\\ \hline -mx^2-(2m^2-3m)x+6m^2\\ -mx^2-(2m^2-3m)x+6m^2\\ \hline\end{array}$$

Thus $x^2+(2m-3)x-6m$ is the G.C.M.; that is $(x-3)(x+2m)$.

94. The given fraction $=\dfrac{a^3(c-b)+b^3(a-c)+c^3(b-a)}{a^2(c-b)+b^2(a-c)+c^2(b-a)}$; the denominator is known to be equal to $(a-b)(b-c)(c-a)$. The numerator is equal to $(a-b)(b-c)(c-a)(a+b+c)$; this may be shewn by actual work, or by the method of Art. 808

95. Clear of fractions; thus we get $4(x-1)^5=0$.

96. Put u for the value of $x-a$, $y-b$, $z-c$; thus $x=u+a$, $y=u+b$, $z=u+c$: substitute in the first equation, and we get

$$(u+a)^3+(u+b)^3+(u+c)^3=3(u+a)(u+b)(u+c);$$

this reduces to $3u(a^2+b^2+c^2-bc-ca-ab)=3abc-a^3-b^3-c^3$;

therefore $$u=-\frac{1}{3}(a+b+c).$$

97. Let x denote the number of sixpences, y the number of shillings, and z the number of half-crowns. Then $x+y+z=102$, $x=2y=5z$.

98. Suppose that the distance from A to B is x miles, and that the distance from B to C is y miles. Then $\frac{x}{3\frac{1}{2}}+\frac{y}{4}$ is the number of hours the person takes to walk; and this is equal to $\frac{x+y}{3\frac{3}{4}}$. And it is also equal to $\frac{y}{3\frac{3}{4}}+\frac{x}{4}+\frac{14}{60}$. Thus $\frac{2x}{7}+\frac{y}{4}=\frac{4(x+y)}{15}=\frac{4y}{15}+\frac{x}{4}+\frac{14}{60}$; therefore $\frac{4x}{15}=\frac{x}{4}+\frac{14}{60}$; therefore $x=14$; &c.

99. $X^3+Y^3+Z^3-3XYZ=(X+Y+Z)(X^2+Y^2+Z^2-YZ-ZX-XY)$. Now $X+Y+Z=(a+b+c)(x+y+z)$; hence by the aid of Example 82 the required result follows.

101. $$(4x+2)^4-(2x+4)^4=(3x-1)^4-(x-3)^4.$$

Now $(4x+2)^4-(2x+4)^4=\{(4x+2)^2+(2x+4)^2\}\{(4x+2)^2-(2x+4)^2\}$

$$=\{(4x+2)^2+(2x+4)^2\}\{4x+2+2x+4\}\{4x+2-2x-4\}$$

$$=\{(4x+2)^2+(2x+4)^2\}\,12(x+1)(x-1).$$

Similarly $(3x-1)^4-(x-3)^4=\{(3x-1)^2+(x-3)^2\}\,8(x+1)(x-1)$.

Hence we have either $(x+1)(x-1)=0$, or

$$12\{(4x+2)^2+(2x+4)^2\}=8\{(3x-1)^2+(x-3)^2\};$$

the latter reduces to $x^2+3x+1=0$.

102. Let x denote the middle number; then $(x-1)x(x+1)=15x$; therefore $x^2-1=15$.

103. The second equation gives $2(x+y)=xy$; thus $xy=18$; substitute $9-x$ for y, &c.

104. We have given that $x=pyz$ and $y=q(x+z)$, where p and q are certain constant quantities: therefore $x=pqz(x+z)$. Now put $x=2$ and $z=2$; thus we obtain $pq=\frac{1}{4}$. Hence the relation between x and z is $4x=z(x+z)$. If we put $x=9$ we shall find $z=3$ or -12.

105. $\frac{18}{2}\left(2-17\times\frac{1}{6}\right)=9\times-\frac{5}{6}=-\frac{15}{2}$.

106. $$14\frac{\left(-\frac{1}{2}\right)^6-1}{-\frac{1}{2}-1}=\frac{28}{3}\left\{1-\left(-\frac{1}{2}\right)^6\right\}=\frac{28}{3}\left(1-\frac{1}{2^6}\right);$$

and $$\frac{14}{1-\left(-\frac{1}{2}\right)}=\frac{14}{1+\frac{1}{2}}=\frac{28}{3}.$$

107. $$\frac{\underline{|2n}}{\underline{|n-1}\,\underline{|n+1}}=\frac{132}{35}\times\frac{\underline{|2n-2}}{\underline{|n}\,\underline{|n-2}};\text{ therefore }\frac{2n(2n-1)}{(n-1)(n+1)}=\frac{132}{35};$$

therefore $35n(2n-1)=66(n^2-1)$; therefore $4n^2-35n+66=0$. The only admissible root of this quadratic is 6.

108. $(2-1)^m = 1^m = 1$.

109. By Art. 528 the general form of the coefficient is $\dfrac{\lfloor n}{\lfloor p \lfloor q \lfloor r \lfloor s \lfloor t \ldots}$ where $p+q+r+s+t+\ldots = n$; and in the present case none of the quantities $p, q, r, \ldots$ is to be greater than unity, so that the coefficient is $\lfloor n$.

110. $(1\cdot08)^x$ is to lie between 1000 and 10000, so that its logarithm must lie between 3 and 4. Now

$$\log(1\cdot08)^x = x\log\frac{108}{100} = x\log\frac{2^2.3^3}{100} = x(2\log 2 + 3\log 3 - 2) = x \times \cdot0334239.$$

Thus x may have any integral value not less than $\dfrac{3}{\cdot0334239}$ and not greater than $\dfrac{4}{\cdot0334239}$. It will be found that x must lie between 90 and 119, both inclusive.

111. Square;

$$a(b+x-a) + b(x+a-b) + 2\{ab(b+x-a)(x+a-b)\}^{\frac{1}{2}} = x(a+b-x);$$

therefore $\quad x^2 - (a-b)^2 = -2\{ab(b+x-a)(x+a-b)\}^{\frac{1}{2}}$;

square; $\quad \{x^2 - (a-b)^2\}^2 = 4ab\{x^2 - (a-b)^2\}$;

therefore $\quad x^2 - (a-b)^2 = 0$ or $x^2 - (a-b)^2 = 4ab$;

therefore $\quad x^2 = (a-b)^2$ or $(a+b)^2$.

112. $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{\alpha^2+\beta^2}{\alpha\beta} = \dfrac{(\alpha+\beta)^2}{\alpha\beta} - 2$; now $\alpha+\beta = -\dfrac{b}{a}$, $\alpha\beta = \dfrac{c}{a}$; see Art. 335: thus $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{b^2}{ac} - 2$. And $\dfrac{\alpha}{\beta} \times \dfrac{\beta}{\alpha} = 1$. Hence the required equation is $x^2 - \left(\dfrac{b^2}{ac} - 2\right)x + 1 = 0$.

113. The first equation gives $x^2 + y^2 = \dfrac{5}{2}xy$; thus $x^2+y^2 = 20$; therefore $x^2 + y^2 - 2xy = 20 - 16$; extract the square root; &c.

114. $(x-2)(x-1) = (x-4)(x+3)$.

115. Let d be the common difference; then the terms are $18-4d$, $18-3d, \ldots\ 18+3d,\ 18+4d$: therefore the sum $= 9 \times 18 = 162$.

116. The common ratio

$$= \frac{1}{3+2\sqrt{2}} \div \frac{1}{1+\sqrt{2}} = \frac{1+\sqrt{2}}{3+2\sqrt{2}} = \frac{(1+\sqrt{2})(3-2\sqrt{2})}{(3+2\sqrt{2})(3-2\sqrt{2})} = \sqrt{2} - 1.$$

The sum to n terms

$$= \frac{1}{1+\sqrt{2}}\,\frac{(\sqrt{2}-1)^n - 1}{\sqrt{2}-1-1} = \frac{1}{1+\sqrt{2}} \cdot \frac{1-(\sqrt{2}-1)^n}{2-\sqrt{2}} = \frac{1-(\sqrt{2}-1)^n}{\sqrt{2}}.$$

117. First consider those cases in which the *last* sign is *positive*. In these cases *every* negative sign has a positive sign after it. Thus we may

consider that there are n inseparable pairs, each composed of a negative sign followed by a positive sign; and $p-n$ single positive signs besides. The number of these cases is $\frac{\lfloor p}{\lfloor n \lfloor p-n}$ by Art. 497. Next consider those cases in which the *last* sign is *negative*. In these cases every negative sign, *except the last*, has a positive sign after it. Thus we may consider that there are $n-1$ inseparable pairs, each composed of a negative sign followed by a positive sign; and $p-n+1$ single positive signs besides. The number of these cases is $\frac{\lfloor p}{\lfloor n-1 \lfloor p-n+1}$. Then the sum of the two numbers

$$=\frac{\lfloor p}{\lfloor n \lfloor p-n}\left\{1+\frac{n}{p-n+1}\right\}=\frac{(p+1)\lfloor p}{(p-n+1)\lfloor n \lfloor p-n}=\frac{\lfloor p+1}{\lfloor n \lfloor p-n+1}.$$

118. First suppose r not greater than m; then the coefficient required is that which arises from the expansion of $\frac{(n-m+1)\,x\,(1-x)}{(1-x)^2}$, that is from the expansion of $(n-m+1)\,x\,(1-x)^{-1}$: the coefficient is therefore $n-m+1$. Next suppose r between $m+1$ and $n+1$, both inclusive; then besides $n-m+1$ we have to find the coefficient which arises from the expansion of $-\frac{x^{m+1}}{(1-x)^2}$, that is from the expansion of $-x^{m+1}(1-x)^{-2}$; that is $-(r-m)$ by Art. 521: therefore the whole coefficient is $n-m+1-(r-m)$, that is $n-r+1$. Last suppose r greater than $n+1$; then besides the $n-r+1$ just obtained, we have to find the coefficient which arises from the expansion of $\frac{x^{n+2}}{(1-x)^2}$, that is from the expansion of $x^{n+2}(1-x)^{-2}$; this is $r-n-1$ by Art. 521: therefore the whole coefficient is $n-r+1+r-n-1$, that is 0.

119. $\sqrt{(n^2+1)}-n=\{\sqrt{(n^2+1)}-n\}\frac{\sqrt{(n^2+1)}+n}{\sqrt{(n^2+1)}+n}=\frac{1}{\sqrt{(n^2+1)}+n}$; and this is greater than $\frac{1}{2n+1}$, and therefore greater than $\frac{1}{2(n+1)}$. Hence the proposed series is greater than $\frac{1}{2}\left\{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\right\}$; and is therefore divergent by Art. 562.

120. $\text{Log}\,\frac{1}{(1\cdot05)^{13}}=-13\log 1\cdot05=-13\times\cdot0211893=-\cdot2754609$

$$=-1+\cdot7245391=\log\cdot5303214.$$

$$\text{Log}\,\frac{1}{(1\cdot05)^{20}}=-20\log 1\cdot05=-20\times\cdot0211893=-\cdot423786$$

$$=-1+\cdot576214=\log\cdot3768894.$$

Then $\frac{1}{\cdot05}\left\{\cdot5303214-\cdot3768894\right\}=20\times\cdot153432=3\cdot06864.$

121. Transpose and square; thus

$$4+5x-x^2-2(4+5x-x^2)^{\frac{1}{2}}(x^2+3x-4)^{\frac{1}{2}}+x^2+3x-4=8x;$$

therefore $$(4+5x-x^2)^{\frac{1}{2}}(x^2+3x-4)^{\frac{1}{2}}=0;$$

therefore either $4+5x-x^2=0$ or $x^2+3x-4=0$.

122. Let β denote one root and 2β the other; thus $a\beta^2+b\beta+c=0$, and $4a\beta^2+2b\beta+c=0$: therefore by subtraction $3a\beta^2+b\beta=0$; therefore $\beta=-\frac{b}{3a}$. Substitute this value for β; thus $\frac{b^2}{9a}-\frac{b^2}{3a}+c=0$; therefore $2b^2=9ac$.

123. $\frac{x+y}{12}=\frac{7}{x+y+5}$; therefore $(x+y)^2+5(x+y)=84$; hence we obtain $x+y=7$ or -12. Again $\frac{1}{x}+\frac{1}{y}=\frac{x+y}{12}$; therefore $xy=12$; &c.

124. Let x, y, z denote the three parts; then $\frac{xy}{4}=\frac{yz}{5}=\frac{zx}{6}$; and $x+y+z=111$. Thus $x=\frac{4z}{5}$, $y=\frac{5x}{6}=\frac{2z}{3}$. Substitute; $\frac{4z}{5}+\frac{2z}{3}+z=111$; &c.

125. See Art. 454. Both values of n must be positive integers. Thus the sum and the difference of the roots must be both even integers, or both odd integers. Therefore $\frac{b-2a}{b}$ must be a positive integer; and $\left(\frac{2a}{b}-1\right)^2+\frac{8s}{b}$ must be a positive integer, and a perfect square; moreover the two integers must both be even, or both be odd, and the square of the first integer greater than the second integer.

126. We have to find the sum of n terms of the series $\frac{1}{a}+\frac{1}{ar}+\frac{1}{ar^2}+\dots$;

the sum $=\frac{1}{a}\,\dfrac{\frac{1}{r^n}-1}{\frac{1}{r}-1}=\dfrac{1-r^n}{ar^{n-1}(1-r)}$.

127. We can give n things to the first person in $\dfrac{mn(mn-1)\dots(mn-n+1)}{\lfloor n}$ ways. Then we have $mn-n$ things left, and so we can give n things to the next person in $\dfrac{(mn-n)(mn-n-1)\dots(mn-2n+1)}{\lfloor n}$ ways. And so on. The product of all these numbers is the whole number of ways in which mn things can be distributed among m persons.

128. $$\frac{(1+x)^n}{(1-x)^3}=\{2-(1-x)\}^n(1-x)^{-3}$$

$$=(1-x)^{-3}\left\{2^n-n2^{n-1}(1-x)+\frac{n(n-1)}{2}2^{n-2}(1-x)^2-\dots\right\}$$

$$=2^n(1-x)^{-3}-n2^{n-1}(1-x)^{-2}+\frac{n(n-1)}{2}2^{n-2}(1-x)^{-1}+\dots$$

Expand each term, and pick out the coefficient of x^{n+r-1}; thus by Art. 521 we obtain $\frac{2^n(n+r)(n+r+1)}{2} - n2^{n-1}(n+r) + \frac{n(n-1)}{2}2^{n-2}$; it will be found that this reduces to $2^{n-3}\{(n+2r)(n+2r+2)+n\}$.

129. As in Art. 529 we have

$$q+2r+3s=4, \quad p+q+r+s=3.$$

p	q	r	s
1	1	0	1
1	0	2	0
0	2	1	0

The coefficient is

$$2\,\lfloor 3 + \frac{\lfloor 3\,(-3)^2}{\lfloor 2} + \frac{\lfloor 3\,.\,2^2(-3)}{\lfloor 2},$$

that is $12+27-36$, that is 3.

130. The n^{th} term of the series is $\frac{n^3}{\lfloor n}$. Now it is easily seen that $n^3 = n(n-1)(n-2)+3n(n-1)+n$; and therefore this term may be written thus: $\frac{1}{\lfloor n-3} + \frac{3}{\lfloor n-2} + \frac{1}{\lfloor n-1}$. If we apply this transformation to every term for which n is not less than 3 we obtain

$$1+\frac{2^3}{2}+1+3+1+5\left\{\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\dots\dots\right\},$$

that is $$5\left\{1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\dots\dots\right\}, \text{ that is } 5e.$$

131. The equation may be written thus:

$$\sqrt{(x-3)(x-5)}+\sqrt{(x-3)(x+5)}=\sqrt{(x-3)(4x-6)}.$$

Therefore either $x-3=0$, or $\sqrt{(x-5)}+\sqrt{(x+5)}=\sqrt{(4x-6)}$. Square the last; $2x+2\sqrt{(x^2-25)}=4x-6$; therefore $x-3=\sqrt{(x^2-25)}$. Square, &c.

132. Let α and β denote the roots; then $\alpha+\beta=\frac{b}{a}$: if α and β have the same signs this sign is that of $\frac{b}{a}$, and if α and β have different signs the sign of the numerically greater of the two is the sign of $\frac{b}{a}$. Similarly $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{b}{c}$: and so the sign of $\frac{b}{c}$ is the same as that of the numerically greater of the two $\frac{1}{\alpha}$ and $\frac{1}{\beta}$, that is the same as that of the numerically less of the two α and β.

133. Put $x-a=t$, $y-b=u$, $z-c=v$; substitute for x, y, and z; thus we get $t+u+v=0$, $\frac{t}{a}+\frac{u}{b}+\frac{v}{c}=0$, $t^2+u^2+v^2+2ta+2ub+2vc=0$. From the first

and second equations, by Art. 385, we have $\dfrac{t}{\frac{1}{c}-\frac{1}{b}}=\dfrac{u}{\frac{1}{a}-\frac{1}{c}}=\dfrac{v}{\frac{1}{b}-\frac{1}{a}}=k$ say; substitute in the last equation; thus

$$k^2\left\{\frac{(b-c)^2}{b^2c^2}+\frac{(c-a)^2}{c^2a^2}+\frac{(a-b)^2}{a^2b^2}\right\}+2k\left\{\frac{a(b-c)}{bc}+\frac{b(c-a)}{ca}+\frac{c(a-b)}{ab}\right\}=0,$$

therefore $k=0$, or $k=-2abc\dfrac{a^2(b-c)+b^2(c-a)+c^2(a-b)}{a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2}$.

134. Suppose that A takes x half-crowns, y shillings, and z sixpences; and that B takes t half-crowns, u shillings, and v sixpences; then

$$5x+2y+z=5t+2u+v;\ \frac{5(x+t)}{15}=\frac{2(y+u)}{4}=\frac{z+v}{1}=k \text{ say};$$

$$x+y+z=60,\quad t+u+v=60,\quad x=t+2.$$

Thus $x+t=3k$, $y+u=2k$, $z+v=k$; therefore $x+y+z+t+u+v=6k$; or $120=6k$, so that $k=20$. Hence $x+t=60$, and $x-t=2$; therefore $x=31$, $t=29$; &c.

135. We have $\dfrac{1}{q}=a+(p-1)b$, $\dfrac{1}{p}=a+(q-1)b$. Hence, by subtraction, $\dfrac{1}{q}-\dfrac{1}{p}=(p-q)b$; therefore $b=\dfrac{1}{pq}$. Substitute the value of b; thus $\dfrac{1}{q}=a+\dfrac{p-1}{pq}$; therefore $a=\dfrac{1}{pq}$. Thus the sum of n terms $=\dfrac{n}{2}(n+1)\dfrac{1}{pq}$; and this $=\dfrac{pq+1}{2}$ if $n=pq$.

136. We have $b=\dfrac{a+c}{2}$, $\beta=\dfrac{2\alpha\gamma}{\alpha+\gamma}$, $\dfrac{\alpha}{\gamma}+\dfrac{\gamma}{\alpha}=\dfrac{a}{c}+\dfrac{c}{a}$.

Hence $(b\beta)^2=\dfrac{(a+c)^2\alpha^2\gamma^2}{(\alpha+\gamma)^2}$. But $\dfrac{\alpha^2+\gamma^2}{\alpha\gamma}=\dfrac{a^2+c^2}{ac}$; therefore $\dfrac{(\alpha+\gamma)^2}{\alpha\gamma}=\dfrac{(a+c)^2}{ac}$; therefore $\dfrac{(a+c)^2}{(\alpha+\gamma)^2}=\dfrac{ac}{\alpha\gamma}$. Thus $(b\beta)^2=ac\,\alpha\gamma$.

137. Take any 2 of the 3 consonants; put either at the beginning and the other at the end of the word; this gives 6 cases. The remaining 6 letters can undergo $\lfloor 6$ permutations. Therefore the required number is $6\lfloor 6$.

138. We have $(1+x)^{2n}=a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots,$

$$\left(1-\frac{1}{x}\right)^{2n}=a_0-\frac{a_1}{x}+\frac{a_2}{x^2}-\frac{a_3}{x^3}+\ldots\ldots;$$

thus $a_0^2-a_1^2+a_2^2-a_3^2+\ldots$ is the term which does not contain x in the product of $(1+x)^{2n}$ and $\left(1-\dfrac{1}{x}\right)^{2n}$.

Now $$(1+x)^{2n}\times\left(1-\frac{1}{x}\right)^{2n}=\frac{(x+1)^{2n}(x-1)^{2n}}{x^{2n}}=\frac{(x^2-1)^{2n}}{x^{2n}}.$$

The term which does not contain x is the coefficient of x^{2n} in the expansion of $(x^2-1)^{2n}$; and this is $\dfrac{(-1)^n\lfloor 2n}{\lfloor n\,\lfloor n}$.

139. The n^{th} term is $\frac{1}{\sqrt[n]{n}}$. Now $\sqrt[n]{n}$ is less than 2; for $(1+1)^n=1+n+\ldots$, so that 2^n is greater than n, and therefore 2 greater than $\sqrt[n]{n}$. Thus every term of the series is greater than $\frac{1}{2}$; and the series is divergent by Art. 557.

140. We have from the given equation $y=\log(1+x)$; therefore $e^y=1+x$; therefore $x=e^y-1=y+\frac{y^2}{\underline{|2}}+\frac{y^3}{\underline{|3}}+\ldots\ldots$

141. $x^2-3x+7-\sqrt{(x^2-3x+7)}+\frac{1}{4}=20\frac{1}{4}$; therefore $\sqrt{(x^2-3x+7)}-\frac{1}{2}=\pm\frac{9}{2}$. The root $\sqrt{(x^2-3x+7)}=-4$ does not strictly apply: take $\sqrt{(x^2-3x+7)}=5$; &c.

142. Suppose that x men stand in the front rank of the solid square; then x^2 is the whole number of men. In a hollow square $24+x$ men stand in the front rank; and as a hollow square is four deep the number of men in it is $(24+x)^2-(24+x-8)^2$. Thus $x^2=4\{(24+x)^2-(16+x)^2\}=4(40+2x)\,8$. Therefore $x^2-64x=1280$. The only admissible root of this quadratic is 80.

143. $6x^2-xy-12y^2=0$. Put $y=vx$; thus $(6-v-12v^2)x^2=0$. The solution $x=0$ will not apply; take then $6-v-12v^2=0$: this gives $v=\frac{2}{3}$ or $-\frac{3}{4}$. Then substitute in the second equation successively $y=\frac{2x}{3}$ and $y=-\frac{3x}{4}$.

144. Let v denote any number greater than 20. Then when the train moves at the rate of v miles per hour instead of 20 miles per hour, the increase of the receipts is $p(v-20)$, and the increase of the expenses is $q(v-20)^2$, where p and q are certain constant quantities: thus the profit is $p(v-20)-q(v-20)^2$. This profit also vanishes when $v=40$: thus $20p-400q=0$; therefore $p=20q$. Hence the profit is $q\{20(v-20)-(v-20)^2\}$, that is $q\{60v-v^2-800\}$, that is $q\{100-(v-30)^2\}$. Hence the profit is greatest when $v=30$.

145. Let $M(13)$ stand for a multiple of 13. Then

$$10=13-3,\quad 100=M(13)-4,\quad 1000=M(13)-1.$$

Hence

$$\begin{aligned}10^4&=10\times10^3=M(13)+3,\\ 10^5&=10^2\times10^3=M(13)+4,\\ 10^6&=10^3\times10^3=M(13)+1,\\ 10^7&=10\times10^6=M(13)-3,\end{aligned}$$

and so on.

Thus

$$\begin{aligned}p_0+10^3p_3+10^6p_6+\ldots&=M(13)+p_0-p_3+p_6-\ldots,\\ 10p_1+10^4p_4+10^7p_7+\ldots&=M(13)-3(p_1-p_4+p_7-\ldots),\\ 10^2p_2+10^5p_5+10^8p_8+\ldots&=M(13)-4(p_2-p_5+p_8-\ldots).\end{aligned}$$

Hence finally the proposed number

$$=M(13)+p_0-p_3+p_6-\ldots-3(p_1-p_4+p_7-\ldots)-4(p_2-p_5+p_8-\ldots)$$

146. Let
$$s = a + ar + ar^2 + \dots + ar^{n-1},$$
$$s' = a - ar + ar^2 - \dots + a(-r)^{n-1};$$
then
$$s = a\frac{r^n - 1}{r-1}, \quad s' = a\frac{(-r)^n - 1}{-r-1} = a\frac{1+r^n}{1+r}, \text{ since } n \text{ is odd;}$$
therefore $ss' = \dfrac{a^2(r^{2n}-1)}{r^2-1}$, and this is equal to the sum of n terms of the series $a^2 + a^2r^2 + a^2r^4 + \dots$

147. Denote the straight lines by the numbers 1, 2, 3,...n. Make a polygon of n sides in the following way: the intersection of 1 and 2, of 2 and 3,... of $n-1$ and n, of n and 1. Thus there is one and only one segment on each straight line. And we see that we must take one segment on each straight line, for otherwise there would be more than one segment on some straight line or straight lines. In this way we see that the Example is equivalent to 12 of Chapter XXXIV.

148. $4^{\frac{1}{8}} = 2^{\frac{2}{8}}$, $8^{\frac{1}{16}} = 2^{\frac{3}{16}}$, $16^{\frac{1}{32}} = 2^{\frac{4}{32}}$, ... thus the expression is equal to 2 raised to the power denoted by $\frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \dots$, that is by $\frac{1}{4}\left(1 - \frac{1}{2}\right)^{-2}$, that is by 1.

149. As in Art. 529 we have
$$q + 2r + 3s + 4t = 4, \quad p + q + r + s + t = -3.$$

The coefficient is

$$(-3)(-1) + (-3)(-4)(-3) + \frac{(-3)(-4)}{\underline{|2}}(-1)^3$$
$$+ \frac{(-3)(-4)(-5)}{\underline{|2}}(-1) + \frac{(-3)(-4)(-5)(-6)}{\underline{|4}},$$

that is $3 - 36 + 6 + 30 + 15$, that is 18.

p	q	r	s	t
-4	0	0	0	1
-5	1	0	1	0
-5	0	2	0	0
-6	2	1	0	0
-7	4	0	0	0

150. Let the bases be a, ar, ar^2,...ar^{n-1}; let the quantities be A_0, A_1, A_2,...A_{n-1}; and let x be the common logarithm. Then
$$A_0 = a^x, \quad A_1 = (ar)^x, \quad A_2 = (ar^2)^x, \quad A_3 = (ar^3)^x, \dots$$
Thus the ratio of A_1 to A_0, of A_2 to A_1,... is equal to r^x; so that x is the logarithm of this ratio to the base r.

151.
$$3 - \frac{11}{2x+1} + 3 - \frac{12}{3x+1} = 3 - \frac{13}{2x+3} + 3 - \frac{3}{x-1};$$
therefore
$$\frac{13}{2x+3} - \frac{11}{2x+1} = \frac{12}{3x+1} - \frac{3}{x-1};$$
therefore
$$\frac{4x-20}{(2x+3)(2x+1)} = \frac{3x-15}{(3x+1)(x-1)};$$
thus either $x = 5$ or $4(3x+1)(x-1) = 3(2x+3)(2x+1)$; the latter reduces to $32x + 13 = 0$.

152. The former part of the Example is contained in Art. 334. Now it is obvious that the equation is satisfied when $x=a$, and when $x=b$, and when $x=c$; hence the equation must be an *identity*. This of course can be verified by simplifying the left-hand member.

153. By division $\frac{(x^2+y^2)x^2}{(x^2-y^2)y^2}=6$; put $y=vx$; thus $1+v^2=6(1-v^2)v^2$: hence $v^2=\frac{1}{2}$ or $\frac{1}{3}$. Then from the first given equation $x^2(1+v^2)=6v$; &c.

154. Suppose that bell metal contains x per cent. of copper, and therefore $100-x$ per cent. of tin. Suppose that the mass which is fused together contains y per cent. of bronze, and therefore $100-y$ per cent. of bell metal.

Then from considering the quantity of copper, zinc, and tin, respectively in the mass, we have

$$\frac{y}{100}\cdot\frac{91}{100}+\frac{100-y}{100}\cdot\frac{x}{100}=\frac{88}{100},$$

$$\frac{y}{100}\cdot\frac{6}{100}=\frac{4\frac{7}{8}}{100},$$

$$\frac{y}{100}\cdot\frac{3}{100}+\frac{100-y}{100}\cdot\frac{100-x}{100}=\frac{7\frac{1}{8}}{100}.$$

From two of these equations the third will follow; for by adding the three we obtain an identity. The second equation gives $y=81\frac{1}{4}$; and substituting in the first equation we get $x=75$.

155. Let P denote the sum of the products of the first n natural numbers taken two and two together. By Art. 225 we have

$$\{1+2+3+\ldots+n\}^2=1^2+2^2+3^2+\ldots+n^2+2P,$$

that is
$$\left\{\frac{n(n+1)}{2}\right\}^2=\frac{n(n+1)(2n+1)}{6}+2P.$$

Hence
$$P=\frac{n^2(n+1)^2}{8}-\frac{n(n+1)(2n+1)}{12}=\frac{n(n+1)}{24}\left\{3n(n+1)-2(2n+1)\right\}$$
$$=\frac{(n-1)n(n+1)(3n+2)}{24}.$$

156. Let the numbers of the first set be denoted by a, ar, ar^2, x; then since the last three are in H.P. we have $\frac{1}{ar}-\frac{1}{ar^2}=\frac{1}{ar^2}-\frac{1}{x}$; thus $x=\frac{ar^2}{2-r}$. Again, let the numbers of the second set be denoted by a, ar, y, z; then, as before, $y=\frac{ar}{2-r}$; and $arz=y^2$; thus $z=\frac{ar}{(2-r)^2}$. We have then to shew that $\frac{ar^2}{2-r}$ is less than $\frac{ar}{(2-r)^2}$, that is, that $r(2-r)$ is less than 1: this is evident for $1-r(2-r)=(1-r)^2$.

157. If there were no restriction the number of different arrangements would be $\underline{|7}$. But we must exclude all the cases in which the blue and the green come together. There are $\underline{|6}$ such cases in which the blue is before the green, and $\underline{|6}$ in which the green is before the blue. Therefore the required number is $\underline{|7}-2\underline{|6}$.

158. Let $(5\sqrt{2}+7)^m=n+a$. Now $(5\sqrt{2}+7)(5\sqrt{2}-7)=50-49=1$: hence $5\sqrt{2}-7$ is a positive proper fraction. Therefore $(5\sqrt{2}-7)^m$ is a positive fraction; and it must be equal to a because $(5\sqrt{2}+7)^m-(5\sqrt{2}-7)^m$ is obviously an integer. Hence $a(n+a)=(5\sqrt{2}-7)^m(5\sqrt{2}+7)^m=(50-49)^m=1$.

159. As in Art. 529 we have

$$q+2r+3s+4t=4, \quad p+q+r+s+t=-\frac{3}{2}.$$

The coefficient is

$$-\frac{3}{2}.5+\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-2)(-4)+\frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{\underline{|2}}3^2$$

$$+\frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{\underline{|2}}(-2)^2.3$$

$$+\frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\left(-\frac{9}{2}\right)}{\underline{|4}}(-2)^4,$$

that is $-\frac{15}{2}+30+\frac{135}{8}-\frac{315}{4}+\frac{315}{8}$, that is 0.

p	q	r	s	t
$-\frac{5}{2}$	0	0	0	1
$-\frac{7}{2}$	1	0	1	0
$-\frac{7}{2}$	0	2	0	0
$-\frac{9}{2}$	2	1	0	0
$-\frac{11}{2}$	4	0	0	0

We may easily verify this result. $1-2x+3x^2-4x^3+\ldots=(1+x)^{-2}$; thus we require the coefficient of x^4 in $\{(1+x)^{-2}\}^{-\frac{3}{2}}$, that is in $(1+x)^3$: it is obvious that this coefficient is 0.

160. Let p denote the number of the population at the beginning of a month; then at the end of that month the number is $p+\frac{p}{480}-\frac{p}{600}$, that is $p+\frac{5p}{2400}-\frac{4p}{2400}$, that is $\frac{2401}{2400}p$. At the end of the second month the number of the population in like manner is $\frac{2401}{2400}\times\frac{2401}{2400}p$, that is $\left(\frac{2401}{2400}\right)^2p$. Similarly at the end of the n^{th} month the number of the population is $\left(\frac{2401}{2400}\right)^np$. Suppose that the population is doubled in n months; then $\left(\frac{2401}{2400}\right)^np=2p$; therefore $\left(\frac{2401}{2400}\right)^n=2$. Take logarithms; thus

$$n\log\frac{2401}{2400}=\log 2, \text{ that is } n(\log 2401-\log 2400)=\log 2,$$

that is $$n\{\log 7^4-\log(3\times 2^3\times 100)\}=\log 2;$$

therefore $$n=\frac{\log 2}{4\log 7-\log 3-3\log 2-2}=\frac{\cdot 3010300}{\cdot 0001807}=1666 \text{ nearly}.$$

161. Here $x^4+1=2(x^4+4x^3+6x^2+4x+1)$; therefore by transposition $x^4+8x^3+12x^2+8x+1=0$; divide by x^2; thus $x^2+\frac{1}{x^2}+8\left(x+\frac{1}{x}\right)+12=0$; that is $\left(x+\frac{1}{x}\right)^2+8\left(x+\frac{1}{x}\right)+10=0$. By solving this quadratic we obtain $x+\frac{1}{x}=-4\pm\sqrt{6}$; hence x can be found.

162. Suppose that in the first race A ran at the rate of x miles an hour, and B at the rate of y miles an hour. Then $\frac{2}{x}-\frac{2}{y}=\frac{2}{60}$, and $\frac{2}{y-2}-\frac{2}{x+2}=\frac{2}{60}$. Therefore $60(y-x)=xy$, and $60(x-y+4)=(y-2)(x+2)$; subtract; thus $120(x-y+2)=-2(x-y+2)$; therefore $x-y+2=0$; &c.

163. Simplify the second equation; thus $2xy+4y=0$; therefore either $y=0$ or $x=-2$. Substitute $y=0$ in the first equation and we obtain $x=-\frac{3}{11}$: substitute $x=-2$ in the first equation, and we obtain $y=\frac{19}{7}$.

164. By Art. 384 we obtain $\frac{x+y+z}{y+z+1+z+x+x+y-1}=x+y+z$; that is $\frac{x+y+z}{2(x+y+z)}=x+y+z$. Thus either $x+y+z=0$ or $x+y+z=\frac{1}{2}$. Taking the former we get $x=0$, $y=0$, $z=0$. Taking the latter we get $x=\frac{1}{2}(y+z+1)$, $y=\frac{1}{2}(z+x)$, $z=\frac{1}{2}(x+y-1)$: these lead to $x=\frac{1}{2}$, $y=\frac{1}{6}$, $z=-\frac{1}{6}$.

165. Let $M(101)$ stand for a multiple of 101. Then $10^2=101-1$, $10^3=M(101)-10$, $10^4=M(101)+1$, $10^5=M(101)+10$; and so on. Thus $p_0+10p_1+10^2p_2+10^3p_3+\ldots=M(101)+p_0+10p_1-p_2-10p_3+p_4+10p_5-\ldots$

166. We have $a=\frac{b+c}{2}$, $c=\frac{2ab}{a+b}$; therefore $c(a+b)=(b+c)b$; therefore $ac=b^2$. And $2a=b+c=b+\frac{2ab}{a+b}$; therefore $(2a-b)(a+b)=2ab$; therefore $b^2+ba-2a^2=0$; therefore $b=a$ or $-2a$. If $b=a$ then also $c=a$; if $b=-2a$ then $c=4a$.

167. Take any two straight lines of the first set, and any straight line out of the other two sets; thus we get $\frac{m(m-1)}{2}(n+p)$ triangles. In like manner take any two straight lines of the second set, and any straight line out of the other two sets; and also take any two straight lines of the third set, and any straight line out of the other two sets. And finally take one straight line from each set; this gives mnp triangles.

168. $(1-x)^{-2}(1-x)^n=(1-x)^{n-2}$. Expand each side and pick out the coefficient of x^{r-1}. On the left-hand side we shall have to take the coefficient from the product

$$\left\{1+2x+3x^2+\ldots+(r+1)x+\ldots\right\}\left\{1-nx+\frac{n(n-1)}{2}x^2-\ldots\right\};$$

hence the coefficient is easily seen to be the quantity denoted by a_r. Thus a_r must be equal to the coefficient of x^{r-1} in the expansion of $(1-x)^{n-2}$. Hence $a_r=0$ if $r-1$ is greater than $n-2$, and $a_r=(-1)^{n-2}=(-1)^n$ if $r-1=n-2$. If $r-1$ is less than $n-2$ we have $a_r=(-1)^{n-2}a_{n-r}$ by Art. 508.

169. As in Art. 529 we have

$$q+2r+4t=6,\quad p+q+r+t=\frac{7}{3}.$$

The coefficient is

$$\frac{7}{3}\cdot\frac{4}{3}(-1)(-3)+\frac{\frac{7}{3}\cdot\frac{4}{3}\cdot\frac{1}{3}}{\underline{|2}}(-1)\,2^2+\frac{\frac{7}{3}\cdot\frac{4}{3}\cdot\frac{1}{3}}{\underline{|3}}(-3)^3$$

$$+\frac{\frac{7}{3}\cdot\frac{4}{3}\cdot\frac{1}{3}\left(-\frac{2}{3}\right)}{\underline{|2}\,\underline{|2}}2^2(-3)^2$$

$$+\frac{\frac{7}{3}\cdot\frac{4}{3}\cdot\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{\underline{|4}}2^4(-3)$$

$$+\frac{\frac{7}{3}\cdot\frac{4}{3}\cdot\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{\underline{|6}}2^6$$

p	q	r	t
$\frac{1}{3}$	0	1	1
$-\frac{2}{3}$	2	0	1
$-\frac{2}{3}$	0	3	0
$-\frac{5}{3}$	2	2	0
$-\frac{8}{3}$	4	1	0
$-\frac{11}{3}$	6	0	0

that is $\frac{28}{3}-\frac{56}{27}-\frac{14}{3}-\frac{56}{9}-\frac{7.5.2^4}{3^5}-\frac{7.2^8}{3^8}$, that is $-\frac{40726}{3^8}$.

170. Let x denote the required logarithm; then $(25)^x=50$, that is $5^{2x}=50$; that is $\left(\frac{10}{2}\right)^{2x}=\frac{100}{2}$. Take the logarithms to the base 10; therefore $2x(\log 10-\log 2)=\log 100-\log 2$; thus $x=\frac{2-\log 2}{2(1-\log 2)}=\frac{1\cdot69897}{2\times\cdot69897}=1\cdot21534$.

171. $x+x^{-1}=(x^{\frac{2}{3}}-1+x^{-\frac{2}{3}})(x^{\frac{1}{3}}+x^{-\frac{1}{3}})$. Thus we have either $x^{\frac{1}{3}}+x^{-\frac{1}{3}}=0$, or $1=\frac{4}{13}(x^{\frac{2}{3}}-1+x^{-\frac{2}{3}})$. The former gives $x^{\frac{2}{3}}+1=0$, so that $x=(-1)^{\frac{3}{2}}=(-1)^{\frac{1}{2}}$. The latter gives $x^{\frac{2}{3}}-1+x^{-\frac{2}{3}}=\frac{13}{4}$, so that $x^{\frac{4}{3}}-\frac{17}{4}x^{\frac{2}{3}}+1=0$, whence we get $x^{\frac{2}{3}}=4$ or $\frac{1}{4}$; therefore $x=4^{\frac{3}{2}}$ or $\left(\frac{1}{4}\right)^{\frac{3}{2}}=\pm8$ or $\pm\frac{1}{8}$.

172. $(\alpha+\beta)^2=\frac{b^2}{a^2}$, $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=\frac{b^2}{a^2}-\frac{4c}{a}$. Hence the required equation is $x^2-\left(\frac{2b^2}{a^2}-\frac{4c}{a}\right)x+\frac{b^2}{a^2}\left(\frac{b^2}{a^2}-\frac{4c}{a}\right)=0$.

173. $64(x^2-y^2)=(x+9y)^2$; therefore $145y^2+18xy-63x^2=0$. Put $y=vx$; then $145v^2+18v-63=0$; this gives $v=\frac{8}{5}$ or $-\frac{21}{29}$. The second given equation may be written thus: $(x^2-x+y)^2-(x^2-x+y)=506$; from this quadratic we obtain $x^2-x+y=23$ or -22. Substitute successively in these equations $y=\frac{8x}{5}$ and $y=-\frac{21}{29}x$.

174. Suppose that A could reap the field alone in x days, B in y days, and C in z days. Then since the whole field is reaped by A working for 12 days, B for 12 days, and C for 6 days $\frac{12}{x}+\frac{12}{y}+\frac{6}{z}=1$. Also $\frac{x}{y}=\frac{2}{3}$. Since A and C together reap $\frac{1}{x}+\frac{1}{z}$ of the field in one day, they could reap the whole field in $\frac{xz}{x+z}$ days; similarly B and C together could reap the whole field in $\frac{yz}{y+z}$ days: therefore $\frac{xz}{x+z}:\frac{yz}{y+z}::7:8$; thus $\frac{8xz}{x+z}=\frac{7yz}{y+z}$. Substitute $\frac{3x}{2}$ for y in the last equation, and simplify; thus we get $3x=5z$. Then substitute for y and z in the first equation. Hence we obtain $x=30$, $y=45$, $z=18$; therefore $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{9}$, so that when they all work together they reap $\frac{1}{9}$ of the field in one day, and therefore reap the whole field in 9 days.

175. By the method of Art. 443 we can shew that *any number* of ounces may be weighed if we have two of each of the following weights: 1 oz., 5 oz., 5^2 oz., 5^3 oz., 5^4 oz.,... If we do not require to go beyond 312 oz. we shall not require any weight beyond the two of 5^3 oz.; for the least weight which would require us to use a weight of 5^4 oz. is $5^4-2.5^3-2.5^2-2.5-2$ ounces, that is 313 ounces.

176. Let the numbers be a, ar, ar^2, ar^3; then $a(1+r+r^2+r^3)=15$, $a^2(1+r^2+r^4+r^6)=85$; that is $a(1+r)(1+r^2)=15$, $a^2(1+r^2)(1+r^4)=85$. Square the first equation and divide it by the second; thus $\frac{(1+r)^2(1+r^2)}{1+r^4}=\frac{45}{17}$; therefore $17(1+2r+2r^2+2r^3+r^4)=45(1+r^4)$; therefore

$$28(1+r^4)-34(r+r^3)-34r^2=0;$$

divide by r^2, therefore
$$28\left(r^2+\frac{1}{r^2}\right)-34\left(r+\frac{1}{r}\right)-34=0;$$

therefore
$$28\left(r+\frac{1}{r}\right)^2-34\left(r+\frac{1}{r}\right)-90=0;$$

therefore $r+\frac{1}{r}=\frac{5}{2}$ or $-\frac{9}{7}$. The latter gives impossible values for r; the former gives $r=2$ or $\frac{1}{2}$, and taking either of these we find that the numbers are 1, 2, 4, 8.

177. Suppose that the p men go to their side of the table; on this side there are $n-p$ places left; these can be filled up by any of the $2n-p-q$ men who will sit on either side, and thus $n-p$ men can be taken in $\dfrac{\lfloor 2n-p-q}{\lfloor n-p \, \lfloor n-q}$ ways. The remaining men will go to the other side. And $\lfloor n$ permutations can be made of the men on each side. Thus the whole number of ways is

$$\frac{\lfloor n \, \lfloor n \, \lfloor 2n-p-q}{\lfloor n-p \, \lfloor n-q}.$$

178. $\dfrac{1}{9a^2+6ax+4x^2}=\dfrac{3a-2x}{27a^3-8x^3}$. Hence we require the coefficient of x^{3r} in the expansion of $\dfrac{3a-2x}{27a^3}\left(1-\dfrac{8x^3}{27a^3}\right)^{-1}$: this coefficient is $\dfrac{3a}{27a^3}\left(\dfrac{8}{27a^3}\right)^r$.

179. Suppose that $(u_n)^{\frac{1}{n}}$ is less than k; then u_n is less than k^n. Hence from and after a certain term the series is less than a Geometrical Progression beginning with that term and having the common ratio k. Therefore if k be less than unity the series is convergent.

180. $$\left(1+\frac{1}{n}\right)^n=\left(\frac{n}{n+1}\right)^{-n}=\left(1-\frac{1}{n+1}\right)^{-n};$$

$$\log\left(1-\frac{1}{n+1}\right)^{-n}=n\left\{\frac{1}{n+1}+\frac{1}{2(n+1)^2}+\frac{1}{3(n+1)^3}+\dots\right\}$$

$$=(n+1)\left\{\frac{1}{n+1}+\frac{1}{2(n+1)^2}+\frac{1}{3(n+1)^3}+\dots\right\}$$

$$-\left\{\frac{1}{n+1}+\frac{1}{2(n+1)^2}+\frac{1}{3(n+1)^3}+\dots\right\}$$

$$=1+\frac{1}{2(n+1)}+\frac{1}{3(n+1)^2}+\frac{1}{4(n+1)^3}+\dots$$

$$-\left\{\frac{1}{n+1}+\frac{1}{2(n+1)^2}+\frac{1}{3(n+1)^3}+\dots\right\}$$

$$=1-\frac{1}{2(n+1)}-\frac{1}{2.3(n+1)^2}-\frac{1}{3.4(n+1)^3}-\dots\dots$$

Hence we see that as n increases this logarithm continually increases; and therefore $\left(1+\dfrac{1}{n}\right)^n$ continually increases with n.

181. It is obvious that $x=1$ is a root of this equation. And we may write the equation thus; $9x^2-1=4x^3(3x-1)$, that is $(3x-1)(3x+1)=4x^3(3x-1)$, so that another root is given by $3x-1=0$. Hence if we bring all the terms of the equation to one side we are sure that $(x-1)(3x-1)$ will divide the terms: see Art. 332. On trial we find that the equation may be written thus: $(x-1)(3x-1)(2x+1)^2=0$, so that the roots are 1, $\dfrac{1}{3}$, and $-\dfrac{1}{2}$.

182. Suppose that on the first occasion the ages of A, B, C are a, ar, ar^2, respectively; and let x denote the number of pounds divided on each occasion. On the first occasion A gets $\frac{ax}{a+ar+ar^2}$ pounds; and on the second occasion he gets $\frac{(a+5)x}{a+5+ar+5+ar^2+5}$ pounds; thus

$$\left\{\frac{a+5}{a(1+r+r^2)+15}-\frac{a}{a(1+r+r^2)}\right\}x=17\tfrac{1}{2}.$$

Similarly $$\left\{\frac{ar+5}{a(1+r+r^2)+15}-\frac{ar}{a(1+r+r^2)}\right\}x=2\tfrac{1}{2}.$$

And $ar^2+5=2(a+5)$; therefore $a(r^2-2)=5$. Divide the first equation by the second; thus $\frac{5(1+r+r^2)-15}{5(1+r+r^2)-15r}=7$, that is $\frac{r^2+r-2}{r^2-2r+1}=7$, that is $\frac{(r+2)(r-1)}{(r-1)^2}=7$; hence $\frac{r+2}{r-1}=7$, and therefore $r=\frac{3}{2}$. Therefore $a=20$. And then we find $x=1045$.

183. $x+y-(xy)^{\frac{1}{2}}=61$, $x^{\frac{1}{4}}y^{\frac{1}{4}}(x^{\frac{1}{2}}+y^{\frac{1}{2}})=78$; therefore $(x^{\frac{1}{2}}+y^{\frac{1}{2}})^2=\frac{(78)^2}{x^{\frac{1}{2}}y^{\frac{1}{2}}}$; subtract the first equation from this and we get $3(xy)^{\frac{1}{2}}=\frac{(78)^2}{(xy)^{\frac{1}{2}}}-61$. From this quadratic we obtain $(xy)^{\frac{1}{2}}=36$ or $-\frac{169}{3}$. The latter gives impossible values: take the former and combine it with the first given equation; hence we get $x=81, 16$; $y=16, 81$.

184. $x=cy+bz=cy+b(bx+ay)$; therefore $x(1-b^2)=y(c+ab)$. Again $y=az+cx=a(bx+ay)+cx$; therefore $y(1-a^2)=x(c+ab)$. Multiply crosswise; thus $x^2(1-b^2)=y^2(1-a^2)$; therefore $\frac{x^2}{1-a^2}=\frac{y^2}{1-b^2}$. Similarly we can shew that each of these fractions also $=\frac{z^2}{1-c^2}$. Also by multiplying two former results we get $xy(1-a^2)(1-b^2)=xy(c+ab)^2$; therefore $(1-a^2)(1-b^2)=(c+ab)^2$; therefore $1=a^2+b^2+c^2+2abc$.

185. Let a denote the last digit of any number; then the cube of the number is equal to a^3 together with some multiple of the radix; and thus the cube ends with the same digit as a^3 ends with. Now suppose the radix to be nine; then 0^3, 3^3 and 6^3 end with the digit 0; 1^3, 4^3 and 7^3 end with the digit 1; and 2^3, 5^3 and 8^3 end with the digit 8. Thus the last digit must be 0 or 1 or 8.

186. By Art. 227, we see that

$$(b+c+d+e)^3=3(b+c+d+e)(b^2+c^2+d^2+e^2)-2(b^3+c^3+d^3+e^3)$$
$$+6(cde+bde+bce+bcd);$$

and a similar result holds for the cube of any multinomial. Hence we have

$$a^3(1+r+r^2+r^3+\dots)^3 = 3a(1+r+r^2+r^3+\dots)\,a^2(1+r^2+r^4+r^6+\dots)$$
$$-2a^3(1+r^3+r^6+r^9+\dots)+6P,$$

where P stands for the required product. Thus

$$a^3\left(\frac{1}{1-r}\right)^3 = 3a^3\frac{1}{1-r}\,\frac{1}{1-r^2} - \frac{2a^3}{1-r^3}+6P;$$

therefore $$P=\frac{a^3}{6}\left\{\frac{1}{(1-r)^3}+\frac{2}{1-r^3}-\frac{3}{(1-r)(1-r^2)}\right\}.$$

If $3P=\dfrac{a^3}{1-r^3}$ we must have $\dfrac{1}{(1-r)^3}=\dfrac{3}{(1-r)(1-r^2)}$; therefore $1+r=3(1-r)$; whence $r=\dfrac{1}{2}$.

187. Suppose that u_n denotes the number of gallons of wine in the second vessel at the end of n operations; then $a-u_n$ is the number of gallons of wine in the first vessel. Take c gallons from each; the c gallons taken from the first contain $\dfrac{c}{a}(a-u_n)$ gallons of wine, and the c gallons taken from the second contain $\dfrac{c}{b}u_n$ gallons of wine. Thus the number of gallons of wine in the second vessel at the end of this operation will be $u_n+\dfrac{c}{a}(a-u_n)-\dfrac{c}{b}u_n$; this we denote by u_{n+1}, so that $u_{n+1}=c+\left(1-\dfrac{c}{a}-\dfrac{c}{b}\right)u_n$. We may write this result thus: $u_{n+1}-\dfrac{ab}{a+b}=\left(1-\dfrac{c}{a}-\dfrac{c}{b}\right)\left(u_n-\dfrac{ab}{a+b}\right)$, or $u_{n+1}-q=p(u_n-q)$, where p stands for $1-\dfrac{c}{a}-\dfrac{c}{b}$ and q for $\dfrac{ab}{a+b}$. Thus we see that u_1-q, u_2-q, u_3-q,... form a Geometrical Progression of which the common ratio is p. Therefore $u_r-q=(u_1-q)p^{r-1}=(c-q)p^{r-1}$, for $u_1=c$. Thus $u_r=\dfrac{ab}{a+b}+\left(c-\dfrac{ab}{a+b}\right)p^{r-1}$; and $c-\dfrac{ab}{a+b}=(1-p)\dfrac{ab}{a+b}-\dfrac{ab}{a+b}=\dfrac{-pab}{a+b}$; therefore $u_r=\dfrac{ab}{a+b}(1-p^r)$.

188. $$\frac{1+2x}{1-x^3}=\frac{1}{1-x}+\frac{x}{1+x+x^2};$$

thus $$(1+2x)(1-x^3)^{-1}=(1-x)^{-1}+\frac{x}{1+x}\left(1+\frac{x^2}{1+x}\right)^{-1}.$$

Expand both sides and pick out the coefficient of x^{3n+2} on both sides. On the left-hand side it is 0. On the right-hand side it is 1 from $(1-x)^{-1}$; while $\dfrac{x}{1+x}\left(1+\dfrac{x^2}{1+x}\right)^{-1}=\dfrac{x}{1+x}-\dfrac{x^3}{(1+x)^2}+\dfrac{x^5}{(1+x)^3}-\dots$, and each of these must be expanded and the coefficient of x^{3n+2} picked out: see Example L. 13. In the identity thus obtained multiply by $(-1)^n$ and transpose, and the required result follows.

189. Here $\frac{u_{n+1}}{u_n}=\frac{n\,n^{n-1}x}{(n+1)^n}=\frac{x}{\left(1+\frac{1}{n}\right)^n}$. The value of $\left(1+\frac{1}{n}\right)^n$ when n is indefinitely great is e: see Art. 552. Thus the series is convergent if x is less than e by Art. 559. By Example 180 the value of $\left(1+\frac{1}{n}\right)^n$ increases with n; and thus $\frac{e}{\left(1+\frac{1}{n}\right)^n}$ is always greater than unity for any assigned value of n however large; hence the series is divergent if $x=e$ by Art. 560. And *a fortiori* if x is greater than e the series is divergent.

190. $\frac{1}{1-x-x^2+x^3}=\frac{1}{(1+x)(1-x)^2}$. Thus the logarithm of this expression $=-2\log(1-x)-\log(1+x)=2\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\dots\right)-\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\dots\right)$. Hence the coefficient of x^n is $\frac{3}{n}$ if n be even and $\frac{1}{n}$ if n be odd.

191. $1+2x^2+x^4=2ax(1-x^2)$; divide by x^2: thus $\frac{1}{x^2}+2+x^2=2a\left(\frac{1}{x}-x\right)$; that is $\left(x-\frac{1}{x}\right)^2+2a\left(x-\frac{1}{x}\right)+4=0$. By solving this quadratic we obtain $x-\frac{1}{x}=-a+\sqrt{(a^2-4)}$; hence x can be found.

192. If $x=y=z$ the expression vanishes. If $x=y$ the expression becomes $z^2(z-x)^2$; similarly it is necessarily positive if $y=z$ or if $z=x$. If x, y, z are all different, suppose this to be their algebraical descending order of magnitude so that $x-y$ and $y-z$ are positive. Thus $x^2(x-y)(x-z)$ is positive; and $z^2(z-x)(z-y)$ is positive, for the factors $z-y$ and $z-x$ are both negative: but $y^2(y-z)(y-x)$ is negative. Now either x^2 or z^2 is greater than y^2; and $x-z$ is greater than $x-y$, and greater than $y-z$; so that at least one of the two $x^2(x-y)(x-z)$ and $z^2(z-x)(z-y)$ is numerically greater than $y^2(y-z)(y-x)$. Thus the whole expression is positive.

193. The equations may be written thus:

$$(x^2+1)(y^2+1)=m^2xy,\quad n^2(x^2+1)\,y=x\,(y^2+1);$$

hence, by multiplication, $n^2(x^2+1)^2(y^2+1)\,y=m^2x^2y\,(y^2+1)$. The solution $y=0$ is inapplicable; $y^2+1=0$ gives impossible values. Take $n^2(x^2+1)^2=m^2x^2$; therefore $x^2+1=\pm\frac{mx}{n}$: hence we can find x. And since $(x^2+1)(y^2+1)=m^2xy$ we get by substitution $\pm\frac{m}{n}(y^2+1)=m^2y$; therefore $y^2+1=\pm mny$: hence we can find y.

194. Put $x=k(1-bv)$, $y=k(1+av)$, $z=k(1+abv^2)$, and substitute in each equation. The first becomes $k(a+b+c+abcv^2)=0$. The second becomes $k^2(1+abv^2)(a+b+c+abcv^2)=0$. Hence both equations are satisfied if $a+b+c+abcv^2=0$.

195. The m^{th} group begins with the $\{(m-1)q+1\}^{\text{th}}$ term of the derived Arithmetical Progression, that is with $\frac{a}{q}-\frac{b}{2q}+\frac{b}{2q^2}+(m-1)q\cdot\frac{b}{q^2}$; and so the sum of the terms in the group is

$$\frac{q}{2}\left\{\frac{2a}{q}-\frac{b}{q}+\frac{b}{q^2}+2(m-1)\frac{b}{q}+(q-1)\frac{b}{q^2}\right\}, \text{ that is } \frac{q}{2}\left\{\frac{2a}{q}+2(m-1)\frac{b}{q}\right\},$$

that is $a+(m-1)b$.

196. Let u_n denote the n^{th} term; then $u_n=\frac{1}{2}(u_{n-1}+u_{n-2})$; therefore $u_n-u_{n-1}=-\frac{1}{2}(u_{n-1}-u_{n-2})$. This shews that the series u_2-u_1, u_3-u_2, u_4-u_3,... is a Geometrical Progression with the common ratio $-\frac{1}{2}$. Therefore $u_n-u_{n-1}=(u_2-u_1)\left(-\frac{1}{2}\right)^{n-2}=(b-a)\left(-\frac{1}{2}\right)^{n-2}$. Thus $u_2-u_1=b-a$, $u_3-u_2=(b-a)\left(-\frac{1}{2}\right)$, $u_4-u_3=(b-a)\left(-\frac{1}{2}\right)^2$, ... $u_n-u_{n-1}=(b-a)\left(-\frac{1}{2}\right)^{n-2}$. Hence, by addition, $u_n-u_1=(b-a)\left\{1+\left(-\frac{1}{2}\right)+\left(-\frac{1}{2}\right)^2+\ldots\ldots+\left(-\frac{1}{2}\right)^{n-2}\right\}$

$$=(b-a)\frac{\left(-\frac{1}{2}\right)^{n-1}-1}{-\frac{1}{2}-1}=\frac{2(b-a)}{3}\left\{1-\left(-\frac{1}{2}\right)^{n-1}\right\}.$$

197. First suppose two things a and b. The permutations are ab and ba. Hence we form $\frac{1}{a(a+b)}$ and $\frac{1}{b(b+a)}$; their sum $=\left(\frac{1}{a}+\frac{1}{b}\right)\frac{1}{a+b}=\frac{1}{ab}$. Next suppose three things a, b, and c. Consider those permutations in which c stands last; they are abc and bac. Hence we form $\frac{1}{a(a+b)(a+b+c)}$ and $\frac{1}{b(b+a)(b+a+c)}$; their sum is $\frac{1}{ab(a+b+c)}$ by *the first case*. Similarly we get $\frac{1}{bc(a+b+c)}$ from the permutations in which a stands last, and $\frac{1}{ca(a+b+c)}$ from the permutations in which b stands last. Hence the whole sum $=\frac{1}{a+b+c}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)=\frac{1}{abc}$. Similarly having thus proved the theorem for *three* things we can prove it for *four*: and so on.

198. The expression on the left-hand side is easily seen to be the term which does not involve y in the expansion of $(1+y)^n\left(1+\frac{x}{y}\right)^n$: now this

$$=\left(1+x+y+\frac{x}{y}\right)^n=(1+x)^n\left\{1+\frac{y+\frac{x}{y}}{1+x}\right\}^n=(1+x)^n\left\{1+nz+\frac{n(n-1)}{2}z^2+\ldots\right\},$$

where z stands for $\dfrac{y+\frac{x}{y}}{1+x}$. We have now to find the term which does not contain y in the expansion of $z, z^2, z^3, \ldots$. We easily see that there is no such a term in $z, z^3, z^5, \ldots$. In z^2 the term is $\dfrac{2x}{(1+x)^2}$; in z^4 the term is $\dfrac{\lfloor 4}{\lfloor 2\,\lfloor 2}\dfrac{x^2}{(1+x)^4}$; and so on.

199. $u_n=\dfrac{1}{v_n{}^{n+1}}$, where $v_n=\left(\dfrac{n+2}{n+1}\right)^{n+1}-\dfrac{n+2}{n+1}=\left(1+\dfrac{1}{n+1}\right)^{n+1}-\dfrac{n+2}{n+1}$; hence when n is taken large enough v_n approaches as closely as we please to the value $e-1$. Hence u_n may be considered to be equal to $\left(\dfrac{1}{e-1}\right)^{n+1}$ when n is large enough; and as $\dfrac{1}{e-1}$ is less than unity the series becomes a Geometrical Progression in which the common ratio is less than unity, and is therefore convergent.

200. By the method of Art. 549 the series is equal to the product of $\lfloor n+3$ into the coefficient of x^{n+3} in the expansion of

$$\left(x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}+\ldots\right)^n;$$

this coefficient is the same as that of x^3 in the expansion of

$$\left(1+\frac{x}{\lfloor 2}+\frac{x^2}{\lfloor 3}+\frac{x^3}{\lfloor 4}+\ldots\right)^n.$$

As in Art. 529, $\quad q+2r+3s=3,\ p+q+r+s=n.$

The coefficient is

$$n\frac{1}{\lfloor 4}+n(n-1)\frac{1}{\lfloor 2}\cdot\frac{1}{\lfloor 3}+\frac{n(n-1)(n-2)}{\lfloor 3}\left(\frac{1}{\lfloor 2}\right)^3,$$

that is $\qquad \dfrac{n}{24}+\dfrac{n(n-1)}{12}+\dfrac{n(n-1)(n-2)}{48}.$

p	q	r	s
$n-1$	0	0	1
$n-2$	1	1	0
$n-3$	3	0	0

Therefore the series $=\lfloor n+3\left\{\dfrac{n}{24}+\dfrac{n(n-1)}{12}+\dfrac{n(n-1)(n-2)}{48}\right\}$.

201. By actual multiplication we obtain for the result of the first part of the Example $1-x-x^2+x^5+x^6+x^7-x^8-x^9-x^{10}+x^{13}+x^{14}-x^{15}$. Now if we multiply out $(1-x^6)(1-x^7)(1-x^8)\ldots$, it is obvious that as far as x^{12} inclusive the product will be $1-x^6-x^7-x^8-x^9-x^{10}-x^{11}-x^{12}$. Multiply this into the former result, and we obtain as far as x^{12} inclusive $1-x-x^2+x^5+x^7-x^{12}$.

202. It is easy to shew by actual multiplication, or by the method of Art. 308, that $(a-b)^3+(b-c)^3+(c-a)^3=-3(b-a)(c-b)(a-c)$;
hence $\quad (a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3=-3(b^2-a^2)(c^2-b^2)(a^2-c^2)$.
Divide the latter by the former, and the required result follows.

203. We have to shew that R^{100} is not less than 50 where $R=1+\frac{4}{100}$.
Now $\log R^{100}=100\log R=100\log\frac{104}{100}=100(\log 104-2)=100(\log 13+\log 8-2)$
$=100(\log 13+3\log 2-2)=1{\cdot}7033$. Also $\log 50=\log\frac{100}{2}=2-\log 2=1{\cdot}69897$.
Thus $\log R^{100}$ is greater than $\log 50$; and therefore R^{100} is greater than 50.

204. Let x denote the value of the continued fraction $\frac{1}{1+}\,\frac{1}{p+}\,\frac{1}{1+}\,\frac{1}{p+}\dots$;
then $\quad x=\dfrac{1}{1+\dfrac{1}{p+x}}=\dfrac{p+x}{p+1+x}$; $\quad$ therefore $x^2+px-p=0$; $\quad$ therefore
$x=\frac{1}{2}(-p+\sqrt{p^2+4p})$; therefore $2x+p=\sqrt{p^2+4p}$: this is the required result.

205. Let x denote the number of weights of 9 lbs., and y the number of weights of 14 lbs.; then $9x+14y=2240$. One solution is $x=0$, $y=160$; the general solution is $x=14t$, $y=160-9t$; thus t may have any value from 0 to 17 both inclusive. Therefore there are 18 solutions.

206. Assume $\quad \dfrac{1}{(1-2x)(1-x)^2}=\dfrac{A}{1-2x}+\dfrac{Bx+C}{(1-x)^2}$; $\quad$ therefore
$1=A(1-x)^2+(Bx+C)(1-2x)$. Since this is to be identically true we may give any value to x; suppose then $x=\frac{1}{2}$: thus $1=\frac{A}{4}$, so that $A=4$. Therefore $1-4(1-x)^2=(Bx+C)(1-2x)$, that is $(2x-3)(1-2x)=(Bx+C)(1-2x)$; therefore $2x-3=Bx+C$; therefore $B=2$, and $C=-3$. Thus

$$\frac{1}{(1-2x)(1-x)^2}=\frac{4}{1-2x}+\frac{2x-3}{(1-x)^2}=4(1-2x)^{-1}+(2x-3)(1-x)^{-2}.$$

Hence the coefficient of x^n in the expansion is $4.2^n+2n-3(n+1)$, that is $2^{n+2}-n-3$.

207. Divide both the proposed expressions by $1-x$; we have then to shew that $\dfrac{1+x+x^2+\dots+x^n}{n+1}$ is less than $\dfrac{1+x+x^2+\dots+x^{n-1}}{n}$, or that nx^n is less than $1+x+x^2+\dots+x^{n-1}$: and this is obvious, for each of the n terms on the right-hand side is greater than x^n.

208. It is obvious that the expression $n^4-4n^3+5n^2-2n$ vanishes when $n=1$; therefore $n-1$ is a factor of the expression, by Art. 332; and n is also a factor. Divide the expression by $n(n-1)$, and we obtain the other factor: in this way we find that the expression $=(n-2)(n-1)^2n$. Now one of the

three factors $n-2$, $n-1$, and n, is divisible by 3; if n be even both n and $n-2$ are divisible by 2; and if n be odd $(n-1)^2$ is divisible by 4. Thus the expression is divisible by 3×4, that is by 12.

209. The number of ways in which 5 counters can be drawn is the number of the combinations of 10 things taken 5 at a time, that is $\frac{|\underline{10}}{|\underline{5}\,|\underline{5}}$. The number of ways in which the 3 marked counters can be drawn among the 5 drawn is the number of combinations of 7 things taken 2 at a time, that is $\frac{|\underline{7}}{|\underline{2}\,|\underline{5}}$. Hence the chance of drawing the 3 marked counters is $\frac{|\underline{7}}{|\underline{2}\,|\underline{5}} \div \frac{|\underline{10}}{|\underline{5}\,|\underline{5}}$, that is $\frac{3.4.5}{8.9.10}$, that is $\frac{1}{12}$. Hence the value of the expectation is $\frac{1}{12}$ of a shilling.

210. The n^{th} term $=\frac{(n-1)^{n-1}}{n^n}=\frac{1}{n-1}\left(\frac{n-1}{n}\right)^n=\frac{1}{n-1}\left(1-\frac{1}{n}\right)^n$. Now $\left(1-\frac{1}{n}\right)^n$ can be brought as near to e^{-1} as we please by taking n large enough: see Art. 552. Thus the series bears a finite ratio to the divergent series $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$; and is therefore itself divergent.

211. Let $a_1, a_2, \dots a_n$ denote the n quantities. Then

$$(a_1+a_2+\dots+a_n)^2=\frac{2n}{n-1}(a_1a_2+a_1a_3+\dots+a_2a_3+\dots).$$

Put for the square on the left-hand side its developed form, multiply by $n-1$, and bring all the terms to the left-hand side: thus we get

$$(a_1-a_2)^2+(a_1-a_3)^2+\dots+(a_2-a_3)^2+\dots=0.$$

Therefore $\qquad a_1-a_2=0,\ a_1-a_3=0, \dots\ a_2-a_3=0, \dots$

212. Put x for $b-c$ and y for $c-a$; then $x+y=-(a-b)$. Hence the first expression becomes $25\{(x+y)^7-x^7-y^7\}\{(x+y)^3-x^3-y^3\}$, that is $25(x^2+xy+y^2)^2 7xy(x+y)\times 3xy(x+y)$: see Example IV. 45. The second expression becomes $21\{(x+y)^5-x^5-y^5\}^2$, that is $21\{5(x+y)(x^2+xy+y^2)xy\}^2$.

213. Suppose that the expectation of life is x years. By Art. 595 we have $1812\frac{4}{5}=150\frac{1-R^{-x}}{R-1}$ where $R=1+\frac{5}{100}=\frac{21}{20}$; therefore $\frac{1}{20}.\frac{1812\frac{4}{5}}{150}=1-R^{-x}$; therefore $R^{-x}=1-\frac{9064}{15000}=\frac{5936}{15000}$; therefore $R^x=\frac{15000}{5936}=\frac{30000}{11872}$. Thus $x\log\frac{21}{20}=\log\frac{3\cdot 0000}{1\cdot 1872}$; therefore $x=\frac{\log 3-\log 1\cdot 1872}{\log 3+\log 7-\log 2-1}=\frac{\cdot 4025974}{\cdot 0211893}$: this is almost exactly 19.

214. $\frac{\sqrt{5}+1}{2}=1+\frac{\sqrt{5}-1}{2}=1+\frac{2}{\sqrt{5}+1}$: thus every quotient is unity. Hence $p_r=p_{r-1}+p_{r-2}$. Thus $p_4=p_3+p_2$, $p_6=p_5+p_4$, ... $p_{2n}=p_{2n-1}+p_{2n-2}$; therefore by addition $p_{2n}=p_3+p_5+p_7+...+p_{2n-1}$.
Similarly $q_r=q_{r-1}+q_{r-2}$; &c.

215. Let x denote the numerator, and y the denominator; then $5x+11y=1031$. Divide by 5; thus $x+2y+\frac{y-1}{5}=206$; therefore $\frac{y-1}{5}$ must be an integer, say $=t$. Thus $y=1+5t$, and $x=204-11t$. As the fraction is to be a *proper* fraction we must have $1+5t$ greater than $204-11t$; this requires t to be greater than 12. Hence we find that t may have any value between 13 and 18 both inclusive.

216. By Art. 648 we know that $\frac{\lfloor n}{(x+1)(x+2)...(x+n+1)}$ is *identically* equal to $\frac{A_1}{x+1}+\frac{A_2}{x+2}+...+\frac{A_{n+1}}{x+n+1}$, where A_1, A_2, ... A_{n+1} are certain constants. Clear of fractions; thus

$$\lfloor n=A_1(x+2)(x+3)...(x+n+1)+A_2(x+1)(x+3)...(x+n+1)$$
$$+A_3(x+1)(x+2)(x+4)...(x+n+1)+...$$

Now since this is an identity it is true for all values of x; put -1 for x; thus every term on the right-hand side vanishes, except that which involves A_1; and we get $\lfloor n=A_1\lfloor n$, so that $A_1=1$. Again in the identity put -2 for x; thus every term on the right-hand side vanishes, except that which involves A_2: and we get $\lfloor n=-A_2\lfloor n-1$, so that $A_2=-n$. Similarly $A_3=\frac{n(n-1)}{2}$. And so on. The theorem may also be demonstrated by induction in the manner of Example 236.

217. Put $\frac{m}{n}$ for p, where m and n are positive integers. We have to shew that $(a+b)^{\frac{m}{n}}a^{\frac{n-m}{n}}$ is less than $\frac{na+mb}{n}$. Suppose that there are m quantities each equal to $a+b$, and $n-m$ quantities each equal to a; then their arithmetical mean is $\frac{m(a+b)+(n-m)a}{n}$, that is $\frac{na+mb}{n}$; and their geometrical mean is $\{(a+b)^m a^{n-m}\}^{\frac{1}{n}}$, that is $(a+b)^{\frac{m}{n}}a^{\frac{n-m}{n}}$: the latter is less than the former by Art. 681.

218. We have $3^4-1=81-1=80$; therefore $3^{n+4}-3^n=80\cdot3^n$; thus if the digit in the tens' place of 3^n is even so also is the digit in the tens' place of 3^{n+4}. But if $n=1$, or 2, or 3, or 4, the digit in the tens' place of 3^n is even; therefore it is even for every value of n. Any power of 9 is some power of 3; and therefore the same statement is true for powers of 9. The statement is also true for powers of 7; it may be established in the same way as for powers of 3, observing that $7^4-1=2400$.

Again, $5^{n+1}-5^n=5^n(5-1)=20.5^{n-1}$; thus if the digit in the tens' place of 5^n is even, so also is the digit in the tens' place of 5^{n+1}. But if $n=1$, or 2, the digit in the tens' place of 5^n is even; therefore it is even for every value of n. And $6^{n+1}-6^n=6^n(6-1)=30.6^{n-1}$; and hence we may shew that the digit in the tens' place of 6^n is odd for every value of n.

219. There are three possible hypotheses, (1) that all the balls are black, (2) that two of the balls are black, (3) that one ball is black. Hence, assuming that before the observed event the three hypotheses were equally probable, the probability of the first hypothesis after the observed event is $1\div\left\{1+\frac{8}{27}+\frac{1}{27}\right\}$, that is $\frac{3}{4}$.

220. By Art. 794 the continued fraction $y+\frac{1}{2y+}\,\frac{1}{2y+}\dots=\sqrt{(y^2+1)}$; thus $x=\sqrt{(y^2+1)}$; by Art. 795 the continued fraction $x-\frac{1}{2x-}\,\frac{1}{2x-}\dots=\sqrt{(x^2-1)}$: since $x=\sqrt{(y^2+1)}$ it follows that $y=\sqrt{(x^2-1)}$.

221. Clear of fractions and bring all the terms to one side; then it will be found that the given relation may be written thus:

$$(a+b-c)(a-b+c)(-a+b+c)=0.$$

Therefore one of the three factors must be zero. Suppose $a+b-c=0$; then

$$\frac{b^2+c^2-a^2}{2bc}=\frac{(a+b)^2+b^2-a^2}{2b(a+b)}=\frac{a+b+b-a}{2b}=1,$$

similarly $\frac{c^2+a^2-b^2}{2ca}=1$; and $\frac{a^2+b^2-c^2}{2ab}=\frac{a^2+b^2-(a+b)^2}{2ab}=-1$.

222. Suppose t the value of each of the equal quantities; then

$$x^2=a^2-t,\quad y^2=b^2-t,\quad z^2=c^2-t;$$

and substituting we get

$$\sqrt{(b^2-t)(c^2-t)}+\sqrt{(c^2-t)(a^2-t)}+\sqrt{(a^2-t)(b^2-t)}=t.$$

Transpose and square;

$$(b^2-t)(c^2-t)+2(c^2-t)\sqrt{(b^2-t)(a^2-t)}+(c^2-t)(a^2-t)$$
$$=t^2-2t\sqrt{(a^2-t)(b^2-t)}+(a^2-t)(b^2-t).$$

Therefore $$2c^2\sqrt{(b^2-t)(a^2-t)}=a^2b^2-b^2c^2-a^2c^2+2c^2t.$$

Square again; &c.

223. By Art. 595 we have $p=\frac{1-R^{-n}}{R-1}$, $q=\frac{1-R^{-2n}}{R-1}$; therefore $\frac{q}{p}=1+R^{-n}$; therefore $(R-1)p+\frac{q}{p}=2$, that is $rp+\frac{q}{p}=2$.

224. $$\sqrt{\left(a^2+\frac{2a}{b}\right)}=a+\sqrt{\left(a^2+\frac{2a}{b}\right)}-a=a+\frac{2a}{\sqrt{(b^2a^2+2ab)}+ab};$$

$$\frac{\sqrt{(b^2a^2+2ab)}+ab}{2a}=b+\frac{\sqrt{(b^2a^2+2ab)}-ab}{2a}=b+\frac{b}{\sqrt{(b^2a^2+2ab)}+ab};$$

$$\frac{\sqrt{(b^2a^2+2ab)}+ab}{b}=2a+\frac{\sqrt{(b^2a^2+2ab)}-ab}{b}=2a+\frac{2a}{\sqrt{(b^2a^2+2ab)}+ab}.$$

Thus it is obvious that the quotients are a, b, $2a$, b, $2a$,...

225. By Art. 337 the proposed *expression* is equal to $2(x-\alpha)(x-\beta)$, where α and β are the roots of the *equation* $2x^2-(21y+1)x-11y^2+34y-3=0$. These roots will be found to be $\frac{21y+1\pm(23y-5)}{4}$, that is $11y-1$ and $-\frac{y}{2}+\frac{3}{2}$. Hence the proposed expression

$$=2(x-11y+1)\left(x+\frac{y}{2}-\frac{3}{2}\right)=(x-11y+1)(2x+y-3).$$

226. By Art. 658 the recurring series can be transformed into two Geometrical Progressions, one arising from the expansion of $\frac{A}{\alpha-x}$ and the other from the expansion of $\frac{B}{\beta-x}$, where A and B are constants, and α and β are the roots of the equation $1-px-qx^2=0$. In order that these Geometrical Progressions may be both convergent x must be numerically less than both α and β.

227. $\frac{p_1a_1^2+p_2a_2^2+\ldots+p_na_n^2}{p_1+p_2+\ldots+p_n}$ is greater than $\left(\frac{p_1a_1+p_2a_2+\ldots+p_na_n}{p_1+p_2+\ldots+p_n}\right)^2$; for if we clear of fractions and bring all the terms to the left-hand side we obtain a series of positive quantities $p_1p_2(a_1-a_2)^2+p_1p_3(a_1-a_3)^2+\ldots$ And $\frac{p_1a_1+p_2a_2+\ldots+p_na_n}{p_1+p_2+\ldots+p_n}$ is greater than $\frac{a_1+a_2+\ldots+a_n}{n}$; for if we clear of fractions and bring all the terms to the left-hand side we obtain a series of positive quantities $(p_1-p_2)(a_1-a_2)+(p_1-p_3)(a_1-a_3)+\ldots$ Hence the required result follows.

228. We may suppose that $a=5p+\alpha$, $b=5q+\beta$, $c=5r+\gamma$; where p, q, r are integers, and α, β, γ represent 0 or ±1 or ±2. Thus

$$(5p+\alpha)^2+(5q+\beta)^2=(5r+\gamma)^2.$$

Therefore $\gamma^2-\beta^2-\alpha^2$ is either 0 or divisible by 5. Now α^2, β^2, γ^2 must be 0 or 1 or 4; and on trial we shall find it impossible to have $\gamma^2-\beta^2-\alpha^2$ equal to 0 or divisible by 5 unless at least one of the three α, β, γ is 0. Thus one of the three a, b, c is divisible by 5.

229. Suppose, for example, that the number is to have 4 digits; then the number is equally likely to be any one of the numbers which lie between 9999 and 1000 both inclusive. Now beginning with 9999 and descending to 1000 we see immediately that one number out of every nine is divisible by 9.

230. This can be verified for the cases $n=1$ and $n=2$, and can be shewn to be universally true by induction For

$$x^n+y^n=(x+y)(x^{n-1}+y^{n-1})-xy(x^{n-2}+y^{n-2});$$

let $x=3+\sqrt{5}$, and $y=3-\sqrt{5}$; then x^n+y^n is the integer next greater than x^n: see Art. 526. Now $x+y$ is divisible by 2, and $xy=4$: thus if $x^{n-1}+y^{n-1}$ is divisible by 2^{n-1}, and $x^{n-2}+y^{n-2}$ by 2^{n-2}, then x^n+y^n is divisible by 2^n.

231. Let $x^2+ax+\beta$ denote the common quadratic factor; and suppose that $x^3+px^2+qx+r=(x^2+ax+\beta)(x+\gamma)$, and $x^3+p'x^2+q'x+r'=(x^2+ax+\beta)(x+\gamma')$. Then equating the coefficients in these identities we get

$$p=a+\gamma, \quad q=a\gamma+\beta, \quad r=\beta\gamma;$$
$$p'=a+\gamma', \quad q'=a\gamma'+\beta, \quad r'=\beta\gamma'.$$

Hence $$p-p'=\gamma-\gamma', \quad q-q'=a(\gamma-\gamma'), \quad r-r'=\beta(\gamma-\gamma');$$
$$p'r-pr'=a\beta(\gamma-\gamma'), \quad q'r-qr'=\beta^2(\gamma-\gamma').$$

Thus the proposed expressions are equal, for each of them is equal to β.

232. It follows from the results in the preceding solution that

$$\frac{p-p'}{r-r'}r=\frac{1}{\beta}\beta\gamma=\gamma, \quad \frac{p-p'}{r-r'}r'=\frac{1}{\beta}\beta\gamma'=\gamma';$$
$$\frac{q-q'}{p-p'}=a, \quad \frac{r-r'}{p-p'}=\beta.$$

233. Since the present value of an annuity of £100 on the life of a person aged 21 is £2150, the present value of the annuity for the child would be £$\frac{2150}{R^{21}}$ if the child were certain to reach the age of 21 years. But as only 6 children out of 10 reach the age of 21 years the present value is £$\frac{6}{10}\times\frac{2150}{R^{21}}$. Now $R=1+\frac{3}{100}$: thus $\log\frac{2150}{R^{21}}=\log\left(43\times\frac{100}{2}\right)-21\log 1{\cdot}03$ $=2+\log 43-\log 2-21\log 1{\cdot}03=3{\cdot}0628=\log 1155$. Therefore the present value $=$ £$\frac{6}{10}\times 1155=$ £693.

234. $$\sqrt{\left(a^2-\frac{a}{n}\right)}=a-1+\sqrt{\left(a^2-\frac{a}{n}\right)}-(a-1)$$
$$=a-1+\frac{(2n-1)a-n}{\sqrt{(n^2a^2-na)}+na-n};$$

$$\frac{\sqrt{(n^2a^2-na)}+na-n}{(2n-1)a-n}=1+\frac{\sqrt{(n^2a^2-na)}-a(n-1)}{(2n-1)a-n}=1+\frac{a}{\sqrt{(n^2a^2-na)}+a(n-1)};$$

$$\frac{\sqrt{(n^2a^2-na)}+a(n-1)}{a}=2(n-1)+\frac{\sqrt{(n^2a^2-na)}-a(n-1)}{a}$$
$$=2(n-1)+\frac{(2n-1)a-n}{\sqrt{(n^2a^2-na)}+na-a}.$$

By proceeding in this way we shall find that the first five quotients are $a-1, 1, 2(n-1), 1, 2(a-1)$; of which all, except the first, recur.

235. Let x denote the digit in the tens' place, and y the digit in the units' place; then the number is $10x+y$: hence $10y+x=\frac{1}{2}(10x+y)-1$; therefore $19y-8x+2=0$. Divide by 8; thus $2y-x+\frac{3y+2}{8}=0$; therefore $\frac{3y+2}{8}$ must be an integer, say $=s$: thus $3y+2=8s$. Divide by 3; thus $y=2s+\frac{2(s-1)}{3}$; therefore $\frac{s-1}{3}$ must be an integer, say $=t$: thus $s=3t+1$. Therefore $y=8t+2$, and $x=19t+5$. Since x and y must each be less than 10, the only admissible solution is $x=5$, and $y=2$.

236. This may be shewn by induction. It is obviously true when $n=1$, whatever be the value of x provided neither $x+1$ nor $x+2$ vanishes. Assume then that it is true for a certain value of n provided no denominator vanishes. Change x into $x+1$, assuming also that $x+n+2$ does not vanish. Thus

$$\frac{1}{x+1}-\frac{n}{(x+1)(x+2)}+\frac{n(n-1)}{(x+1)(x+2)(x+3)}-\ldots=\frac{1}{x+n+1},$$

and $$\frac{1}{x+2}-\frac{n}{(x+2)(x+3)}+\frac{n(n-1)}{(x+2)(x+3)(x+4)}-\ldots=\frac{1}{x+n+2}.$$

Subtract the second result from the first: thus

$$\frac{1}{(x+1)(x+2)}-\frac{2n}{(x+1)(x+2)(x+3)}+\frac{3n(n-1)}{(x+1)\ldots(x+4)}-\ldots=\frac{1}{x+n+1}-\frac{1}{x+n+2}$$

Again, subtract this from the first result: thus

$$\frac{1}{x+1}-\frac{n+1}{(x+1)(x+2)}+\frac{(n+1)n}{(x+1)(x+2)(x+3)}-\ldots=\frac{1}{x+n+2}.$$

This is what we should get by changing n into $n+1$ in the first result: thus if the theorem is true for a specific value of n it is true for the next value. Hence as it is true when $n=1$, it is universally true.

The theorem may also be demonstrated in the manner of Example 216; as we may assume that the expression on the left-hand side is identically equal to $\frac{A_1}{x+1}+\frac{A_2}{x+2}+\ldots+\frac{A_{n+1}}{x+n+1}$ where $A_1, A_2 \ldots A_{n+1}$ are constants.

237. We have to shew that x^n-1 is greater than $n(x-1)x^{\frac{n-1}{2}}$; divide by $x-1$; then we have to shew that $\frac{x^{n-1}+x^{n-2}+\ldots+1}{n}$ is greater than $x^{\frac{n-1}{2}}$. The left-hand member is the arithmetical mean of a set of n quantities, and the right-hand member is the geometrical mean of the same: the former is the greater by Art. 681.

238. All these statements can be verified by trial for low powers and shewn to be universally true by induction.

For $4^3-4=60$; thus $4^{n+2}-4^n=60.4^{n-1}$; therefore the digit in the tens' place of 4^{n+2} is even or odd according as that in the tens' place of 4^n is even or odd.

Also $2^5-2=30$; thus $2^{n+4}-2^n=30\,.\,2^{n-1}$; therefore the digit in the tens' place of 2^{n+4} is even or odd according as that in the tens' place of 2^n is even or odd. We suppose that n is not less than 4.

And $8^5-8=32760$; thus $8^{n+4}-8^n=32760\,.\,8^{n-1}$; therefore the digit in the tens' place of 8^{n+4} is even or odd according as that in the tens' place of 8^n is even or odd.

239. The chance that the digit taken is 2 is $\frac{1}{8}$; and if the digit is 2 the chance is $\frac{1}{2}$ that the digit in the tens' place of the power is odd: thus there is a chance of $\frac{1}{16}$ relative to 2. Similarly there is a chance of $\frac{1}{16}$ relative to 4 and to 8. And by Example 218 there is a chance $\frac{1}{8}$ relative to 6. By Example 218 the chance is zero relative to 3, 5, 7, or 9. Thus on the whole the chance is $\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{8}$, that is $\frac{5}{16}$.

240. Here $u_n=\dfrac{2n^2+3n+2}{(n+1)(n+2)(n+3)}$; $\dfrac{u_n}{u_{n+1}}=\dfrac{(n+4)(2n^2+3n+2)}{(n+1)\{2(n+1)^2+3(n+1)+2\}}$

$$=\frac{(n+4)(2n^2+3n+2)}{(n+1)(2n^2+7n+7)}=\frac{2n^3+11n^2+14n+8}{2n^3+9n^2+14n+7}=\frac{n^3+\frac{11}{2}n^2+7n+4}{n^3+\frac{9}{2}n^2+7n+\frac{7}{2}}.$$

Thus with the notation of Art. 776 we have $a-A-1=0$; and the series is divergent.

Or we may proceed thus: By the method of Art. 647 we have

$$\frac{2n^2+3n+2}{(n+1)(n+2)(n+3)}=\frac{1}{2}\left(\frac{1}{n+1}-\frac{8}{n+2}+\frac{11}{n+3}\right);$$

and by transforming every term we find that the proposed series

$$=-\frac{11}{12}+2\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots+\frac{1}{n}+\ldots\right),$$

and is therefore divergent by Art. 562.

241. Here $a_2=\frac{1}{2}(a_1+b_1)$, and $b_2=\dfrac{2a_1b_1}{a_1+b_1}$; therefore $a_2b_2=a_1b_1$; thus the theorem is true when $n=2$. We shall now shew by induction that it is universally true. Assume that for any value of n we have $a_nb_n=a_1b_1$: now $a_{n+1}=\frac{1}{2}(a_1+b_n)$, and $b_{n+1}=\dfrac{2b_1a_n}{b_1+a_n}$; therefore

$$a_{n+1}b_{n+1}=\frac{(a_1+b_n)\,b_1a_n}{b_1+a_n}=\frac{\left(a_1+\dfrac{a_1b_1}{a_n}\right)b_1a_n}{b_1+a_n}=a_1b_1.$$

This shews that if the theorem is true for a specific value of n it is true for the next value. Hence as it is true when $n=2$, it is universally true.

242. Add the first and the second equations; thus $x^2+y^2-a^2-b^2=2xy$; therefore $x-y=\pm\sqrt{(a^2+b^2)}$: similarly $y-z=\pm\sqrt{(b^2+c^2)}$, and $z-x=\pm\sqrt{(c^2+a^2)}$. Therefore, by addition, we see that the equations are inconsistent unless $\pm\sqrt{(a^2+b^2)}\pm\sqrt{(b^2+c^2)}\pm\sqrt{(c^2+a^2)}=0$. If this relation does hold the three given equations are equivalent to *two*, and are therefore insufficient to find the values of the *three* unknown quantities.

243. Let P denote the original capital; then the person spends every year $2Pr$, when $r=\frac{4}{100}$: and we have to determine n so that P may be the present worth of an annuity of $2Pr$ continued for n years. Thus $P=2Pr\frac{1-R^{-n}}{r}$; therefore $1-R^{-n}=\frac{1}{2}$; therefore $R^n=2$, that is $(1+r)^n=2$;

therefore $n\log\frac{104}{100}=\log 2$; therefore $n=\frac{\log 2}{\log 13+\log 8-2}=18$ nearly.

244. $$\sqrt{\left(a^2+\frac{4a+2}{3}\right)}=a+\sqrt{\left(a^2+\frac{4a+2}{3}\right)}-a$$
$$=a+\frac{4a+2}{\sqrt{(9a^2+12a+6)}+3a}.$$

Put N for $9a^2+12a+6$.

$$\frac{\sqrt{N}+3a}{4a+2}=1+\frac{\sqrt{N}-(a+2)}{4a+2}=1+\frac{2a+1}{\sqrt{N}+a+2},$$
$$\frac{\sqrt{N}+a+2}{2a+1}=2+\frac{\sqrt{N}-3a}{2a+1}=2+\frac{6}{\sqrt{N}+3a},$$
$$\frac{\sqrt{N}+3a}{6}=a+\frac{\sqrt{N}-3a}{6}=a+\frac{2a+1}{\sqrt{N}+3a},$$
$$\frac{\sqrt{N}+3a}{2a+1}=2+\frac{\sqrt{N}-(a+2)}{2a+1}=2+\frac{4a+2}{\sqrt{N}+a+2},$$
$$\frac{\sqrt{N}+a+2}{4a+2}=1+\frac{\sqrt{N}-3a}{4a+2}=1+\frac{3}{\sqrt{N}+3a},$$
$$\frac{\sqrt{N}+3a}{3}=2a+\frac{\sqrt{N}-3a}{3}=2a+\frac{4a+2}{\sqrt{N}+3a};$$

the quotients will now recur.

245. Suppose that the farmer bought x sheep and y bullocks; then $\frac{3x}{2}+5y=25$; therefore $3x+10y=50$. Divide by 3; thus $x+3y+\frac{y-2}{3}=16$; therefore $\frac{y-2}{3}$ must be an integer, say $=t$: thus $y=2+3t$, and $x=10-10t$. Either $x=10$, and $y=2$; or $x=0$, and $y=5$.

246. If each term on the left-hand side is expanded the whole coefficient of x^r will be found to be $\frac{(-1)^r\lfloor n}{\lfloor r\lfloor n-r}\left\{\frac{1}{m}+\frac{n-r}{m(m-1)}+\frac{(n-r)(n-r-1)}{m(m-1)(m-2)}+\ldots\right\}$ and

the coefficient of x^r on the right-hand side is $(-1)^r \frac{\lfloor n}{\lfloor r \lfloor n-r} \cdot \frac{1}{m-n+r}$. We have to shew that these are equal. Put $n-r=s$; thus we have to shew that $\frac{1}{m}+\frac{s}{m(m-1)}+\frac{s(s-1)}{m(m-1)(m-2)}+\ldots=\frac{1}{m-s}$. This may be shewn by induction. It is obviously true when $s=1$, whatever be the value of m provided neither m nor $m-1$ vanishes. Assume then that it is true for a certain value of s provided no denominator vanishes. Change m into $m-1$ assuming also that $m-s-1$ does not vanish. Thus

$$\frac{1}{m}+\frac{s}{m(m-1)}+\frac{s(s-1)}{m(m-1)(m-2)}+\ldots=\frac{1}{m-s},$$

$$\frac{1}{m-1}+\frac{s}{(m-1)(m-2)}+\frac{s(s-1)}{(m-1)(m-2)(m-3)}+\ldots=\frac{1}{m-s-1}.$$

Subtract the first result from the second: thus

$$\frac{1}{m(m-1)}+\frac{2s}{m(m-1)(m-2)}+\frac{3s(s-1)}{m(m-1)(m-2)(m-3)}+\ldots=\frac{1}{m-s-1}-\frac{1}{m-s},$$

add this to the first result; thus

$$\frac{1}{m}+\frac{s+1}{m(m-1)}+\frac{(s+1)s}{m(m-1)(m-2)}+\ldots=\frac{1}{m-s-1}.$$

This is what we should get by changing s into $s+1$ in the first result: thus if the theorem is true for a specific value of s it is true for the next value. Hence as it is true when $s=1$, it is universally true.

247. Transfer all the terms to the left-hand side; then omitting those which cancel we have a series in which each term is of the form $(a^p-b^p)(a^q-b^q)$: these terms are positive, for both factors are positive, or both negative.

248. By Fermat's Theorem $N^{n-1}=1+kn$ where k is some integer; raise both sides to the power n^{r-1}; then on the left-hand side we have N^m; and on the right-hand side we have by expansion $1+pn^r$, where p is some integer.

249. There are six hypotheses with respect to the notes to be regarded as equally probable before the observed event. (1) Three of £5. (2) Two of £5, one of £20. (3) Two of £5, one of £10. (4) One of £5, two of £10. (5) One of £5, two of £20. (6) One of £5, one of £10, one of £20. The probability of the observed event on these hypotheses is respectively 1, $\frac{2}{3}$, $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{3}$. Hence after the observed event the probabilities are respectively $\frac{3}{10}$, $\frac{2}{10}$, $\frac{2}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{1}{10}$. The value of the next draw in pounds is therefore $\frac{1}{3}\times\frac{1}{10}\left\{3\times 3\times 5+2\times 30+2\times 20+25+45+35\right\}$, that is, $\frac{250}{30}$, that is $\frac{25}{3}$, that is $8\frac{1}{3}$.

250.
$$\begin{array}{r|l} & 1+0+\ 3-12+\ \ 4 \\ 4 & \quad 4+16+28-128-832 \\ -12 & \qquad -12-48-\ 84+384+2496 \\ \hline & 1+4+\ 7-32-208-448 \end{array}$$

Thus $\dfrac{x^4+3x^2-12x+4}{x^2-4x+12}=x^2+4x+7-32x^{-1}-208x^{-2}-\dfrac{448x^{-1}-2496x^{-2}}{x^2-4x+12}$

251. The first equation may be written: $x^2y(y-1)+x(y-1)-4(y-1)=0$. Thus either $y-1=0$ or $x^2y+x-4=0$. Take $y=1$ and substitute in the second equation; thus we find $x=1$. Then take $x^2y+x-4=0$; the second equation is $x^3y^3+x(y-3y^2)+1=0$: substituting for x^2y from the former equation we have $(4-x)y^2+x(y-3y^2)+1=0$; therefore $4y^2(1-x)+xy+1=0$; therefore $4y^2(1-x)+\dfrac{4}{x}=0$; therefore $y^2=\dfrac{1}{x(x-1)}$. And $y=\dfrac{4-x}{x^2}$; therefore $\left(\dfrac{4-x}{x^2}\right)^2=\dfrac{1}{x(x-1)}$; therefore $(4-x)^2(x-1)=x^3$; therefore $16-24x+9x^2=0$; therefore $x=\dfrac{4}{3}$: and then $y=\dfrac{3}{2}$.

252. We have $a_1-a=a_1-a$, $a_2-a_1=a_1-a+d$, $a_3-a_2=a_1-a+2d,\ldots$ $a_r-a_{r-1}=a_1-a+(r-1)d$; therefore, by addition, $a_r-a=r(a_1-a)+\dfrac{r(r-1)}{2}d$. And as a_{n+1} is the same as b we get $b-a=(n+1)(a_1-a)+\dfrac{(n+1)nd}{2}$; therefore $a_1-a=\dfrac{b-a}{n+1}-\dfrac{n}{2}d$: this determines a_1-a, and then $a_2, a_3,\ldots$ become known. We thus find that $a_1=a+\dfrac{b-a}{n+1}-\dfrac{nd}{2}$, and that $a_n=b-\left\{\dfrac{b-a}{n+1}+\dfrac{nd}{2}\right\}$; we will suppose b greater than a: then in order that $a_1, a_2,\ldots a_n$ may lie between a and b we must have a_1 greater than a, and a_n less than b; thus $\dfrac{b-a}{n+1}-\dfrac{nd}{2}$ and $\dfrac{b-a}{n+1}+\dfrac{nd}{2}$ must both be positive, so that d must lie between $-\dfrac{2(b-a)}{n(n+1)}$ and $\dfrac{2(b-a)}{n(n+1)}$.

253. The value of r is here $\dfrac{3}{88}$. Suppose that the terminable annuity is £A per annum; then $100=\dfrac{3}{r}+\dfrac{A}{r}(1-R^{-30})$; therefore $12=\dfrac{A}{r}(1-R^{-30})$; therefore $A=12\times\dfrac{3}{88}\times\dfrac{1}{1-R^{-30}}$. Now $R=1+r=1+\dfrac{3}{88}=\dfrac{91}{88}$; therefore $\log R^{-30}=-30\log\dfrac{13\times 7}{11\times 8}=-30(\log 13+\log 7-\log 11-3\log 2)=-\cdot 43680$ $=-1+\cdot 56320=\log\cdot 3658$. Thus $A=12\times\dfrac{3}{88}\times\dfrac{1}{1-\cdot 3658}=\cdot 645$ nearly.

254. When $\sqrt{(a^2+1)}$ is converted into a continued fraction every quotient after the first is $2a$; thus $p_n=2ap_{n-1}+p_{n-2}$, $q_n=2aq_{n-1}+q_{n-2}$. The series (1) of Art. 614 becomes $0, a, a, a,\ldots$; and the series (2) becomes

1, 1, 1, ...: therefore by that Article $(a^2+1)\,q_n = ap_n + p_{n-1}$, $p_n = aq_n + q_{n-1}$. From the first of these four equations $p_{n-1} = \frac{p_n - p_{n-2}}{2a}$; change n into $n+1$, and substitute in the third equation: thus $(a_2+1)\,q_n = \frac{p_{n+1} - p_{n-1}}{2} + p_{n-1}$, therefore $2\,(a^2+1)\,q_n = p_{n+1} + p_{n-1}$. Again, from the second equation $q_{n-1} = \frac{q_n - q_{n-2}}{2a}$; change n into $n+1$ and substitute in the fourth equation: thus $p_n = \frac{q_{n+1} - q_{n-1}}{2} + q_{n-1}$; therefore $2p_n = q_{n+1} + q_{n-1}$.

255. Suppose that the boy bought x apples, y pears, and z peaches; then $x+y+z=12$, and $\frac{x}{5}+y+2z=12$. By subtraction $\frac{4x}{5}-z=0$: substitute for z in either equation; thus $\frac{9x}{5}+y=12$. Hence $\frac{x}{5}$ must be an integer; say $=t$; therefore $x=5t$, $y=12-9t$, and $z=4t$. If $t=0$ we get $x=0$, $y=12$, $z=0$; if $t=1$ we get $x=5$, $y=3$, $z=4$.

256. Assume $$\frac{a+bx}{(1-cx)\left(1-\frac{x}{c}\right)} = \frac{A}{1-cx} + \frac{B}{1-\frac{x}{c}};$$ therefore $a+bx = A\left(1-\frac{x}{c}\right) + B\,(1-cx)$. Since this is to be identically true we may give any value to x, suppose then $x=\frac{1}{c}$: thus $a+\frac{b}{c} = A\left(1-\frac{1}{c^2}\right)$, so that $A = -\frac{c\,(ac+b)}{1-c^2}$. Again, suppose that $x=c$: thus $a+bc = B\,(1-c^2)$, so that $B = \frac{a+bc}{1-c^2}$. Thus $$\frac{a+bx}{(1-cx)\left(1-\frac{x}{c}\right)} = \frac{a+bc}{1-c^2}\,\frac{1}{1-\frac{x}{c}} - \frac{c\,(ac+b)}{1-c^2}\,\frac{1}{1-cx};$$ and therefore the coefficient of x^n in the expansion is $\frac{a+bc}{1-c^2}\,\frac{1}{c^n} - \frac{c\,(ac+b)}{1-c^2}\,c^n$.

257. Consider the n numbers $1^2, 2^2, 3^2, \ldots n^2$; their arithmetical mean is $\frac{1^2+2^2+3^2+\ldots+n^2}{n}$, that is $\frac{(n+1)(2n+1)}{6}$; their geometrical mean is $\{1^2 . 2^2 . 3^2 \ldots n^2\}^{\frac{1}{n}}$, that is $\{\lfloor n\}^{\frac{2}{n}}$; the former is the greater by Art. 681. Again consider the n numbers $1^3, 2^3, 3^3, \ldots n^3$; and proceed as before.

258. By Fermat's Theorem $N^{n-1} = 1 + kn$ where k is some integer; raise both sides to the power n: thus $N^{2m} = (1+kn)^n = 1 + pn^2$ where p is some integer. Therefore $(N^m+1)\,(N^m-1) = pn^2$. Now N^m+1 and N^m-1 cannot both be divisible by n; for if they could, their difference, which is 2, would be divisible by n: and this is impossible except $n=2$. Thus when n is not equal to 2, we see that N^m+1 or N^m-1 must be divisible by n^2. Moreover if $n=2$ then N must be an odd number, so that N^m+1 and N^m-1 are both even; and therefore one of them must be divisible by 4, that is by 2^2.

259. It is an even chance whether the number taken is odd or even; in the former case the digit in the units' place of the square is odd, and in the latter case it is even.

Again, 0^2, 1^2, 2^2, 3^2, 5^2, 7^2, 8^2 and 9^2 may all be considered to have an even digit in the tens' place; and 4^2 and 6^2 have an odd digit. Also, N and k being any integer, N^2 and $(N+10k)^2$ have both an even digit or both an odd digit in the tens' place; so that it is sufficient to consider the cases $0^2, 1^2, 2^2, \dots 9^2$. Hence we see that $\frac{2}{10}$ or $\frac{1}{5}$ is the chance that the digit in the tens' place of the square is odd.

Similarly with respect to the digit in the next place it will be sufficient to consider the cases $0^2, 1^2, 2^2, \dots (99)^2$: moreover since $(50+a)^2-(50-a)^2=200a$ the results for the numbers between 51 and 99, both inclusive, are respectively the same as those for the numbers from 49 to 1, both inclusive. On trial it will be found that there are 59 cases in which the digit is even, and 41 cases in which the digit is odd. Hence we see that $\frac{41}{100}$ is the chance that the digit in the hundreds' place of the square is odd.

260. Suppose that the expression $=1+A_1x+A_2x^2+A_3x^3+\dots$, where A_1, A_2, $A_3, \dots$ do not contain x. Change x into cx; then we can infer that $\frac{1+cx}{1-cx}(1+A_1cx+A_2c^2x^2+A_3c^3x^3+\dots)=1+A_1x+A_2x^2+\dots$ Clear of fractions; then equate the coefficients of x^r: thus $A_rc^r+A_{r-1}c^r=A_r-A_{r-1}c$; therefore $A_{r-1}(c+c^r)=A_r(1-c^r)$; so that $A_r=\frac{c\,(1+c^{r-1})}{1-c^r}A_{r-1}$. This will determine in succession every coefficient: thus A_0 denotes 1; therefore $A_1=\frac{c\,(1+1)}{1-c}$; $A_2=\frac{c\,(1+c)}{1-c^2}A_1=\frac{c^2\,(1+1)\,(1+c)}{(1-c)(1-c^2)}$; and so on.

261. $\alpha^4+\alpha^2\beta^2+\beta^4=(\alpha^2+\beta^2)^2-\alpha^2\beta^2=\{(\alpha+\beta)^2-2\alpha\beta\}^2-\alpha^2\beta^2$

$$=\left(\frac{b^2}{a^2}-\frac{2c}{a}\right)^2-\left(\frac{c}{a}\right)^2=\frac{b^4}{a^4}-\frac{4b^2c}{a^3}+\frac{3c^2}{a^2}.$$

262. Let a be the first term, and b the common difference of the corresponding Arithmetical Progression; then $\frac{1}{n}=a+(m-1)\,b$, $\frac{1}{m}=a+(n-1)b$; hence we get $a=b=\frac{1}{mn}$. Let x denote the r^{th} term of the H.P., then $\frac{1}{x}=a+(r-1)\,b=\frac{r}{mn}$; therefore $x=\frac{mn}{r}$.

263. Let u_n denote the n^{th} term; then $u_n=\sqrt{u_{n-1}\,u_{n-2}}$; therefore $\log u_n=\frac{1}{2}(\log u_{n-1}+\log u_{n-2})$. Let $\log u_n=v_n$; thus $v_n=\frac{1}{2}(v_{n-1}+v_{n-2})$. Hence, as in Example 196, we find that $v_n=v_1+\frac{2}{3}(v_2-v_1)\left\{1-\left(-\frac{1}{2}\right)^{n-1}\right\}$; and so when n is large enough we have v_n differing as little as we please from

$v_1 + \frac{2}{3}(v_2 - v_1)$, that is from $\frac{2v_2 + v_1}{3}$: thus when n is very large we may put $\log u_n = \frac{2\log b + \log a}{3} = \log (b^2 a)^{\frac{1}{3}}$, so that $u_n = (b^2 a)^{\frac{1}{3}}$.

264. Suppose that on dividing b by a we have a quotient p and a remainder r; then $b = pa + r$. Therefore $\frac{a}{b}$ is greater than $\frac{1}{p+1}$; assume $\frac{a}{b} = \frac{1}{p+1} + \frac{a'}{b'}$; then $\frac{a'}{b'} = \frac{a(p+1) - b}{b(p+1)} = \frac{a-r}{b(p+1)} = \frac{a-r}{bq_1}$ where q_1 stands for $p+1$. In the same manner we can put $\frac{a'}{b'}$ in the form $\frac{1}{q_1 q_2} + \frac{a''}{b''}$ where $b'' = bq_1 q_2$, and a'' is less than $a - r$. Proceeding in this way we obtain the required form. If $b = 7$ and $a = 5$ we have $q_1 = 2$, $q_2 = 3$, $q_3 = 4$, $q_4 = 7$.

265. Suppose that x coins of the first kind and y of the second are taken, then $\frac{81x}{100} + \frac{666y}{1000} = 108$; therefore $90x + 74y = 12000$; therefore $45x + 37y = 6000$. Divide by 37; thus $x + y + \frac{2(4x-3)}{37} = 162$; therefore $\frac{4x-3}{37}$ must be an integer, say $= s$; thus $4x = 3 + 37s$. Divide by 4; thus $x = 9s + \frac{s+3}{4}$; therefore $\frac{s+3}{4}$ must be an integer, say $= t$; thus $s = 4t - 3$. Therefore $x = 37t - 27$ and $y = 195 - 45t$. The solutions are obtained by putting $t = 1, 2, 3$, or 4; so that we have $x = 10$, $y = 150$; $x = 47$, $y = 105$; $x = 84$, $y = 60$; $x = 121$, $y = 15$. The first solution gives the largest number of coins, and the smallest value in money; the last solution gives the smallest number of coins, and the largest value in money.

266. We have $p_n = 2ap_{n-1} + p_{n-2}$, $q_n = 2aq_{n-1} + q_{n-2}$; thus

$$\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = \frac{2a(p_{n-1}q_{n-2} - p_{n-2}q_{n-1})}{q_n q_{n-2}},$$

so that the factor $2a$ occurs in the numerator. This is true whether n is even or odd; there is however a difference between these two cases. We have $q_1 = 1$; hence the relation $q_n = 2aq_{n-1} + q_{n-2}$ shews in succession that $q_3, q_5, q_7, \ldots$ are not divisible by $2a$: on the other hand $q_2 = 2a$, and the same relation shews in succession that $q_4, q_6, q_8, \ldots$ are divisible by $2a$. Moreover $p_{n-1}q_{n-2} - p_{n-2}q_{n-1}$ is numerically equal to unity; so that when the fraction is reduced to its lowest terms if n is even the numerator is 1, but if n is odd the numerator is $2a$.

267. Let n be the number of terms: we have to shew that $\frac{a(1+r^{n-1})}{2}$ is greater than $\frac{a(1+r+r^2+\ldots+r^{n-1})}{n}$, or that $n(1+r^{n-1})$ is greater than $2(1+r+r^2+\ldots+r^{n-1})$. Now the right-hand member may be arranged into n pairs of the form $r^{m-1} + r^{n-m}$ where m takes in succession all the integral values from 1 to n inclusive. And $1 + r^{n-1}$ is greater than $r^{m-1} + r^{n-m}$; for $1 + r^{n-1} - (r^{m-1} + r^{n-m}) = (1 - r^{m-1})(1 - r^{n-m})$, and the two factors are both positive or both negative.

268. It was shewn in Example 228 that one of the three a, b, c is divisible by 5; and in a similar manner we can shew that either a or b must be divisible by 3. Now it is impossible that a and b can both be odd; for then a^2+b^2 would be divisible by 2 but not by 4, whereas c^2 is divisible by 4 if c is even, and is not divisible by 2 if c is odd. Hence either a and b are both even, or one of the two is even and the other odd. If a and b are even then c is even, and abc is divisible by 8. If a is odd and b even, then c is odd; let $a=2p+1$ and $c=2q+1$; then

$$b^2=(2q+1)^2-(2p+1)^2=4q(q+1)-4p(p+1);$$

thus b^2 is divisible by 8, therefore b must be divisible by 4. Thus in this case abc is divisible by 4. Therefore in both cases abc is divisible by $5\times3\times4$.

If a is a prime number greater than 3 it cannot be divisible by 3 or by 4; so that b must be divisible by both 3 and by 4, and therefore divisible by 3×4.

269. The chance of drawing two specified numbers is $\dfrac{2}{n(n-1)}$: hence the expectation in shillings is $\dfrac{2}{n(n-1)}\times P$, where P denotes the product of the first n natural numbers taken two and two together. Hence by Example 155 the value of the expectation in shillings is $\dfrac{2}{n(n-1)}\times\dfrac{(n-1)n(n+1)(3n+2)}{24}$, that is $\dfrac{(n+1)(3n+2)}{12}$.

270. Let p_r denote the probability that the individual will die during the r^{th} year; let q_r denote the probability that he will be alive at the end of the r^{th} year. Suppose £x the payment to be made immediately and repeated. Then the present value of £P to be received at the death of the individual is $P\left(\dfrac{p_1}{R}+\dfrac{p_2}{R^2}+\dfrac{p_3}{R^3}+\dots\right)$; and the present value of all the payments is $x\left(1+\dfrac{q_1}{R}+\dfrac{q_2}{R^2}+\dfrac{q_3}{R^3}+\dots\right)$: these two present values then must be equal. Moreover we have $p_r=q_{r-1}-q_r$, observing that q_0 is equivalent to 1, and $A=\dfrac{q_1}{R}+\dfrac{q_2}{R^2}+\dfrac{q_3}{R^3}+\dots$ Thus $x(1+A)=P\left(\dfrac{1}{R}+\dfrac{A}{R}-A\right)$, so that $x=\dfrac{P}{R}-\dfrac{PA}{1+A}$.

271. Let each of the fractions $=k$; then $k\log x=x(y+z-x)$, and $k\log y=y(z+x-y)$; multiply the former by y, and the latter by x, and add; thus $y\log x+x\log y=\dfrac{2xyz}{k}$; therefore $y^x x^y=e^{\frac{2xyz}{k}}$. Similarly $x^z z^x$ and $y^z z^y$ may each be shewn equal to the same.

272. Transpose the first equation and square; thus

$$(x^2+a^2)(y^2+b^2)=(x^2+b^2)(y^2+a^2)-2(a+b)^2\sqrt{(x^2+b^2)(y^2+a^2)}+(a+b)^4;$$

therefore $\quad (x^2-y^2)(a^2-b^2)-2(a+b)^2\sqrt{(x^2+b^2)(y^2+a^2)}+(a+b)^4=0.$

But $x^2-y^2=(x+y)(x-y)=(a+b)(x-y)$; therefore dividing by $(a+b)^2$ and transposing we get $(x-y)(a-b)+(a+b)^2=2\sqrt{(x^2+b^2)(y^2+a^2)}$. The left-hand member of the last equation $=(x-y)(a-b)+(x+y)(a+b)=2(xa+yb)$ therefore $xa+yb=\sqrt{(x^2+b^2)(y^2+a^2)}$; square thus $2xyab=x^2y^2+a^2b^2$; therefore $(xy-ab)^2=0$; therefore $xy=ab$. Combine this with $x+y=a+b$.

273. Let x^2 denote one of the square numbers, and y the corresponding quotient; then $x^2=7y+4$; therefore $y=\frac{x^2-4}{7}=\frac{(x+2)(x-2)}{7}$. Thus either $x+2$ or $x-2$ must be divisible by 7, so that we must have $x=7t\pm 2$, where t is an integer; and $y=7t^2\pm 4t$.

274. We have $p_n=2ap_{n-1}+p_{n-2}$. Thus $p_1+p_2x+p_3x^2+\dots$ is a *recurring* series; and its sum by the method of Art. 656 is $\frac{p_1+(p_2-2ap_1)x}{1-2ax-x^2}$. Hence p_n is the coefficient of x^{n-1} in the expansion of this expression. Moreover $p_1=a$, $p_2=2a^2+1$; hence the expression becomes $\frac{a+x}{1-2ax-x^2}$. Now by Art. 337 we have $1-2ax-x^2=-(x-\alpha)(x-\beta)$ where α and β are the roots of the equation $x^2+2ax-1=0$, so that $\alpha+\beta=-2a$. Hence the expression becomes $-\frac{x-\frac{1}{2}(\alpha+\beta)}{(x-\alpha)(x-\beta)}$, that is $\frac{1}{2}\left(\frac{1}{\alpha-x}+\frac{1}{\beta-x}\right)$, that is $-\frac{1}{2}\left(\frac{\beta}{1+\beta x}+\frac{\alpha}{1+\alpha x}\right)$, since $\alpha\beta=-1$. Hence the coefficient of x^{n-1}, or p_n, is $-\frac{1}{2}(\beta^n+\alpha^n)(-1)^{n-1}$, that is $\frac{1}{2}\{(-\alpha)^n+(-\beta)^n\}$, that is $\frac{1}{2}\{(a+\sqrt{a^2+1})^n+(a-\sqrt{a^2+1})^n\}$.

Similarly q_n is the coefficient of x^{n-1} in the expansion of $\frac{q_1+(q_2-2aq_1)x}{1-2ax-x^2}$, that is of $\frac{1}{1-2ax-x^2}$, that is of $-\frac{1}{(x-\alpha)(x-\beta)}$, that is of $\frac{1}{\beta-\alpha}\left(\frac{1}{x-\alpha}-\frac{1}{x-\beta}\right)$, that is of $\frac{1}{\beta-\alpha}\left(\frac{\beta}{1+\beta x}-\frac{\alpha}{1+\alpha x}\right)$: so that we have $q_n=\frac{(-1)^n}{\alpha-\beta}(\beta^n-\alpha^n)$ $=\frac{1}{\alpha-\beta}\{(-\beta)^n-(-\alpha)^n\}=\frac{1}{2\sqrt{(a^2+1)}}\{(a+\sqrt{a^2+1})^n-(a-\sqrt{a^2+1})^n\}$.

275. Assume $\frac{1+7x-x^2}{(1+3x)^2(1-10x)}=\frac{A}{1-10x}+\frac{Bx+C}{(1+3x)^2}$; then

$$1+7x-x^2=A(1+3x)^2+(Bx+C)(1-10x).$$

Since this is to be identically true we may give any value to x; suppose then $x=\frac{1}{10}$: thus $1+\frac{7}{10}-\frac{1}{100}=\frac{169}{100}A$, so that $A=1$. Therefore

$$1+7x-x^2-(1+3x)^2=(Bx+C)(1-10x),$$

that is $x(1-10x)=(Bx+C)(1-10x)$; therefore $x=Bx+C$; therefore $B=1$ and $C=0$. Thus $\frac{1+7x-x^2}{(1+3x)^2(1-10x)}=\frac{1}{1-10x}+\frac{x}{(1+3x)^2}=(1-10x)^{-1}+x(1+3x)^{-2}$. Hence the coefficient of x^n in the expansion is $10^n+n(-3)^{n-1}$.

276. Assume that $18=4p+q$, $80=18p+4q$; hence we obtain $p=4$, $q=2$: and these values verify $356=80p+18q$. Assume that $8=3p+2q+r$, $13=8p+3q+2r$, $30=13p+8q+3r$, hence we obtain $p=0$, $q=3$, $r=2$: and these values verify $55=30p+13q+8r$.

277. I. If $m=1$ the inequality becomes an equality. II. If $m=2$ we have to shew that $(n-1)(a^2+b^2+c^2+\ldots+k^2)$ is greater than $2(ab+ac+\ldots+bc+\ldots)$: this coincides with Example LI. 20. III. If $m=3$ we have to shew that $(n-1)(n-2)(a^3+b^3+c^3+\ldots+k^3)$ is greater than $2.3.(abc+abd+\ldots+bcd+\ldots)$. The right-hand member $=2(aA+bB+cC+\ldots)$ where A denotes the product of all the letters except a, taken two together; B denotes the product of all the letters except b, taken two together; and so on. For instance, the term abc will occur in aA, in bB, and in cC; so that it will occur 3 times on the whole. Now by II. we know that $2A$ is less than $(n-2)$ times the sum of the squares of all the letters except a; so that if we denote the sum of the squares of all the letters by Σ_2 we have $2A$ less than Σ_2-a^2: similarly $2B$ is less than Σ_2-b^2, $2C$ is less than Σ_2-c^2, and so on. Thus the right-hand member is less than $(n-2)\{a(\Sigma_2-a^2)+b(\Sigma_2-b^2)+c(\Sigma_2-c^2)+\ldots\}$, that is the right-hand member is less than $(n-2)\{\Sigma_1\Sigma_2-\Sigma_3\}$, where Σ_1 stands for $a+b+c+\ldots+k$, and Σ_3 for $a^3+b^3+c^3+\ldots+k^3$. And $\Sigma_1\Sigma_2$ is less than $n\Sigma_3$ by Example 247; therefore finally the right-hand member is less than $(n-1)(n-2)\Sigma_3$. IV. Suppose $m=4$. This case is established by the aid of III. and Example 247, in the same way as III. was established by the aid of II. and Example 247. And so on.

278. By Fermat's Theorem $a^{n-1}-1$ is a multiple of n. Now either a or $a^{n-1}-1$ must be an even number, and so be a multiple of 2; hence since n is not equal to 2 we have $a(a^{n-1}-1)$ a multiple of $2n$, that is a^n-a is a multiple of $2n$. Therefore a^n and a must end with the same digit when expressed in the scale whose radix is $2n$.

279. There are four hypotheses with respect to the coins to be regarded as equally probable before the observed event. (1) 5 shillings. (2) 4 shillings and 1 sovereign. (3) 3 shillings and 2 sovereigns. (4) 2 shillings and 3 sovereigns. The probability of the observed event on these hypotheses is respectively $1, \frac{6}{10}, \frac{3}{10}, \frac{1}{10}$. Hence after the observed event the probabilities of the hypotheses are respectively $\frac{10}{20}, \frac{6}{20}, \frac{3}{20}, \frac{1}{20}$. Now on the first hypothesis there are 3 shillings left in the bag, and so the value of a draw is 2 shillings. On the second hypothesis there are 2 shillings and 1 sovereign left in the bag, and the value of a draw is $\frac{2+2\times 21}{3}$ shillings, that is $\frac{44}{3}$ shillings. On the third hypothesis there are 1 shilling and 2 sovereigns left in the bag, and the value of a draw is $\frac{2\times 20+2\times 21}{3}$ shillings, that is $\frac{82}{3}$ shillings. On the last hypothesis there are 3 sovereigns left in the bag, and the value of a draw is 40 shillings. Hence the whole value of another draw in shillings is $\frac{10}{20}\times 2+\frac{6}{20}\times\frac{44}{3}+\frac{3}{20}\times\frac{82}{3}+\frac{1}{20}\times 40$, that is $\frac{1}{10}\left\{10+44+41+20\right\}$, that is $\frac{115}{10}$, that is $11\frac{1}{2}$.

280. As in Example 246 we can shew that

$$\frac{1}{m}(1+x)^n - \frac{n}{m(m+1)}(1+x)^{n-1} + \frac{n(n-1)}{m(m+1)(m+2)}(1+x)^{n-2} - \ldots$$

$$= \frac{1}{m+n} + n\frac{x}{m+n-1} + \frac{n(n-1)}{1.2}\,\frac{x^2}{(m+n-2)} + \ldots$$

Change the sign of x, and subtract; thus we obtain the required result.

281. Let $\frac{x^2-2x-3}{2x^2+2x+1}=y$; then $x^2-2x-3=y(2x^2+2x+1)$; therefore $x^2(2y-1)+2x(y+1)+y+3=0$; by solving this quadratic we obtain $x=\frac{-(y+1)\pm\sqrt{-(y+4)(y-1)}}{2y-1}$: hence y must lie between 1 and -4 in order that x may be real.

282. Square the first equation; thus $x+y=a-2(xy)^{\frac{1}{2}}$; square again; thus $x^2+y^2=a^2-4a(xy)^{\frac{1}{2}}+2xy$. Transpose the second equation and square; thus $x^2+y^2=b^2-2b(2xy)^{\frac{1}{2}}+2xy$. Hence, by using the former result we have $a^2-4a(xy)^{\frac{1}{2}}+2xy=b^2-2b(2xy)^{\frac{1}{2}}+2xy$; therefore $(xy)^{\frac{1}{2}}=\frac{a^2-b^2}{4a-2b\sqrt{2}}$. Square the first equation, and subtract four times the last; thus $\left(x^{\frac{1}{2}}-y^{\frac{1}{2}}\right)^2 = a-\frac{4(a^2-b^2)}{4a-2b\sqrt{2}}=\frac{2b^2-ab\sqrt{2}}{2a-b\sqrt{2}}$. Since $x^{\frac{1}{2}}+y^{\frac{1}{2}}$ and $x^{\frac{1}{2}}-y^{\frac{1}{2}}$ are thus known we can find $x^{\frac{1}{2}}$ and $y^{\frac{1}{2}}$, and thence x and y.

283. We have $\frac{p_1}{q_1}=\frac{1}{a}$, $\frac{p_2}{q_2}=\frac{b}{ab+1}$; and so on: thus we shall find that if n is even $p_n=bp_{n-1}+p_{n-2}$, and that if n is odd $p_n=ap_{n-1}+p_{n-2}$. Suppose n even; then $p_n=bp_{n-1}+p_{n-2}$, $p_{n+1}=ap_n+p_{n-1}$, $p_{n+2}=bp_{n+1}+p_n$; thus from the first of these equations $p_{n-1}=\frac{p_n-p_{n-2}}{b}$; therefore $p_{n+1}=ap_n+\frac{p_n-p_{n-2}}{b}$, therefore $p_{n+2}=abp_n+p_n-p_{n-2}+p_n$; that is $p_{n+2}=(ab+2)p_n-p_{n-2}$. And we shall find that this relation also holds when n is odd. Thus we see that $p_1+p_2x+p_3x^2+\ldots$ is a *recurring* series; and its sum by the method of Art. 656 is $\frac{p_1+p_2x+\{p_3-p_1(ab+2)\}x^2+\{p_4-p_2(ab+2)\}x^3}{1-(ab+2)x^2+x^4}$. Hence p_n is the coefficient of x^{n-1} in the expansion of this expression. Moreover $p_1=1$, $p_2=b$, $p_3=ab+1$, $p_4=ab^2+2b$; hence the expression becomes $\frac{1+bx-x^2}{1-(ab+2)x^2+x^4}$.

Similarly we proceed with respect to q_n, observing that $q_1=a$, $q_2=ab+1$, $q_3=a^2b+2a$, $q_4=a^2b^2+3ab+1$.

284. $\dfrac{1+bx-x^2}{1-(ab+2)x^2+x^4}=\dfrac{1+bx-x^2}{(x^2-\lambda)(x^2-\mu)}$; assume that this $=\dfrac{Ax+B}{x^2-\lambda}+\dfrac{Cx+D}{x^2-\mu}$; then $1+bx-x^2=(Ax+B)(x^2-\mu)+(Cx+D)(x^2-\lambda)$. As this is to be identically true we may equate the coefficients of the various powers of x; thus

$$0=A+C,\ -1=B+D,\ b=-A\mu-C\lambda,\ 1=-B\mu-D\lambda;$$

therefore $$A=\frac{b}{\lambda-\mu},\ B=\frac{1-\lambda}{\lambda-\mu},\ C=-\frac{b}{\lambda-\mu},\ D=-\frac{1-\mu}{\lambda-\mu}.$$

Thus $$\frac{1+bx-x^2}{1-(ab+2)x^2+x^4}=\frac{1}{\lambda-\mu}\left\{\frac{bx+1-\lambda}{x^2-\lambda}-\frac{bx+1-\mu}{x^2-\mu}\right\}$$

$$=\frac{1}{\lambda-\mu}\left\{\frac{\lambda bx+\lambda(1-\mu)}{1-\lambda x^2}-\frac{\mu bx+\mu(1-\lambda)}{1-\mu x^2}\right\},\ \text{since } \lambda\mu=1.$$

Hence the coefficient of $x^{2n-1}=\dfrac{b}{\lambda-\mu}(\lambda^n-\mu^n)$; and the coefficient of x^{2n} $=\dfrac{1}{\lambda-\mu}\left\{\lambda^{n+1}(1-\mu)-\mu^{n+1}(1-\lambda)\right\}=\dfrac{1}{\lambda-\mu}\left\{\lambda^{n+1}-\mu^{n+1}-\lambda^n+\mu^n\right\}$. The former coefficient $=p_{2n}$, and the latter $=p_{2n+1}$ by Example 283.

In like manner we find that

$$\frac{a+(ab+1)x-x^3}{1-(ab+2)x^2+x^4}=\frac{1}{\lambda-\mu}\left\{\frac{a\lambda+(\lambda-1)\lambda x}{1-\lambda x^2}-\frac{a\mu+(\mu-1)\mu x}{1-\mu x^2}\right\}.$$

Hence the coefficient of $x^{2n-2}=\dfrac{a}{\lambda-\mu}(\lambda^n-\mu^n)$; and the coefficient of x^{2n-1} $=\dfrac{1}{\lambda-\mu}\left\{\lambda^n(\lambda-1)-\mu^n(\mu-1)\right\}$. The former coefficient $=q_{2n-1}$, and the latter $=q_{2n}$ by Example 283.

285. Let x denote the digit in the tens' place of the second number, and y the digit in the units' place; then the second number is $10x+y$. The first number $=xy$ and also $=2(10x+y)-100$. Therefore $xy=2(10x+y)-100$; thus $x(20-y)=100-2y$, therefore $x=\dfrac{100-2y}{20-y}=2+\dfrac{60}{20-y}$. Therefore $20-y$ must be a divisor of 60; and neither x nor y can be greater than 9; hence we shall obtain as the only admissible solutions $20-y=12$, so that $y=8$ and $x=7$; or $20-y=15$, so that $y=5$ and $x=6$. Thus the numbers are 56 and 78, or 30 and 65.

286. Let x_r denote the probability that a person of the given age will be alive at the end of r years; then x_r^m is the probability that the m persons of the given age will be alive at the end of r years: therefore if B be the annual payment $A_m=B\left(\dfrac{x_1^m}{R}+\dfrac{x_2^m}{R^2}+\dfrac{x_3^m}{R^3}+\dots\right)$. Now $1-x_r$ is the probability that a person of the given age will be dead at the end of r years; therefore $(1-x_r)^n$ is the probability that the n persons will all be dead at the

end of r years; and therefore $1-(1-x_r)^n$ is the probability that they will not all be dead. Hence the value of the annuity to continue as long as there is a survivor out of the n persons is

$$B\left\{\frac{1-(1-x_1)^n}{R}+\frac{1-(1-x_2)^n}{R^2}+\frac{1-(1-x_3)^n}{R^3}+\ldots\right\}.$$

Expand the binomials; thus we get a set of terms of which the general form is $-(-1)^r B\frac{n(n-1)\ldots(n-r+1)}{\underline{|r}}\left\{\frac{x_1^r}{R}+\frac{x_2^r}{R^2}+\frac{x_3^r}{R^3}+\ldots\right\}$, and from what has been shewn above this is $-(-1)^r\frac{n(n-1)\ldots(n-r+1)}{\underline{|r}}A_r$.

287. If x, y, z are all equal the expression vanishes; if two of them are equal the expression is obviously positive; if they are all unequal let y be that which is algebraically intermediate between the other two. Let $b^2=(a+c)^2-\lambda$, so that λ is necessarily positive. Substitute for b^2 and the expression becomes $\{a(y-x)+c(y-z)\}^2-\lambda(y-x)(y-z)$: this is positive since $y-x$ and $y-z$ are of different signs.

288. Every number is of one of the forms $5n$, $5n\pm1$, $5n\pm2$. Now the square of $5n$ is $25n^2$; the square of $5n\pm1$ is of the form $5p+1$; the square of $5n\pm2$ is of the form $5q+4$, that is of the form $5(q+1)-1$. Hence every square number is of one of the forms $5m$, $5m+1$, $5m-1$.

Now $n^5-n=n(n^4-1)=n(n-1)(n+1)(n^2+1)$. The product $(n-1)n(n+1)$ is divisible by $\underline{|3}$. If neither $n-1$, nor n, nor $n+1$ is divisible by 5, then n is of the form $5m+2$ or $5m-2$; and then n^2+1 is divisible by 5. If n be odd then $n-1$, $n+1$, and n^2+1 are all even; and either $n-1$ or $n+1$ is divisible by 4: thus $(n-1)(n+1)(n^2+1)$ is divisible by $2\times4\times2$, that is by 16. Hence $n(n-1)(n+1)(n^2+1)$ is divisible by $16\times3\times5$, that is by 240.

289. There are n hypotheses which are to be regarded as equally probable before the observed event; namely that all the balls are black, that all but one are black, that all but two are black, and so on. The probability of the observed event on these hypotheses is respectively

$$1,\ \left(\frac{n-1}{n}\right)^2,\ \left(\frac{n-2}{n}\right)^2,\ \ldots\ \left(\frac{1}{n}\right)^2.$$

Hence after the observed event the probabilities of the hypotheses are respectively $\frac{n^2}{S}$, $\frac{(n-1)^2}{S}$, $\frac{(n-2)^2}{S}$, ... $\frac{1^2}{S}$, where S stands for

$$n^2+(n-1)^2+(n-2)^2+\ldots+1^2,\text{ that is for }\frac{n(n+1)(2n+1)}{6}.$$

The probability that a third draw will give a black ball is

$$\frac{n^2}{S}\times1+\frac{(n-1)^2}{S}\times\frac{n-1}{n}+\frac{(n-2)^2}{S}\times\frac{n-2}{n}+\ldots\ldots+\frac{1^2}{S}\times\frac{1}{n},\text{ that is}$$

$$\frac{1}{nS}\left\{n^3+(n-1)^3+(n-2)^3+\ldots+1^3\right\},\text{ that is }\frac{1}{nS}\left\{\frac{n(n+1)}{2}\right\}^2,\text{ that is }\frac{3n+3}{4n+2}.$$

290. Put $x=\frac{1}{y}$, thus we get $\frac{n^p}{y^n}$. Now, by Art. 542,

$$y^n=1+(\log y)\,n+\frac{(\log y)^2n^2}{\lfloor 2}+\ldots+\frac{(\log y)^rn^r}{\lfloor r}+\ldots.$$

It is obvious that by taking n large enough y^n can be made greater than any assigned multiple of n^p. In fact if r be greater than p the single term $\frac{(\log y)^rn^r}{\lfloor r}$ can be made greater than any assigned multiple of n^p. Thus $\frac{n^p}{y^n}$ is indefinitely small when n is indefinitely large.

291. We have $x=\frac{2x^3}{1+x^3}$; therefore $1+x^3=2x^2$; therefore $1-x^2-x^2(1-x)=0$. Dividing by $1-x$ we get $1+x-x^2=0$; and by solving this quadratic we have $x=\frac{1\pm\sqrt{5}}{2}$. Again $y^2=\frac{2y^3}{1+y^3}$; therefore $1+y^3=2y$; therefore $1-y-y(1-y^2)=0$. Dividing by $1-y$ we get $1-y(1+y)=0$; and solving this quadratic we have $y=\frac{-1\pm\sqrt{5}}{2}$. Since $x+y$ is not to be zero we must take the upper sign for both x and y or the lower sign for both. Hence $-y^2=\frac{-3\pm\sqrt{5}}{2}$, $y=\frac{-1\pm\sqrt{5}}{2}$, $x=\frac{1\pm\sqrt{5}}{2}$, $x^2=\frac{3\pm\sqrt{5}}{2}$: thus these four terms are in Arithmetical Progression, the common difference being 1. The sum of the four terms is $\pm2\sqrt{5}$; and it will be found that x^3+y^3 also$=\pm2\sqrt{5}$.

292. Proceed as in Art. 473.

Let $\qquad s=1^2r+3^2r^2+5^2r^3+\ldots+(2n-1)^2r^n$; then

$$sr=\qquad 1^2r^2+3^2r^3+\ldots+(2n-3)^2r^n+(2n-1)^2r^{n+1};$$

therefore by subtraction

$$s(1-r)=r+8r\{r+2r^2+3r^3+\ldots+(n-1)r^{n-1}\}-(2n-1)^2r^{n+1};$$

Then, by Example XXXI. 18 we have $s(1-r)$

$$=r+8r\left\{\frac{r-(n-1)r^n}{1-r}+\frac{r^2(1-r^{n-2})}{(1-r)^2}\right\}-(2n-1)^2r^{n+1};$$

and it will be found that this reduces to

$$\frac{1}{(1-r)^3}\left[r(1+6r+r^2)-\{(2n-1)(1-r)+2\}^2r^{n+1}-4r^{n+2}\right].$$

293. Here $\frac{u_{n+1}}{u_n}=\left(\frac{2n+1}{2n-1}\right)^2r=\left(1+\frac{2}{2n-1}\right)^2r$; this can be brought as near to r as we please by taking n large enough: thus if r is less than unity the series is convergent by Art. 559. Now when n is made indefinitely great r^{n+2} is indefinitely small; and so is $\{(2n-1)(r-1)+2\}^2r^{n+1}$, as we see by Example 290. Hence the sum of the series continued to infinity is $\frac{r(1+6r+r^2)}{(1-r)^3}$.

294. If $\sqrt{7}$ is converted into a continued fraction the quotients are 2, 1, 1, 1, 4, 1, 1, 1, 4,...; the first two convergents preceding those formed with the quotient 4 are $\frac{8}{3}$ and $\frac{127}{48}$. Thus $x=8$, and $y=3$ is one solution; and $x=127$, and $y=48$ is another.

295. Since the first quotient is 5 we have $N=25+x$, where x is either 1 or 2; for since the second quotient is 5 we have $\sqrt{N}$ less than $5+\frac{1}{5}$, and therefore N less than $27+\frac{1}{25}$. Thus $\sqrt{25+x}=5+\sqrt{25+x}-5=5+\frac{x}{\sqrt{25+x}+5}$. Then $\frac{\sqrt{25+x}+5}{x}=5+$ a proper fraction, by supposition. If we put for x in succession 1 and 2, we find that the greatest integer in $\frac{\sqrt{25+x}+5}{x}$ is respectively 10 and 5: thus $x=2$ is the only admissible value, and $N=27$.

296. We have to shew that if x is positive $\frac{x^3+2a^3}{x}$ is greater than $\frac{a^3+2a^3}{a}$: now it will be found that $\frac{x^3+2a^3}{x}-3a^2=\frac{(x-a)^2(x+2a)}{x}$, which is necessarily positive: this establishes the theorem.

297. Suppose the steamer to move at the rate of v miles per hour: then the voyage lasts $\frac{2000}{v}$ hours. The number of tons of coal consumed in an hour is Av^3, where A is some constant; but when $v=15$ the amount is 1·5 tons: therefore $A(15)^3=1\cdot5$, so that $A=\frac{1}{2250}$. Hence $\frac{v^3}{2250}$ tons of coal are consumed per hour; and the cost of these in shillings is $\frac{18v^3}{2250}$. Therefore the whole cost for the voyage is $\frac{2000}{v}\left(\frac{18v^3}{2250}+16\right)$ shillings, that is $16\left(v^2+\frac{2000}{v}\right)$. By Example 296 the least cost is when $v^3=1000$, that is when $v=10$, and it is 4800 shillings.

298. Since the number is odd and has an even digit in the tens' place it will be of the form $20p+q$, where p is some integer, and q stands for 1, 3, 5, 7, or 9. If this be raised to the power n the result consists of q^n+ a multiple of 20. Thus the digit in the tens' place will be even provided the digit in the tens' place of q^n is even; and this is known to be the case: see Example 218.

299. The sum of the coefficients of the *even* powers of x in the expansion of $(x+x^2+x^3)^4$ gives the number of cases in which the sum of the numbers drawn is *even*; and a similar result is true for the *odd* powers of x. Now $(x+x^2+x^3)^4=(x+x^3)^4+4x^2(x+x^3)^3+6x^4(x+x^3)^2+4x^6(x+x^3)+x^8$. Hence the sum of the even coefficients is $(1+1)^4+6(1+1)^2+1$, that is 41; and the sum of the odd coefficients is $4(1+1)^3+4(1+1)$, that is 40.

800. This may be demonstrated by induction. We can easily verify that it holds when $n=1$, or when $n=2$. Let $\frac{p_n}{q_n}$ denote the value of the continued fraction when n components are used. Then, by Art. 781, we have $\frac{p_{n+1}}{q_{n+1}} = \frac{\frac{2}{n+1} p_n + p_{n-1}}{\frac{2}{n+1} q_n + q_{n-1}}$. We shall now shew that we may always take $q_n = n+1$ and $p_n = (n+1) S_n$, where S_n denotes $\frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \dots + \frac{(-1)^{n+1}}{n(n+1)}$. For assume that these relations hold up to p_n and q_n inclusive; then

$$\frac{p_{n+1}}{q_{n+1}} = \frac{2S_n + nS_{n-1}}{2+n} = \frac{2S_n + n\left\{S_n - \frac{(-1)^{n+1}}{n(n+1)}\right\}}{n+2} = S_n + \frac{(-1)^{n+2}}{(n+1)(n+2)} = S_{n+1}.$$

Hence we may take $p_{n+1} = (n+2) S_{n+1}$ and $q_{n+1} = n+2$; so that the relations which hold up to p_n and q_n hold also when n is changed into $n+1$. Thus they hold universally. And

$$S = 1 - \frac{1}{2} - \left(\frac{1}{2} - \frac{1}{3}\right) + \frac{1}{3} - \frac{1}{4} - \left(\frac{1}{4} - \frac{1}{5}\right) + \dots$$

$$= -1 + 2\left\{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\right\};$$

thus when n is infinite we have $S = 2 \log 2 - 1$.

THE END.

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